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ON MINIMAL AND MAXIMAL SOLUTIONS OF TWO–POINT SINGULAR BOUNDARY VALUE PROBLEMS

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We consider the differential equation

$$u'' = f(t, u, u') \tag{1}$$

with the boundary conditions

$$u(a+) = c_1, \quad u(b-) = c_2, \tag{2_1}$$

or

$$u(a+) = c_1, \quad u'(b-) = c_2, \tag{2_2}$$

where $-\infty < a < b < +\infty$, $c_i \in R$ $(i = 1, 2)$, and $f :]a, b[\times R^2 \to R$ satisfies the local Carathéodory conditions.

In the case, where the function

$$f^*_r(t) = \max\{|f(t, x, y)| : \ |x| + |y| \le r\}$$

is Lebesgue integrable on $[a, b]$ for an arbitrary $r > 0$, problems $(1), (2_1)$ and $(1), (2_2)$ are called regular, otherwise they are called singular.

The basis for the theory of regular problems of the type $(1), (2_1)$ and $(1), (2_2)$ were laid in the classical works by S. N. Bernshtein [4], M. Nagumo [15] and H. Epheser [5].

From these works originates the tradition of formulation of theorems on solvability of the above-mentioned problems in terms of so-called lower and upper functions of Eq. (1). Precisely, these theorems contain sufficient conditions for existence of a solution of problem $(1), (2_1)$ or $(1), (2_2)$, satisfying the inequalities

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \quad \text{for} \quad a < t < b, \tag{3}$$

where σ_1 and σ_2 are, respectively, the lower and upper functions of Eq. (1) such that

$$\sigma_1(t) \le \sigma_2(t) \quad \text{for} \quad a < t < b. \tag{4}$$

Nowadays there exists a complete enough theory for solvability of the singular boundary value problems $(1), (2_k), (3)$ $(k = 1, 2)$. The results, obtained in this direction, are contained, e.g., in [1]–[3], [6]–[14], [16]. However, the question for these problems to have minimal and maximal solutions remains so far open. We made an attempt to fill to some extent the existing gap.

To formulate our results, we use the following notations.

$$R =]-\infty, +\infty[\,, \quad R_+ = [0, +\infty[\,.$$

$u(t+)$ and $u(t-)$ are, respectively, the right and the left limits of the function u at the point t.

$\widetilde{C}^1([t_1, t_2])$ is the space of functions $u : [t_1, t_2] \to R$ which are absolutely continuous together with their first derivatives.

$L([t_1, t_2])$ is the space of Lebesgue integrable functions $u : [t_1, t_2] \to R$.

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$\tilde{C}^1_{loc}(I)$ and $L_{loc}(I)$, where $I \subset R$ is an open or a half-open interval, is the set of functions $u : I \to R$ whose restrictions to any closed interval $[t_1, t_2] \subset I$ belong to the class $\tilde{C}^1([t_1, t_2])$ and $L([t_1, t_2])$, respectively.

The function $u \in \tilde{C}^1_{loc}(]a, b[)$ is said to be a solution of Eq. (1) if it satisfies this equation almost everywhere on $]a, b[$.

The solution u of Eq. (1), satisfying conditions (2_k) and (3), is said to be a solution of problem $(1), (2_k), (3)$.

The solution $\overline{u}$ (the solution $\underline{u}$) of problem $(1), (2_k), (3)$ is said to be **a maximal solution** (**a minimal solution**) if an arbitrary solution u of this problem satisfies the inequality

$$u(t) \leq \overline{u}(t) \quad \big(u(t) \geq \underline{u}(t)\big) \quad \text{for} \quad a < t < b.$$

Following [6], let us introduce the definition.

Definition 1. A function $\sigma :]a, b[\to R$ is said to be **a lower (**resp. **an upper) function** of Eq. (1) if:

(i) σ is locally absolutely continuous and σ' admits the representation

$$\sigma'(t) = \gamma(t) + \gamma_0(t),$$

where $\gamma :]a, b[\to R$ is a locally absolutely continuous function, while $\gamma_0 :]a, b[\to R$ is a non-decreasing (resp. non-increasing) function whose derivative is equal to zero almost everywhere on $]a, b[$;

(ii) the inequality

$$f\big(t, \sigma(t), \sigma'(t)\big) \leq \sigma''(t) \quad \Big(\text{resp. } f\big(t, \sigma(t), \sigma'(t)\big) \geq \sigma''(t)\Big)$$

holds almost everywhere on $]a, b[$.

Throughout the paper it is supposed that $f(\cdot, x, y) :]a, b[\to R$ is measurable for any $(x, y) \in R^2$ and $f(t, \cdot, \cdot) : R^2 \to R$ is continuous for almost all $t \in]a, b[$. Moreover, the functions $\sigma_1 :]a, b[\to R$ and $\sigma_2 :]a, b[\to R$ are, respectively, the lower and the upper functions of Eq. (1), satisfying condition (4).

Problem $(1), (2_1), (3)$ is investigated under the assumptions that

$$\begin{gathered} \text{there exist finite limits } \sigma_i(a+), \quad \sigma_i(b-) \quad (i = 1, 2), \\ \text{and } c_1 \in \big[\sigma_1(a+), \sigma_2(a+)\big], \quad c_2 \in \big[\sigma_1(b-), \sigma_2(b-)\big], \end{gathered} \tag{5_1}$$

and

$$f^*_r \in L_{loc}(]a, b[) \quad \text{for} \quad r > 0. \tag{6_1}$$

As for problem $(1), (2_2), (3)$, it is investigated under the assumptions that

$$\begin{gathered} \text{there exist finite limits } \sigma_i(a+), \quad \sigma'_i(b-) \quad (i = 1, 2), \quad \sigma'_1(b-) \leq \sigma'_2(b-), \\ \text{and } c_1 \in \big[\sigma_1(a+), \sigma_2(a+)\big], \quad c_2 \in \big[\sigma'_1(b-), \sigma'_2(b-)\big], \end{gathered} \tag{5_2}$$

and

$$f^*_r(\cdot) \in L_{loc}(]a, b]) \quad \text{for} \quad r > 0. \tag{6_2}$$

Definition 2. A function f belongs to the class $B_1(\sigma_1, \sigma_2)$ if there exist numbers $a_0 \in]a, b[$, $b_0 \in]a_0, b[$, and a continuous function $\rho \in]a, b[\to R_+$ such that $\rho \in L([a, b])$, and for any $t_1 \in]a, a_0[$, $t_2 \in]b_0, b[$ and a continuous function $\eta :]a, b[\to [0, 1]$, an arbitrary solution $u :]t_1, t_2[\to R$ of the differential equation

$$u'' = \eta(t) f(t, u, u'), \tag{7}$$

satisfying the condition

$$\sigma_1(t) \leq u(t) \leq \sigma_2(t) \quad \text{for} \quad t_1 < t < t_2,$$

admits the estimate

$$|u'(t)| \leq \rho(t) \quad \text{for} \quad t_1 \leq t \leq t_2.$$

Definition 3. A function f belongs to the class $B_2(\sigma_1,\sigma_2)$ if there exist $a_0 \in]a,b[$ and a continuous function $\rho \in]a,b[\to R_+$ such that $\rho \in L([a,b])$ and for any $t_0 \in]a,a_0[$ and a continuous function $\eta :]a,b[\to [0,1]$, an arbitrary solution $u : [t_0,b] \to R$ of the differential equation (7), satisfying the conditions

$$\sigma_1(t) \le u(t) \le \sigma_2(t) \ \text{ for } \ t_0 \le t \le b, \ \ \sigma_1(b-) \le u'(b-) \le \sigma_2(b-),$$

admits the estimate

$$|u'(t)| \le \rho(t) \quad \text{for} \quad t_0 \le t \le b.$$

Everywhere below the function $\omega : R \to]0,+\infty[$ is called the **Nagumo function** if it is continuous and

$$\int_{-\infty}^{0} \frac{dy}{\omega(y)} = +\infty, \quad \int_{0}^{+\infty} \frac{dy}{\omega(y)} = +\infty.$$

Theorem 1. *If conditions* (5_1), (6_1) *hold and*

$$f \in B_1(\sigma_1,\sigma_2),$$

then problem (1), (2_1), (3) *has a minimal and a maximal solutions.*

Theorem 2. *If conditions* (5_2), (6_2) *hold and*

$$f \in B_2(\sigma_1,\sigma_2),$$

then problem (1), (2_2), (3) *has a minimal and a maximal solutions.*

From these theorems several effective conditions for the existence of extremal solutions of problems (1), (2_1), (3) and (1), (2_2), (3) are obtained.
In particular, the following statements are valid.

Corollary 1_1. *Let conditions* (5_1), (6_1) *hold and let there exist numbers* $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, *a non-negative function* $h \in L([a,b])$ *and a Nagumo function* ω *such that*

$$f(t,x,y)\operatorname{sgn} y \ge -\omega(y)\big(h(t)+|y|\big) \ \ \textit{for} \ \ a<t<b_0, \ \ \sigma_1(t) \le x \le \sigma_2(t), \ \ y \in R$$

and

$$f(t,x,y)\operatorname{sgn} y \le \omega(y)\big(h(t)+|y|\big) \ \ \textit{for} \ \ a_0<t<b, \ \ \sigma_1(t) \le x \le \sigma_2(t), \ \ y \in R.$$

Then problem (1), (2_1), (3) *has a minimal and a maximal solutions.*

Corollary 1_2. *Let conditions* (5_2), (6_2) *hold and let there exist a non-negative function* $h \in L([a,b])$ *and a Nagumo function* ω *such that*

$$f(t,x,y)\operatorname{sgn} y \ge -\omega(y)\big(h(t)+|y|\big) \ \ \textit{for} \ \ a<t<b, \ \ \sigma_1(t) \le x \le \sigma_2(t), \ \ y \in R.$$

Then problem (1), (2_2), (3) *has a minimal and a maximal solutions.*

Corollary 2_1. *Let conditions* (5_1), (6_1) *hold and let there exist numbers* $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, $\lambda \in]0,b-a[$, $h_2 \in R_+$, *and non-negative functions* $h_0 \in L_{loc}(]a,b[)$ *and* $h_1 \in L([a,b])$ *such that*

$$\int_a^b (t-a)(b-t)h_0(t)\,dt < +\infty$$

and the inequalities

$$f(t,x,y)\operatorname{sgn} y \ge -h_0(t) - \left[\frac{\lambda}{(t-a)(b-t)} + h_1(t)\right]|y| - h_2 y^2$$
$$\textit{for} \ \ a<t<b_0, \ \ \sigma_1(t) \le x \le \sigma_2(t), \ \ y \in R$$

and

$$f(t,x,y)\operatorname{sgn} y \le h_0(t) + \left[\frac{\lambda}{(t-a)(b-t)} + h_1(t)\right]|y| + h_2 y^2$$

$$\text{for } a_0 < t < b, \ \ \sigma_1(t) \le x \le \sigma_2(t), \ \ y \in R$$

are fulfilled. Then problem (1), (2_1), (3) *has a minimal and a maximal solutions.*

Corollary 2_2**.** *Let conditions* (5_2), (6_2) *hold and let there exist numbers* $\lambda \in]0,1[$, $h_2 \in R_+$, *and non-negative functions* $h_0 \in L_{loc}(]a,b])$ *and* $h_1 \in L([a,b])$ *such that*

$$f(t,x,y)\operatorname{sgn} y \ge -h_0(t) - \left[\frac{\lambda}{t-a} + h_1(t)\right]|y| - h_2 y^2$$

$$\text{for } a < t < b, \ \ \sigma_1(t) \le x \le \sigma_2(t), \ \ y \in R.$$

Let, moreover,

$$\int\limits_a^b (t-a)h_0(t)\,dt < +\infty.$$

Then problem (1), (2_2), (3) *has a minimal and a maximal solutions.*

As an example, we consider the differential equation

$$u'' = f_0(t,u,u') + f_1(t,u,u')u', \tag{1'}$$

where $f_0 :]a,b[\times R^2 \to R$ and $f_1 :]a,b[\times R^2 \to R$ are functions satisfying the local Carathéodory conditions, and there exists a positive constant r_0 such that

$$f_0(t,x,y)x \ge 0 \quad \text{for} \quad a < t < b, \ \ |x| \ge r_0, \ \ y \in R.$$

Then arbitrary constants

$$\sigma_1 \in]-\infty, -r_0[\quad \text{and} \quad \sigma_2 \in]r_0, +\infty[$$

are, respectively, the lower and the upper functions of Eq. ($1'$). Moreover, it is obvious that

$$\text{if } c_1 \in [\sigma_1,\sigma_2], \ \ c_2 \in [\sigma_1,\sigma_2] \ \ \big(\text{if } c_1 \in [\sigma_1,\sigma_2]\big),$$

then an arbitrary solution of problem ($1'$), (2_1) (of problem ($1'$), (2_2)) admits the estimate

$$\sigma_1 \le u(t) \le \sigma_2 \quad \text{for} \quad a < t < b. \tag{3'}$$

Therefore, in the sequel we consider not problems ($1'$), (2_1), ($3'$) and ($1'$), (2_2), ($3'$) but problems ($1'$), (2_1) and ($1'$), (2_2).

Set

$$f_{ir}^*(\cdot) = \sup\{|f_0(\cdot,x,y)| : \ |x| \le r, \ y \in R\} \quad \text{for } r > 0 \quad (i = 0,1).$$

Corollaries 2_1 and 2_2 imply the following propositions.

Corollary 3_1**.** *Let*

$$\int\limits_a^b (t-a)(b-t)f_{0r}^*(t)\,dt < +\infty, \quad f_{1r}^* \in L_{loc}(]a,b[) \quad \text{for } r > 0$$

and there exist numbers $a_0 \in]a,b[$, $b_0 \in]0,b[$, *continuous functions* $\lambda : R \to [0, b-a[$, $\ell : R \to R_+$ *and a non-negative function* $h \in L([a,b])$ *such that the inequalities*

$$f_1(t,x,y) \ge -\frac{\lambda(x)}{(t-a)(b-t)} - \ell(x)\big(h(t) + |y|\big) \ \ \text{for } a < t < b_0, \ \ (x,y) \in R^2$$

and

$$f_1(t,x,y) \le \frac{\lambda(x)}{(t-a)(b-t)} + \ell(x)\big(h(t) + |y|\big) \ \ \text{for } a_0 < t < b, \ \ (x,y) \in R^2$$

are fulfilled. Then problem $(1')$, (2_1) *has a minimal and a maximal solutions.*

Corollary 3_2**.** *Let*

$$\int_a^b (t-a)f_{0r}^*(t)\,dt < +\infty, \quad f_{1r}^* \in L_{loc}(]a,b]) \quad \textit{for } \; r>0$$

and there exist continuous functions $\lambda : R \to [0,1[$ *and* $\ell : R \to R_+$ *and a non-negative function* $h \in L([a,b])$ *such that on* $]a,b[\,\times R^2$ *the inequality*

$$f_1(t,x,y) \geq -\frac{\lambda(x)}{t-a} - \ell(x)\big(h(t)+|y|\big)$$

hold. Then problem $(1')$, (2_2) *has a minimal and a maximal solutions.*

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