Memoirs on Differential Equations and Mathematical Physics
Volume 35, 2005, 129–146

J. Shabani and A. Vyabandi

ON A NEW TWO PARAMETER MODEL OF RELATIVISTIC POINT INTERACTIONS IN ONE DIMENSION

Abstract. We introduce and study a new 2-parameter model of relativistic point interactions in one dimension formally given by

$$D_{\underline{\underline{\alpha}},y} = D + \underline{\underline{\alpha}}\delta(x-y); x \in \mathbb{R}, \quad y > 0,$$

where D is the free Dirac Hamiltonian and $\underline{\underline{\alpha}}$ is a 2×2 matrix. $D_{\underline{\underline{\alpha}},y}$ provides a generalization of two models of relativistic point interactions discussed in [Lett. Math. Phys. **13** (1987), 345–358].

We define $D_{\underline{\underline{\alpha}},y}$ using the theory of self-adjoint extensions of symmetric closed operators in Hilbert spaces, derive its resolvent equation, analyze its spectral properties and discuss scattering theory for the pair $(D_{\underline{\underline{\alpha}},y}, D)$. We also study the nonrelativistic limit of $D_{\underline{\underline{\alpha}},y}$ which provides a special 2-parameter model of the one-dimensional generalized point interactions introduced in [1].

2000 Mathematics Subject Classification. 81Q10, 47N50, 81Q05.
Key words and phrases. boundary conditions problem, one-dimensional Dirac operator, self-adjoint extensions, resolvent equation, spectral properties, nonrelativistic limit, scattering theory.

რეზიუმე. ნაშრომში ჩვენ ვმარტავთ და შევისწავლით ერთ განზომილებაში რელატივისტური წერტილოვანი ურთიერთქმედების ერთ ახალ 2-პარამეტრიან მოდელს, რომელიც ფორმალურად

$$D_{\underline{\underline{\alpha}},y} = D + \underline{\underline{\alpha}}\delta(x-y); x \in \mathbb{R}, \quad y > 0,$$

ტოლობითაა მოცემული, სადაც D დირაკის თავისუფალი ჰამილტონიანია, ხოლო $\underline{\underline{\alpha}}$ – 2×2 მატრიცი. $D_{\underline{\underline{\alpha}},y}$ წარმოადგენს [Lett. Math. Phys. **13** (1987), 345–358]-ში განხილული რელატივისტური წერტილოვანი ურთიერთქმედების ორი მოდელის განზოგადებას.

$D_{\underline{\underline{\alpha}},y}$-ს ჩვენ განვსაზღვრავთ ჰილბერტის სივრცეებში სიმეტრიული ჩაკეტილი ოპერატორების თვითშეუღლებული გაფართოებების თეორიის გამოყენებით, გამოგვყავს მისი რეზოლვენტური განტოლება, ვაანალიზებთ მის სპექტრულ თვისებებს და განვიხილავთ გაფანტვის თეორიას $(D_{\underline{\underline{\alpha}},y}, D)$ წყვილისათვის. ჩვენ აგრეთვე შევისწავლით $D_{\underline{\underline{\alpha}},y}$-ის არარელატივისტურ ზღვარს, რომელიც წარმოადგენს [1]-ში შემოტანილ ერთგანზომილებიანი განზოგადებული წერტილოვანი ურთიერთქმედებების კერძო სახის 2-პარამეტრიან მოდელს.

1. Introduction

Relativistic point interactions in one dimension have been discussed for a long time in various areas of physics, in particular in connection with the Kronig–Penney type models and Saxon–Hutner conjecture (see, e.g., [2–10] and references therein).

The first rigorous mathematical formulation of these interactions was given in [10] using the theory of self-adjoint extensions of symmetric closed operators in Hilbert spaces.

Indeed [10] defines two models $D_{\alpha,y}$ and $T_{\beta,y}$ of relativistic point interactions which provide natural generalisation of nonrelativistic one-dimensional δ-interactions of the first and the second type [11].

This paper considers a 2-parameter model $D_{\underline{\underline{\alpha}},y}$ of relativistic point interactions in one dimension formally given by

$$D_{\underline{\underline{\alpha}},y} = D + \underline{\underline{\alpha}}\delta(x-y), \quad x \in \mathbb{R}, \quad y > 0,$$

where D is the free Dirac Hamiltonian and $\underline{\underline{\alpha}}$ is a 2×2 matrix of the form

$$\underline{\underline{\alpha}} = \begin{pmatrix} \alpha & 0 \\ 0 & \tilde{\alpha} \end{pmatrix}, \ \ \alpha,\ \tilde{\alpha} \in \mathbb{R}.$$

To the best of our knowledge, this model is new. It provides a straightforward generalisation of the models $D_{\alpha,y}$ and $T_{\beta,y}$ discussed in [10] which correspond to the special cases $\alpha \neq 0$, $\tilde{\alpha} = 0$ and $\alpha = 0$, $\tilde{\alpha} = -c^2\beta \neq 0$, respectively.

The paper is organized as follows. In Section 2, we define the quantum Hamiltonian $D_{\underline{\underline{\alpha}},y}$ following the strategy used in [12, 14] in the case of relativistic δ-sphere interactions. We also derive the resolvent equation of $D_{\underline{\underline{\alpha}},y}$, analyse its spectral properties and carry out a systematic study of the scattering theory for the pair $(D_{\underline{\underline{\alpha}},y}, D)$.

The nonrelativistic limit corresponding to $D_{\underline{\underline{\alpha}},y}$ defines a 2-parameter model $\Delta_{\alpha,\beta,y}$ of nonrelativistic point interactions in one dimension. It turns out that this model is a special case of the one dimensional generalized point interactions introduced in [1].

Section 3 is devoted to the study of $\Delta_{\alpha,\beta,y}$. In a forthcoming paper [15] we generalize the results of section 2 and 3 to finitely and infinitely many relativistic point interactions.

2. The Relativistic Point Interaction

A. Definition of the Hamiltonian. The quantum Hamiltonian describing a relativistic point interaction is formally given by

$$H = D + \underline{\underline{\alpha}}\delta(x-y), \quad x \in \mathbb{R}, \quad y > 0, \tag{1}$$

where $\underline{\underline{\alpha}}$ is a 2×2 matrix of the form

$$\underline{\underline{\alpha}} = \begin{pmatrix} \alpha & 0 \\ 0 & \tilde{\alpha} \end{pmatrix}, \quad \alpha,\ \tilde{\alpha} \in \mathbb{R}, \tag{2}$$

and the one-dimension free Dirac operator in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \bigotimes \mathbb{C}^2$ is defined by [10]

$$D = -\mathrm{i}c\frac{d}{dx} \bigotimes \sigma_1 + \left(\frac{c^2}{2}\right) \bigotimes \sigma_3 = \begin{pmatrix} \frac{c^2}{2} & -\mathrm{i}c\frac{d}{dx} \\ -\mathrm{i}c\frac{d}{dx} & -\frac{c^2}{2} \end{pmatrix}, \quad \mathcal{D}(D) = H^{2,1}(\mathbb{R}) \bigotimes \mathbb{C}^2, \tag{3}$$

where

(i) $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli matrices in $\mathbb{C}^2$;

(ii) c is the velocity of light;

(iii) $H^{m,n}(\Omega)$ is the Sobolev space of indices(m, n).

We consider the symmetric closed operator $\dot{D}_y$ defined by

$$\dot{D}_y = D, \quad \mathcal{D}(\dot{D}_y) = \left\{ g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in H^{2,1}(\mathbb{R}) \bigotimes \mathbb{C}^2 | g(y\pm) = 0 \right\}.$$

The adjoint $\dot{D}_y^*$ of $\dot{D}_y$ reads

$$\dot{D}_y^* = D, \quad \mathcal{D}(\dot{D}_y^*) = \left\{ g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in H^{2,1}(\mathbb{R}) \bigotimes \mathbb{C}^2 | g \in \mathrm{AC}_{loc}(\mathbb{R} - \{y\}) \right\}.$$

$\mathrm{AC}_{loc}(\Omega)$ denotes the set of locally absolutely continuous functions on Ω.

A straightforward computation shows that the equation

$$\dot{D}_y^* g(z) = zg(z), \quad g \in \mathcal{D}(\dot{D}_y^*), \quad z \in \mathbb{C} - \left\{ \Big(-\infty, -\frac{c^2}{2} \Big] \bigcup \Big[\frac{c^2}{2}, \infty \Big) \right\}$$

has the solutions

$$g^{(1)}(z, x) = \frac{\mathrm{i}}{2c} \begin{cases} \binom{\zeta}{1} \mathrm{e}^{ik'(x-y)}, & x > y, \\ \binom{0}{0}, & x < y, \end{cases}$$

$$g^{(2)}(z, x) = \frac{\mathrm{i}}{2c} \begin{cases} \binom{0}{0}, & x > y, \\ \binom{\zeta}{-1} \mathrm{e}^{ik'(y-x)}, & x < y, \end{cases} \quad \mathrm{Im}\, k' > 0,$$

where

$$k' = \frac{1}{c}\sqrt{z^2 - \frac{c^4}{4}} \equiv k'(z), \tag{4}$$

$$\zeta = \frac{1}{ck'(z)}[z + \frac{c^2}{2}], \quad \mathrm{Im}\, k'(z) \geq 0, \quad z \in \mathbb{C}. \tag{5}$$

Thus $\dot{D}_y$ has deficiency indices (2,2) and hence it has a four-parameter family of self-adjoint extensions. Let us now construct the self-adjoint extension corresponding to the free Dirac operator with the potential

$$V(x) = \underline{\underline{\alpha}}\delta(x-y), \quad \underline{\underline{\alpha}} = \begin{pmatrix} \alpha & 0 \\ 0 & \tilde{\alpha} \end{pmatrix}, \quad \alpha, \tilde{\alpha} \in \mathbb{R}.$$

Assume that g satisfies the equation

$$\begin{aligned} &[D + \underline{\underline{\alpha}}\delta(x-y)]g = zg, \\ &D = \begin{pmatrix} \frac{c^2}{2} & -\mathrm{i}c\frac{d}{dx} \\ -\mathrm{i}c\frac{d}{dx} & -\frac{c^2}{2} \end{pmatrix}, \quad \underline{\underline{\alpha}} = \begin{pmatrix} \alpha & 0 \\ 0 & \tilde{\alpha} \end{pmatrix}, \quad \alpha, \tilde{\alpha} \in \mathbb{R}, \end{aligned} \tag{6}$$

and the limits $g(y\pm)$ exist. Integrating the equation (6) over $(y-\epsilon, y+\epsilon)$ and taking the limit as $\epsilon \to 0$, we get the following boundary conditions

$$\begin{cases} g_2(y+) - g_2(y-) = -\frac{\mathrm{i}\alpha}{2c}[g_1(y+) + g_1(y-)], \\ g_1(y+) - g_1(y-) = -\frac{\mathrm{i}\tilde{\alpha}}{2c}[g_2(y+) + g_2(y-)]. \end{cases} \tag{7}$$

As indicated in [12], the boundary conditions in (7) defines a self-adjoint extension of $\dot{D}_y$ iff $\underline{\underline{\alpha}} = \underline{\underline{\alpha}}^+$.

Consider in $L^2(\mathbb{R}) \bigotimes \mathbb{C}^2$ the operator $D_{\underline{\underline{\alpha}},y}$ defined by

$$D_{\underline{\underline{\alpha}},y} = \begin{pmatrix} \frac{c^2}{2} - ic\frac{d}{dx} \\ -ic\frac{d}{dx} - \frac{c^2}{2} \end{pmatrix}, \tag{8}$$

$$\mathcal{D}(D_{\underline{\underline{\alpha}},y}) = \left\{ g \in \mathcal{D}(\dot{D}_y^*) \;\middle|\; \begin{matrix} g_2(y+) - g_2(y-) = -\frac{\mathrm{i}\alpha}{2c}[g_1(y+) + g_1(y-)] \\ g_1(y+) - g_1(y-) = -\frac{\mathrm{i}\tilde{\alpha}}{2c}[g_2(y+) + g_2(y-)] \end{matrix} \right\}.$$

According to [12], the operator $D_{\underline{\underline{\alpha}},y}$ provides the mathematical definition of the formal expression (1).

The case $\underline{\underline{\alpha}} = 0$ (i.e., $\alpha = \tilde{\alpha} = 0$) in the equation (8) yields the free Dirac Hamiltonian $D_{0,y} \equiv D$.

The case $\alpha \neq 0$, $\tilde{\alpha} = 0$ in the equation (8) yields the Hamiltonian $D_{\alpha,y}$ which describes the relativistic δ-point interaction of the first type centered at $y \in \mathbb{R}$ defined by [10]:

$$\begin{aligned} D_{\alpha,y} &= D, \\ \mathcal{D}(D_{\alpha,y}) &= \big\{ g \in H^{2,1}(\mathbb{R} - \{y\}) \bigotimes \mathbb{C}^2 | g_2 \in \mathrm{AC}_{loc}(\mathbb{R}), \\ &\quad g_1 \in \mathrm{AC}_{loc}(\mathbb{R} - \{y\}); \;\; g_2(y+) - g_2(y-) = -(i\alpha/c)g_1(y) \big\}, \\ &\quad -\infty < \alpha \leq \infty. \end{aligned}$$

The case $\alpha = 0$, $\tilde{\alpha} = -c^2\beta \neq 0$ in the equation (8) yields the Hamiltonian $T_{\beta,y}$ which describes the relativistic δ-point interaction of the second type centered at $y \in \mathbb{R}$ defined by [10]:

$$\begin{aligned} T_{\beta,y} &= D, \\ \mathcal{D}(T_{\beta,y}) &= \big\{ g \in H^{2,1}(\mathbb{R} - \{y\}) \bigotimes \mathbb{C}^2 | g_2 \in \mathrm{AC}_{loc}(\mathbb{R} - \{y\}), \\ & g_1 \in \mathrm{AC}_{loc}(\mathbb{R});\ \ g_1(y+) - g_1(y-) = i\beta c g_2(y) \big\}, \\ & -\infty < \beta \leq \infty. \end{aligned}$$

Following [12], we note that all the results corresponding to $D_{\underline{\underline{\alpha}},y}$ could be generalized to the model $D_{\hat{\alpha},y}$ formally given by

$$H = D + \hat{\alpha}\delta(x-y), \quad x \in \mathbb{R}, \quad y > 0,$$

where $\hat{\alpha}$ is a non-diagonal 2×2 matrix with $\hat{\alpha} = \hat{\alpha}^+$.

B. The resolvent equation. From the Krein resolvent formula [16], after a straightforward computation (see, e.g., [11]) we obtain

$$\begin{aligned} (D_{\underline{\underline{\alpha}},y} - z)^{-1} &= (D - z)^{-1} - \\ & - \frac{1}{(2c + i\alpha\zeta)(2c + i\tilde{\alpha}\zeta^{-1})} \big\{ \alpha(\overline{\tilde{f}_{k'}(.-y)}, .) f_{k'}(.-y) + \\ & + \tilde{\alpha}(\overline{\tilde{g}_{k'}(.-y)}, .) g_{k'}(.-y) + i\frac{\alpha\tilde{\alpha}}{2c}(\overline{\tilde{f}_{k'}(.-y)}, .)\hat{f}_{k'}(.-y) + \\ & + i\frac{\alpha\tilde{\alpha}}{2c}(\overline{\tilde{g}_{k'}(.-y)}, .)\hat{g}_{k'}(.-y) \big\}, \quad z \in \rho(D_{\underline{\underline{\alpha}},y}), \ \ \mathrm{Im}\, k' > 0, \end{aligned} \tag{9}$$

where $R_{k'} = (D - z)^{-1}$, $z \in \mathbb{C} - \Big\{ (-\infty, -\frac{c^2}{2}] \bigcup [\frac{c^2}{2}, \infty) \Big\}$ is the free Dirac resolvent with integral kernel [10]

$$R_{k'}(x - x') = \frac{i}{2c} \begin{pmatrix} \zeta & \mathrm{sgn}(x - x') \\ \mathrm{sgn}(x - x') & \zeta^{-1} \end{pmatrix} \mathrm{e}^{ik'|x - x'|}$$

and

$$f_{k'}(x-y) = \begin{cases} \binom{\zeta}{1} \mathrm{e}^{ik'(x-y)}, & x > y, \\ \binom{\zeta}{-1} \mathrm{e}^{ik'(y-x)}, & x < y, \end{cases}$$

$$\tilde{f}_{k'}(x-y) = \begin{cases} \binom{-\zeta}{1} \mathrm{e}^{ik'(x-y)}, & x > y, \\ \binom{-\zeta}{-1} \mathrm{e}^{ik'(y-x)}, & x < y, \end{cases}$$

$$g_{k'}(x-y) = \begin{cases} \binom{1}{\zeta^{-1}} \mathrm{e}^{ik'(x-y)}, & x > y, \\ \binom{-1}{\zeta^{-1}} \mathrm{e}^{ik'(y-x)}, & x < y, \end{cases}$$

$$\tilde{g}_{k'}(x-y) = \begin{cases} \binom{1}{-\zeta^{-1}} \mathrm{e}^{ik'(x-y)}, & x > y, \\ \binom{-1}{-\zeta^{-1}} \mathrm{e}^{ik'(y-x)}, & x < y, \end{cases}$$

$$\hat{f}_{k'}(x-y) = \begin{cases} \binom{1}{\zeta^{-1}} \mathrm{e}^{ik'(x-y)}, & x > y, \\ \binom{1}{-\zeta^{-1}} \mathrm{e}^{ik'(y-x)}, & x < y, \end{cases}$$

$$\hat{g}_{k'}(x-y) = \begin{cases} \binom{\zeta}{1} \mathrm{e}^{ik'(x-y)}, & x > y, \\ \binom{-\zeta}{1} \mathrm{e}^{ik'(y-x)}, & x < y, \end{cases}$$

$$z \in \mathbb{C} - \left\{ \Big(-\infty, -\frac{c^2}{2} \Big] \cup \Big[\frac{c^2}{2}, \infty \Big) \right\}, \ \ \operatorname{Im} k' > 0.$$

Remark 1. From the equation (9), a straightforward computation shows

(i) As $\tilde{\alpha} \to 0$, the Hamiltonian $D_{\underline{\underline{\alpha}},y}$ converges in the norm resolvent sense to $D_{\alpha,y}$:

$$n \cdot \lim_{\tilde{\alpha} \to 0} (D_{\underline{\underline{\alpha}},y} - z)^{-1} = (D_{\alpha,y} - z)^{-1}, \ \ z \in \rho(D_{\underline{\underline{\alpha}},y}) \cap \rho(D_{\alpha,y}),$$

where [10]

$$(D_{\alpha,y} - z)^{-1} = (D - z)^{-1} - \frac{\alpha}{2c(2c + i\alpha\zeta)} (\overline{\tilde{f}_{k'}(.-y)}, .) f_{k'}(.-y),$$
$$z \in \rho(D_{\alpha,y}), \quad \operatorname{Im} k' > 0.$$

(ii) Let $\tilde{\alpha} = -\beta c^2$, $\beta \in \mathbb{R}$. Then as $\alpha \to 0$, the Hamiltonian $D_{\underline{\underline{\alpha}},y}$ converges in the norm resolvent sense to $T_{\beta,y}$:

$$n. \lim_{\alpha \to 0} (D_{\underline{\underline{\alpha}},y} - z)^{-1} = (T_{\beta,y} - z)^{-1}, \quad z \in \rho(D_{\underline{\underline{\alpha}},y}) \cap \rho(T_{\beta,y}),$$

where [10]

$$(T_{\beta,y} - z)^{-1} = (D - z)^{-1} + \frac{\beta}{2(2 - i\beta c\zeta^{-1})} (\overline{\tilde{g}_{k'}(.-y)}, .) g_{k'}(.-y),$$
$$z \in \rho(T_{\beta,y}), \quad \operatorname{Im} k' > 0.$$

The following theorem gives the additional information on the domain of $D_{\underline{\underline{\alpha}},y}$.

Theorem 2.1. *The domain* $\mathcal{D}(D_{\underline{\underline{\alpha}},y})$, $-\infty < \alpha, \tilde{\alpha} \le \infty$, $y \in \mathbb{R}$, *consists of all elements* $\psi_{\underline{\underline{\alpha}}}$ *of the type*

$$\psi_{\underline{\underline{\alpha}}}(x) = \phi_{k'}(x) - \frac{2ic}{(2c + i\alpha\zeta)(2c + i\tilde{\alpha}\zeta^{-1})} \times$$
$$\times \Big\{ \alpha\phi_{k',1}(y) f_{k'}(x-y) + \tilde{\alpha}\phi_{k',2}(y) g_{k'}(x-y) +$$
$$+ i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',1}(y)\hat{f}_{k'}(x-y) + i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',2}(y)\hat{g}_{k'}(x-y) \Big\}, \ \ x \neq y, \quad (10)$$

where $\phi_{k'} = \binom{\phi_{k',1}}{\phi_{k',2}} \in \mathcal{D}(D) = H^{2,1}(\mathbb{R}) \bigotimes \mathbb{C}^2$ *and* $\operatorname{Im} k' > 0$. *The decomposition* (10) *is unique and with* $\psi_{\underline{\underline{\alpha}}}$ *of this form we obtain*

$$(D_{\underline{\underline{\alpha}},y} - z)\psi_{\underline{\underline{\alpha}}} = (D - z)\phi_{k'}. \quad (11)$$

Let $\psi_{\underline{\underline{\alpha}}} \in \mathcal{D}(D_{\underline{\underline{\alpha}},y})$ *and assume that* $\psi_{\underline{\underline{\alpha}}} = 0$ *in an open set* $\vartheta \in \mathbb{R}$. *Then* $D_{\underline{\underline{\alpha}},y}\psi_{\underline{\underline{\alpha}}} = 0$ *in* ϑ, *i.e.,* $D_{\underline{\underline{\alpha}},y}$ *describes a local interaction.*

Proof. The following relation

$$\begin{aligned}\mathcal{D}(D_{\underline{\underline{\alpha}},y}) &= (D_{\underline{\underline{\alpha}},y}-z)^{-1}(D-z)\mathcal{D}(D) = \\ &= \Big\{R_{k'} - \frac{1}{(2c+i\alpha\zeta)(2c+i\tilde{\alpha}\zeta^{-1})}[\alpha(\tilde{f}_{k'}(.-y),.)f_{k'}(.-y)+ \\ &\quad + \tilde{\alpha}(\tilde{g}_{k'}(.-y),.)g_{k'}(.-y) + i\frac{\alpha\tilde{\alpha}}{2c}(\tilde{f}_{k'}(.-y),.)\hat{f}_{k'}(.-y)+ \\ &\quad + i\frac{\alpha\tilde{\alpha}}{2c}(\tilde{g}_{k'}(.-y),.)\hat{g}_{k'}(.-y)]\Big\}(D-z)\mathcal{D}(D), \\ &\quad z\in\rho(D_{\underline{\underline{\alpha}},y}), \quad \operatorname{Im} k'>0,\end{aligned}$$

proves (10).

Next let $\psi_{\underline{\underline{\alpha}}} = 0$. Then

$$\begin{aligned}\phi_{k'}(x) = \frac{2ic}{(2c+i\alpha)(2c+i\tilde{\alpha}\zeta^{-1})}&\Big\{\alpha\phi_{k',1}(y)f_{k'}(x-y)+ \\ &+ \tilde{\alpha}\phi_{k',2}(y)g_{k'}(x-y) + i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',1}(y)\hat{f}_{k'}(x-y)+ \\ &+ i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',2}(y)\hat{g}_{k'}(x-y)\Big\}\end{aligned}$$

and $\phi_{k'} \in C^0(\mathbb{R})$, implies $\phi_{k'} = 0$ which proves the uniqueness of (10). The relation (11) follows from

$$\begin{aligned}(D_{\underline{\underline{\alpha}},y}-z)^{-1}(D-z)\phi_{k'} = \phi_{k'} - &\frac{1}{(2c+i\alpha\zeta)(2c+i\tilde{\alpha}\zeta^{-1})}\times \\ \times &\Big\{\alpha(\tilde{f}_{k'}(.-y),(D-z)\phi_{k'})f_{k'}(.-y)+ \\ &+ \tilde{\alpha}(\tilde{g}_{k'}(.-y),(D-z)\phi_{k'})g_{k'}(.-y)+ \\ &+ i\frac{\alpha\tilde{\alpha}}{2c}(\tilde{f}_{k'}(.-y),(D-z)\phi_{k'})\hat{f}_{k'}(.-y)+ \\ &+ i\frac{\alpha\tilde{\alpha}}{2c}(\tilde{g}_{k'}(.-y),(D-z)\phi_{k'})\hat{g}_{k'}(.-y)\Big\} = \\ &= \psi_{\underline{\underline{\alpha}}}, z\in\rho(D_{\underline{\underline{\alpha}},y}), \quad \operatorname{Im} k'>0.\end{aligned}$$

Let us now prove locality. We assume first $y \notin \vartheta$. Then

$$\begin{aligned}((D-z)(\alpha\phi_{k',1}(y)f_{k'}(.-y) + \tilde{\alpha}\phi_{k',2}(y)g_{k'}(.-y)+ \\ + i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',1}(y)\hat{f}_{k'}(.-y) + i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',1}(y)\hat{g}_{k'}(.-y)))(x) = 0\end{aligned}$$

implies that

$$\begin{aligned}(D_{\underline{\underline{\alpha}},y}\psi_{\underline{\underline{\alpha}}})(x) &= z\psi_{\underline{\underline{\alpha}}}(x) + ((D-z)\phi_{k'})(x) = \\ &= \frac{2ic}{(2c+i\alpha\zeta)(2c+i\tilde{\alpha}\zeta^{-1})}((D-z)(\alpha\phi_{k',1}(y)f_{k'}(.-y)+ \\ &\quad + \tilde{\alpha}\phi_{k',2}(y)g_{k'}(.-y) + i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',1}(y)\hat{f}_{k'}(.-y)+\end{aligned}$$

$$+ i\frac{\alpha\tilde{\alpha}}{2c}\phi_{k',1}(y)\hat{g}_{k'}(. - y)))(x) = 0, \quad x \in \vartheta.$$

Second, if $y \in \vartheta$, then $\psi_{\underline{\underline{\alpha}}}(y) = 0$ and $\phi_{k'} \in C^0(\mathbb{R})$ implies $\phi_{k'} = 0$, and hence

$$(D_{\underline{\underline{\alpha}},y}\psi_{\underline{\underline{\alpha}}})(x) = z\psi_{\underline{\underline{\alpha}}}(x) = 0, \quad x \in \vartheta. \qquad \square$$

C. Spectral properties. The spectral properties of $D_{\underline{\underline{\alpha}},y}$ follow from (9). For $\alpha, \tilde{\alpha} \in \mathbb{R}$ the essential spectrum is purely absolutely continuous and coincides with $(-\infty, -\frac{c^2}{2}] \bigcup [\frac{c^2}{2}, \infty)$. The point spectrum of $D_{\underline{\underline{\alpha}},y}$ in $[-\frac{c^2}{2}, \frac{c^2}{2}]$ contains the poles of the resolvent equation (9). Then $D_{\underline{\underline{\alpha}},y}$ has two eigenvalues in $[-\frac{c^2}{2}, \frac{c^2}{2}]$ iff $\alpha, \tilde{\alpha} < 0$:

$$\sigma_p(D_{\underline{\underline{\alpha}},y}) = \begin{cases} \left\{\frac{c^2(4c^2-\alpha^2)}{2(4c^2+\alpha^2)}, \frac{c^2(\tilde{\alpha}^2-4c^2)}{2(4c^2+\tilde{\alpha}^2)}\right\}, & \alpha, \tilde{\alpha} < 0 \\ \varnothing, & \alpha, \tilde{\alpha} \geq 0, \quad \alpha = \tilde{\alpha} = \infty, \end{cases}$$

and two resonances iff $\alpha, \tilde{\alpha} > 0$.

Following the strategy of [10], one proves that the operator $(D_{\underline{\underline{\alpha}},y} - \frac{c^2}{2})$ converges in the norm resolvent sense to the Schrödinger operator $\Delta_{\alpha,\beta,y}$

$$n - \lim_{c\to\infty}(D_{\underline{\underline{\alpha}},y} - \frac{c^2}{2} - z)^{-1} = (\Delta_{\alpha,\beta,y} - z)^{-1} \bigotimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \alpha, \beta \in \mathbb{R},$$

where

$$\begin{aligned} &\Delta_{\alpha,\beta,y} = -\frac{d^2}{dx^2}, \\ &\mathcal{D}(\Delta_{\alpha,\beta,y}) = \\ &\quad = \left\{ g \in H^{2,2}(\mathbb{R} - \{y\}) \;\middle|\; \begin{array}{l} g'(y+) - g'(y-) = \frac{\alpha}{2}[g(y+) + g(y-)] \\ g(y+) - g(y-) = \frac{\beta}{2}[g'(y+) + g'(y-)] \end{array} \right\}, \\ &\quad -\infty < \alpha, \beta \leq \infty. \end{aligned} \tag{12}$$

The Hamiltonian $\Delta_{\alpha,\beta,y}$ defines a special exactly solvable model of nonrelativistic point interaction. In section 2.A we will discuss the properties of the above Hamiltonian.

In particular, as $c \to \infty$, the two eigenvalues of $D_{\underline{\underline{\alpha}},y}$ (rest energy subtracted) $(E_\alpha - \frac{c^2}{2})$, $(E_{\tilde{\alpha}} - \frac{c^2}{2})$ give their respective nonrelativistic limits

$$\begin{aligned} \lim_{c\to\infty}\left(E_\alpha - \frac{c^2}{2}\right) &= \lim_{c\to\infty}\left(\frac{c^2(4c^2 - \alpha^2)}{2(4c^2 + \alpha^2)} - \frac{c^2}{2}\right) \\ &= -\frac{\alpha^2}{4}\lim_{c\to\infty}\left[1 + \frac{\alpha^2}{4c^2}\right]^{-1} \\ &= -\frac{\alpha^2}{4}, \\ \lim_{c\to\infty}\left(E_{\tilde{\alpha}} - \frac{c^2}{2}\right) &= \lim_{c\to\infty}\left(\frac{c^2(\tilde{\alpha}^2 - 4c^2)}{2(4c^2 + \tilde{\alpha}^2)} - \frac{c^2}{2}\right) \end{aligned}$$

$$= -\frac{4}{\beta^2}\lim_{c\to\infty}\left[1+\frac{4}{\beta^2 c^2}\right]^{-1}, \quad \beta = -\frac{\tilde\alpha}{c^2}$$
$$= -\frac{4}{\beta^2}.$$

In Section 2.C we will show that $-\frac{\alpha^2}{4}$ and $-\frac{4}{\beta^2}$ are the two eigenvalues of $\Delta_{\alpha,\beta,y}$ [see the equation (27)].

D. Scattering theory of the pair $(D_{\underline{\underline{\alpha}},y}, D)$**.** From Theorem 2.1, the scattering wave functions of $D_{\underline{\underline{\alpha}},y}$ are defined by

$$\begin{aligned}
\psi_{\underline{\underline{\alpha}},y}(k,\sigma,x) = &\begin{pmatrix} e^{ik'\sigma x} \\ \sigma\zeta^{-1}e^{ik'\sigma x}\end{pmatrix} - \\
&- \frac{2ice^{ik'\sigma y}}{(2c+i\alpha\zeta)(2c+i\tilde\alpha\zeta^{-1})}\left\{\alpha\left\{\begin{array}{ll}\binom{\zeta}{1}e^{ik'(x-y)}, & x>y\\ \binom{\zeta}{-1}e^{ik'(y-x)}, & x<y\end{array}\right\}+\right.\\
&+\tilde\alpha\sigma\zeta^{-1}\left\{\begin{array}{ll}\binom{1}{\zeta^{-1}}e^{ik'(x-y)}, & x>y\\ \binom{-1}{\zeta^{-1}}e^{ik'(y-x)}, & x<y\end{array}\right\}+\\
&+i\frac{\alpha\tilde\alpha}{2c}\left\{\begin{array}{ll}\binom{1}{\zeta^{-1}}e^{ik'(x-y)}, & x>y\\ \binom{1}{-\zeta^{-1}}e^{ik'(y-x)}, & x<y\end{array}\right\}+\\
&\left.+i\frac{\sigma\zeta^{-1}\alpha\tilde\alpha}{2c}\left\{\begin{array}{ll}\binom{\zeta}{1}e^{ik'(x-y)}, & x>y\\ \binom{-\zeta}{1}e^{ik'(y-x)}, & x<y\end{array}\right\}\right\},\\
&x,y\in\mathbb{R},\quad k'\geq 0,\quad \sigma=\pm1, -\infty<\alpha,\tilde\alpha\leq\infty.
\end{aligned}$$

A straightforward computation shows that $\psi_{\underline{\underline{\alpha}},y}(k,\sigma)$ are eigenfunctions associated with $D_{\underline{\underline{\alpha}},y}$ corresponding to left ($\sigma = +1$) and right ($\sigma = -1$) incidence [11].

The asymptotic forms of $\psi_{\underline{\underline{\alpha}},y}$ are defined by [11, 17]

$$\begin{aligned}
\psi_{\underline{\underline{\alpha}},y}(z,+1,x) &= \begin{cases}\mathcal{T}^l_{\underline{\underline{\alpha}},y}(z)\psi(z,+1,x) & \text{as } x\to\infty,\\ \psi(z,+1,x)+\mathcal{R}^l_{\underline{\underline{\alpha}},y}(z)\psi(z,-1,x) & \text{as } x\to-\infty,\end{cases}\\
\psi_{\underline{\underline{\alpha}},y}(z,-1,x) &= \begin{cases}\psi(z,-1,x)+\mathcal{R}^r_{\underline{\underline{\alpha}},y}(z)\psi(z,+1,x) & \text{as } x\to\infty,\\ \mathcal{T}^r_{\underline{\underline{\alpha}},y}(z)\psi(z,-1,x) & \text{as } x\to-\infty,\end{cases}
\end{aligned}\tag{13}$$

where $\psi(z,\sigma,x)$ is the solution of $D\psi = z\psi$ given by

$$\psi(z,\sigma,x) = \begin{pmatrix} e^{i\sigma k'x} \\ \sigma\zeta^{-1}e^{i\sigma k'x}\end{pmatrix}, \quad \sigma=\pm1,$$

with k' and ζ defined by (4) and (5), respectively. Then the reflection and transmission coefficients from the left ($\sigma = +1$) and the right ($\sigma = -1$) are defined by

$$\mathcal{R}^l_{\underline{\underline{\alpha}},y}(z) = \lim_{x\to-\infty} \frac{1}{2}\Big(\mathrm{e}^{ik'x}, -\zeta \mathrm{e}^{ik'x}\Big)\left[\psi_{\underline{\underline{\alpha}},y}(z,+1,x) - \begin{pmatrix} \mathrm{e}^{ik'x} \\ \zeta^{-1}\mathrm{e}^{ik'x}\end{pmatrix}\right],$$

$$\mathcal{R}^r_{\underline{\underline{\alpha}},y}(z) = \lim_{x\to+\infty} \frac{1}{2}\Big(\mathrm{e}^{-ik'x}, \zeta \mathrm{e}^{-ik'x}\Big)\left[\psi_{\underline{\underline{\alpha}},y}(z,-1,x) - \begin{pmatrix} \mathrm{e}^{-ik'x} \\ -\zeta^{-1}\mathrm{e}^{-ik'x}\end{pmatrix}\right],$$

$$\mathcal{T}^l_{\underline{\underline{\alpha}},y}(z) = \lim_{x\to+\infty} \frac{1}{2}\Big(\mathrm{e}^{-ik'x}, \zeta \mathrm{e}^{-ik'x}\Big)\psi_{\underline{\underline{\alpha}},y}(z,+1,x),$$

$$\mathcal{T}^r_{\underline{\underline{\alpha}},y}(z) = \lim_{x\to-\infty} \frac{1}{2}\Big(\mathrm{e}^{ik'x}, -\zeta \mathrm{e}^{ik'x}\Big)\psi_{\underline{\underline{\alpha}},y}(z,-1,x),$$

$$k' \geq 0, \quad -\infty < \alpha, \tilde{\alpha} \leq \infty, \quad y \in \mathbb{R}.$$

After a straightforward computation, one obtains

Theorem 2.2. *Let $\alpha, \tilde{\alpha} \in \mathbb{R} - \{0\}$, $y \in \mathbb{R}$. Then the unitary on-shell scattering matrix $\mathcal{S}_{\underline{\underline{\alpha}},y}(z)$ in $\mathbb{C}^2$ associated with the pair $(D_{\underline{\underline{\alpha}},y}, D)$ reads*

$$\mathcal{S}_{\underline{\underline{\alpha}},y}(z) = \begin{bmatrix} \mathcal{T}^l_{\underline{\underline{\alpha}},y}(z) & \mathcal{R}^r_{\underline{\underline{\alpha}},y}(z) \\ \mathcal{R}^l_{\underline{\underline{\alpha}},y}(z) & \mathcal{T}^r_{\underline{\underline{\alpha}},y}(z) \end{bmatrix}, \; k' \geq 0, \; -\infty < \alpha, \tilde{\alpha} \leq \infty, \; y \in \mathbb{R},$$

with

$$\mathcal{T}^l_{\underline{\underline{\alpha}},y}(z) = 1 - \frac{2ic}{\Big(2c + i\alpha\zeta\Big)\Big(2c + i\tilde{\alpha}\zeta^{-1}\Big)} \times$$
$$\times \Big(\alpha\zeta + \tilde{\alpha}\zeta^{-1} + i\frac{\alpha\tilde{\alpha}}{c}\Big) = \mathcal{T}^r_{\underline{\underline{\alpha}},y}(z),$$

$$\mathcal{R}^l_{\underline{\underline{\alpha}},y}(z) = -\frac{2ic}{\Big(2c + i\alpha\zeta\Big)\Big(2c + i\tilde{\alpha}\zeta^{-1}\Big)}\Big(\alpha\zeta - \tilde{\alpha}\zeta^{-1}\Big)e^{2ik'y},$$

$$\mathcal{R}^r_{\underline{\underline{\alpha}},y}(z) = -\frac{2ic}{\Big(2c + i\alpha\zeta\Big)\Big(2c + i\tilde{\alpha}\zeta^{-1}\Big)}\Big(\alpha\zeta - \tilde{\alpha}\zeta^{-1}\Big)e^{-2ik'y}.$$

In particular, as $c \to \infty$, the unitary on-shell scattering matrix $\mathcal{S}_{\underline{\underline{\alpha}},y}(k^2 + \frac{c^2}{2})$ gives its nonrelativistic limit $\mathcal{S}_{\alpha,\beta,y}(k)$ [see the equation (28)]. Indeed,

$$\lim_{c\to\infty} \mathcal{T}^l_{\underline{\underline{\alpha}},y}(z) =$$
$$= \lim_{c\to\infty}\left\{1 - \frac{2ic}{(2c + i\alpha\zeta)(2c + i\tilde{\alpha}\zeta^{-1})}\Big(\alpha\zeta + \tilde{\alpha}\zeta^{-1} + i\frac{\alpha\tilde{\alpha}}{c}\Big)\right\}. \tag{14}$$

Let $z = k^2 + \frac{c^2}{2}$, $k > 0$ and $\tilde{\alpha} = -\beta c^2, \beta \in \mathbb{R}$, then after a straightforward computation (2) reads

$$\lim_{c\to\infty} \mathcal{T}^l_{\underline{\underline{\alpha}},y}\left(k^2 + \frac{c^2}{2}\right) = i\frac{\left(\frac{\alpha\beta}{4} - 1\right)}{4k^2\left(\frac{\alpha}{4k} - \frac{i}{2}\right)\left(\frac{1}{2k^2} - i\frac{\beta}{4k}\right)} =$$

$$= \lim_{c\to\infty} \mathcal{T}^r_{\underline{\underline{\alpha}},y}(k^2 + \frac{c^2}{2}), \tag{15}$$

$$\lim_{c\to\infty} \mathcal{R}^l_{\underline{\underline{\alpha}},y}(k^2 + \frac{c^2}{2}) =$$

$$= \lim_{c\to\infty} \left\{ -\frac{2ic}{(2c+i\alpha\zeta)(2c+i\tilde{\alpha}\zeta^{-1})} \left(\alpha\zeta - \tilde{\alpha}\zeta^{-1}\right) e^{2ik'y} \right\} =$$

$$= -\frac{\left(\frac{\alpha}{k} + \beta k\right)}{8k^2\left(\frac{\alpha}{4k} - \frac{i}{2}\right)\left(\frac{1}{2k^2} - i\frac{\beta}{4k}\right)} e^{2iky} \tag{16}$$

and

$$\lim_{c\to\infty} \mathcal{R}^r_{\underline{\underline{\alpha}},y}\left(k^2 + \frac{c^2}{2}\right) =$$

$$= \lim_{c\to\infty} \left\{ -\frac{2ic}{(2c+i\alpha\zeta)(2c+i\tilde{\alpha}\zeta^{-1})} \left(\alpha\zeta - \tilde{\alpha}\zeta^{-1}\right) e^{-2ik'y} \right\} =$$

$$= -\frac{\left(\frac{\alpha}{k} + \beta k\right)}{8k^2\left(\frac{\alpha}{4k} - \frac{i}{2}\right)\left(\frac{1}{2k^2} - i\frac{\beta}{4k}\right)} e^{-2iky}. \tag{17}$$

We note that in the low-energy limit $k \to 0$ ($k' \to 0$ and $z = \frac{c^2}{2}$)

$$S_{\underline{\underline{\alpha}},y}\left(k^2 + \frac{c^2}{2}\right) \xrightarrow[k\to 0]{} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

$$y \in \mathbb{R}, \quad -\infty < \alpha, \beta \le \infty, \quad \underline{\underline{\alpha}} \ne \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In the high energy we obtain

$$S_{\underline{\underline{\alpha}},y} \xrightarrow[k\to\infty]{} \frac{1}{(2c+i\alpha)(2-i\beta c)} \begin{pmatrix} 4c\left(1 - \frac{\alpha\beta}{4}\right) & -2i(\alpha + \beta c^2) \\ -2i(\alpha + \beta c^2) & 4c\left(1 - \frac{\alpha\beta}{4}\right) \end{pmatrix},$$

$$-\infty < \alpha, \beta \le \infty, \quad y \in \mathbb{R},$$

and

$$S_{\underline{\underline{\alpha}},y} \xrightarrow[k\to\infty]{c\to\infty} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Remark 2. It turns out that the equation (19) can be obtained as a special case of the one-dimensional generalized point interactions introduced in [1].

We note that the pole of $S_{\underline{\underline{\alpha}},y}(z)$ coincides with the bound state ($\alpha, \tilde{\alpha} < 0$) or resonance ($\alpha, \tilde{\alpha} > 0$) of $D_{\underline{\underline{\alpha}},y}$.

3. The Nonrelativistic Point Interaction

A. Basic properties. Consider in the Hilbert space $L^2(\mathbb{R})$ the closed and nonnegative operator $\tilde{H}_y$ defined by

$$\tilde{H}_y = -\frac{d^2}{dx^2},$$
$$\mathcal{D}(\tilde{H}_y) = \{g \in H^{2,2}(\mathbb{R}) | g(y) = g'(y) = 0\}.$$

The adjoint $\tilde{H}_y^*$ of $\tilde{H}_y$ is defined by

$$\tilde{H}_y^* = -\frac{d^2}{dx^2},$$
$$\mathcal{D}(\tilde{H}_y^*) = H^{2,2}(\mathbb{R} - \{y\}), \quad y \in \mathbb{R}.$$

Hence the equation

$$\tilde{H}_y^* f(k) = k^2 f(k), \quad f(k) \in \mathcal{D}(\tilde{H}_y^*), \quad k^2 \in \mathbb{C} - \mathbb{R}, \quad \operatorname{Im} k > 0,$$

has two linearly independent solutions

$$f_1(k,x) = \begin{cases} \mathrm{e}^{ik(x-y)}, & x > y, \\ 0, & x < y, \end{cases}$$
$$f_2(k,x) = \begin{cases} 0, & x > y, \\ \mathrm{e}^{ik(y-x)}, & x < y, \end{cases} \quad \operatorname{Im} k > 0. \tag{18}$$

Therefore $\tilde{H}_y$ has deficiency indices (2,2) and hence it has a four-parameter family of self-adjoint extensions. We consider in $L^2(\mathbb{R})$ the operator $\Delta_{\alpha,\beta,y}$ defined by the equation (12)

$$\begin{aligned} &\Delta_{\alpha,\beta,y} = -\frac{d^2}{dx^2}, \\ &\mathcal{D}(\Delta_{\alpha,\beta,y}) = \\ &= \left\{ g \in H^{2,2}(\mathbb{R} - \{y\}) \;\middle|\; \begin{array}{l} g'(y+) - g'(y-) = \frac{\alpha}{2}[g(y+) + g(y-)] \\ g(y+) - g(y-) = \frac{\beta}{2}[g'(y+) + g'(y-)] \end{array} \right\}, \\ &-\infty < \alpha, \beta \leq \infty. \end{aligned} \tag{19}$$

Let $\alpha\beta - 4 = 0$, $\alpha, \beta \in \mathbb{R}$. Then the integration by parts shows that $\Delta_{\alpha,\beta,y}$ is symmetric and since $\tilde{H}_y$ has deficiency indices (2,2) and the 2-boundary conditions in (19) are symmetric and linearly independent, it follows that $\Delta_{\alpha,\beta,y}$ is self-adjoint ([18], Theorem XII.4.30). We will accept those α, β which satisfy the condition $\alpha\beta - 4 = 0,\ \alpha, \beta \in \mathbb{R}$.

The case $\alpha = 0$, $\beta = 0$ in the equation (19) yields the kinetic energy Hamiltonian Δ_0 in $L^2(\mathbb{R})$

$$\Delta_0 = -\frac{d^2}{dx^2}, \quad \mathcal{D}(\Delta_0) = H^{2,2}(\mathbb{R}).$$

The case $\alpha \neq 0$, $\beta = 0$ in the equation (19) gives the δ-point interaction of the first type, whereas $\alpha = 0$, $\beta \neq 0$ leads to a δ-point interaction of the second type [11].

B. Resolvent equation. The resolvent of $\Delta_{\alpha,\beta,y}$ is given by the following theorem.

Theorem 3.1. *The resolvent of* $\Delta_{\alpha,\beta,y}$ *is given by*

$$\begin{aligned}&(\Delta_{\alpha,\beta,y}-k^2)^{-1}=\\ &=G_k+\frac{1}{2\left(\frac{\alpha}{4k}-i\frac{1}{2}\right)\left(\frac{1}{2k^2}-i\frac{\beta}{4k}\right)}\Big\{i\frac{\alpha}{2k^2}(\overline{G_k(.-y)},.)G_k(.-y)+\\ &+i\frac{\beta}{2}(\overline{\tilde{\tilde{G}}_k(.-y)},.)\tilde{\tilde{G}}_k(.-y)+\frac{\alpha\beta}{4k}(\overline{G_k(.-y)},.)G_k(.-y)-\\ &-\frac{\alpha\beta}{4k}(\overline{\tilde{\tilde{G}}_k(.-y)},.)\tilde{\tilde{G}}_k(.-y)\Big\}, \qquad (20)\\ &k^2\in\rho(\Delta_{\alpha,\beta}),\quad \operatorname{Im}k>0,\quad -\infty<\alpha,\beta\le\infty,\quad y\in\mathbb{R},\end{aligned}$$

where

$$G_k(x-y)=\frac{i}{2k}\begin{cases}e^{ik(x-y)}, & x>y,\\ e^{ik(y-x)}, & x<y,\quad \operatorname{Im}k>0,\end{cases} \qquad (21)$$

$$\tilde{\tilde{G}}_k(x-y)=\frac{i}{2k}\begin{cases}e^{ik(x-y)}, & x>y,\\ -e^{ik(y-x)}, & x<y,\quad \operatorname{Im}k>0.\end{cases} \qquad (22)$$

Proof. We use the resolvent formula

$$(\Delta_{\alpha,\beta,y}-k^2)^{-1}=G_k-\frac{1}{4k^2}\sum_{i,j=1}^{2}\lambda_{ij}(k)(f_j(-\bar{k}),.)f_i(k), \qquad (23)$$

where f_j, $j=1,2$, are defined by (18).

Next consider $h\in L^2(\mathbb{R})$ and define the function $g\in\mathcal{D}(\Delta_{\alpha,\beta,y})$ by

$$g(k,x)=((\Delta_{\alpha,\beta,y}-k^2)^{-1}h)(x).$$

After imposing the boundary conditions in (19), one obtains

$$\lambda(k)=\frac{1}{2\left[\frac{\alpha}{4k}-i\frac{1}{2}\right]\left[\frac{\alpha}{4k}-i\frac{1}{2}\right]}\begin{pmatrix} i\frac{\alpha}{2k^2}+i\frac{\beta}{2} & i\frac{\alpha}{2k^2}+\frac{\alpha\beta}{2k}-i\frac{\beta}{2}\\ i\frac{\alpha}{2k^2}+\frac{\alpha\beta}{2k}-i\frac{\beta}{2} & i\frac{\alpha}{2k^2}+i\frac{\beta}{2}\end{pmatrix}. \qquad (24)$$

Inserting (24) in (23), one obtains (20).

Remark 3. From (20) one obtains the following results.

(i) As $\beta\to 0$, the Hamiltonian $\Delta_{\alpha,\beta,y}$ converges in the norm resolvent sense to $-\Delta_{\alpha,y}$:

$$n\cdot\lim_{\beta\to 0}(\Delta_{\alpha,\beta,y}-z)^{-1}=(-\Delta_{\alpha,y}-z)\ \ z\in\rho(\Delta_{\alpha,\beta,y})\cap\rho(-\Delta_{\alpha,y}),$$

where [11]

$$\begin{aligned}&(-\Delta_{\alpha,y}-k^2)^{-1}=G_k-\frac{2\alpha k}{i\alpha+2k}(\overline{G_k(.-y)},.)G_k(.-y),\\ &k^2\in\rho(-\Delta_{\alpha,y}),\quad \operatorname{Im}k>0,\quad -\infty<\alpha\le\infty,\quad y\in\mathbb{R},\end{aligned}$$

with G_k defined by (21).

(ii) As $\alpha \to 0$, the Hamiltonian $\Delta_{\alpha,\beta,y}$ converges in the norm resolvent sense to $-\Delta_{\beta,y}$:

$$n \cdot \lim_{\alpha \to 0} (\Delta_{\alpha,\beta,y} - z)^{-1} = (-\Delta_{\beta,y} - z) \;\; z \in \rho(\Delta_{\alpha,\beta,y}) \cap \rho(-\Delta_{\beta,y}),$$

where [11]

$$(-\Delta_{\beta,y}) - k^2)^{-1} = G_k - \frac{2\beta k^2}{2 - i\beta k} (\overline{\tilde{\tilde{G}}_k(.-y), .}) \tilde{\tilde{G}}_k(.-y),$$
$$k^2 \in \rho(-\Delta_{\alpha,y}), \quad \operatorname{Im} k > 0, \quad -\infty < \beta \leq \infty, \quad y \in \mathbb{R},$$

with $\tilde{\tilde{G}}_k$ defined by (22). The additional information on the domain of $\Delta\alpha, \beta, y$ is given by the following theorem.

Theorem 3.2. *The domain $\mathcal{D}(\Delta_{\alpha,\beta,y})$, $-\infty < \alpha, \beta \leq \infty$, $y \in \mathbb{R}^3$, consists of all elements $\psi_{\alpha,\beta}$ of the type*

$$\psi_{\alpha,\beta}(x) = \varphi_k(x) +$$
$$+ \frac{1}{2\left(\frac{\alpha}{4k} - i\frac{1}{2}\right)\left(\frac{1}{2k^2} - i\frac{\beta}{4k}\right)} \left\{ i\frac{\alpha}{2k^2}\varphi_k(y) G_k(x-y) - \frac{\beta}{2k}\varphi'(y)\tilde{\tilde{G}}_k(x-y) + \right.$$
$$\left. + \frac{\alpha\beta}{4k}\varphi_k(y) G_k(x-y) - i\frac{\alpha\beta}{4k^2}\varphi'_k(y)\tilde{\tilde{G}}_k(x-y) \right\}, \;\; x \neq y, \tag{25}$$

where $\varphi_k \in \mathcal{D}(\Delta_0) = H^{2,2}(\mathbb{R})$ and $\operatorname{Im}\, k > 0$. *The decomposition* (3.2) *is unique and with $\psi_{\alpha,\beta}$ of this form we obtain*

$$(\Delta_{\alpha,\beta,y} - z)\psi_{\alpha,\beta} = (\Delta_0 - z)\varphi_k. \tag{26}$$

Let $\psi_{\alpha,\beta} \in \mathcal{D}(\Delta_{\alpha,\beta,y})$ and assume that $\psi_{\alpha,\beta} = 0$ in an open set $\tilde{\vartheta} \in \mathbb{R}^3$. Then $\Delta_{\alpha,\beta,y}\psi_{\alpha,\beta} = 0$ in $\tilde{\vartheta}$, i.e., $\Delta_{\alpha,\beta,y}$ describes a local interaction.

Proof. Similar to the proof of Theorem 2.1.

C. Spectral properties. For $\alpha, \beta \in \mathbb{R}$, the essential spectrum of $\Delta_{\alpha,\beta,y}$ is purely absolutely continuous and coincides with $[0, \infty)$, while the singular spectrum is empty. The point spectrum of $\Delta_{\alpha,\beta,y}$ is given as the poles of the resolvent equation (20). One obtains

$$\sigma_p = \begin{cases} \left\{-\frac{\alpha^2}{4}, -\frac{4}{\beta^2}\right\}, & \alpha, \beta < 0, \\ 0, & \alpha, \beta \geq 0. \end{cases} \tag{27}$$

For $\alpha, \beta > 0$, $\Delta_{\alpha,\beta,y}$ has two resonances at $k_1 = -\frac{2i}{\beta}$ and $k_2 = -\frac{i\alpha}{2}$ with resonance functions respectively given by

$$\psi_{k_1}(x) = \begin{cases} \mathrm{e}^{\frac{\alpha}{2}(x-y)}, & x > y, \\ \mathrm{e}^{\frac{\alpha}{2}(y-x)}, & x < y, \end{cases} \qquad \alpha > 0,$$
$$\psi_{k_2}(x) = \begin{cases} \mathrm{e}^{\frac{2}{\beta}(x-y)}, & x > y, \\ -\mathrm{e}^{\frac{2}{\beta}(y-x)}, & x < y, \end{cases} \qquad \beta > 0.$$

D. Scattering theory of the pair $(\Delta_{\alpha,\beta,y}, \Delta_0)$**.** From (3.2) one can define the generalized function associated with $\Delta_{\alpha,\beta,y}$ by

$$\psi_{\alpha,\beta,y} =$$
$$= e^{ik\sigma x} + \frac{1}{4k\left(\frac{\alpha}{4k} - i\frac{1}{2}\right)\left(\frac{1}{2k^2} - i\frac{\beta}{4k}\right)} \left\{ -\frac{\alpha}{2k^2} e^{ik\sigma y} \left\{ \begin{array}{ll} e^{ik(x-y)}, & x > y \\ e^{ik(y-x)}, & x < y \end{array} \right\} + \right.$$
$$+ \frac{\sigma\beta}{2} e^{ik\sigma y} \left\{ \begin{array}{ll} e^{ik(x-y)}, & x > y \\ -e^{ik(y-x)}, & x < y \end{array} \right\} + i\frac{\alpha\beta}{4k} e^{ik\sigma y} \left\{ \begin{array}{ll} e^{ik(x-y)}, & x > y \\ e^{ik(y-x)}, & x < y \end{array} \right\} +$$
$$\left. + \, i\frac{\sigma\alpha\beta}{4k} e^{ik\sigma y} \left\{ \begin{array}{ll} e^{ik(x-y)}, & x > y \\ -e^{ik(y-x)}, & x < y \end{array} \right\} \right\},$$
$$x, y \in \mathbb{R},\ k > 0, \sigma = \pm 1, -\infty < \alpha, \beta \le \infty.$$

The corresponding reflection and transmission coefficients from the left ($\sigma = +1$) and the right ($\sigma = -1$) are defined by [11]

$$\begin{aligned}
\mathcal{R}^l_{\alpha,\beta,y}(k) &= \lim_{x \to -\infty} e^{ikx} \left[\psi_{\alpha,\beta,y}(z, +1, x) - e^{ikx} \right], \\
\mathcal{R}^r_{\alpha,\beta,y}(k) &= \lim_{x \to +\infty} e^{-ikx} \left[\psi_{\alpha,\beta,y}(k, -1, x) - e^{-ikx} \right], \\
\mathcal{T}^l_{\alpha,\beta,y}(k) &= \lim_{x \to +\infty} e^{-ikx} \psi_{\alpha,\beta,y}(k, +1, x), \\
\mathcal{T}^r_{\alpha,\beta,y}(k) &= \lim_{x \to -\infty} e^{ikx} \psi_{\alpha,\beta,y}(k, -1, x), \\
& k \ge 0, \quad -\infty < \alpha, \beta \le \infty, \quad y \in \mathbb{R}.
\end{aligned}$$

After a straightforward computation, one obtains

Theorem 3.3. *Let* $\alpha, \beta \in \mathbb{R} - \{0\}$, $y \in \mathbb{R}$. *Then the unitary on-shell scattering matrix* $\mathcal{S}_{\alpha,\beta,y}(k)$ *in* $\mathbb{C}^2$ *associated with the pair* $(\Delta_{\alpha,\beta,y}, \Delta_0)$ *reads*

$$\mathcal{S}_{\alpha,\beta,y}(k) = \left[\begin{array}{cc} \mathcal{T}^l_{\alpha,\beta,y}(k) & \mathcal{R}^r_{\alpha,\beta,y}(k) \\ \mathcal{R}^l_{\alpha,\beta,y}(k) & \mathcal{T}^r_{\alpha,\beta,y}(k) \end{array} \right], \qquad (28)$$
$$k \ge 0, \quad -\infty < \alpha, \beta \le \infty, \quad y \in \mathbb{R},$$

with

$$\mathcal{T}^l_{\alpha,\beta,y}(k) = i \frac{\left(\frac{\alpha\beta}{4} - 1\right)}{4k^2 \left(\frac{\alpha}{4k} - \frac{i}{2}\right)\left(\frac{1}{2k^2} - i\frac{\beta}{4k}\right)} = \mathcal{T}^r_{\alpha,\beta,y}(k),$$

$$\mathcal{R}^l_{\alpha,\beta,y}(k) = -\frac{\left(\frac{\alpha}{k} + \beta k\right)}{8k^2\left(\frac{\alpha}{4k} - \frac{i}{2}\right)\left(\frac{1}{2k^2} - i\frac{\beta}{4k}\right)} e^{2iky},$$

$$\mathcal{R}^r_{\alpha,\beta,y}(k) = -\frac{\left(\frac{\alpha}{k} + \beta k\right)}{8k^2\left(\frac{\alpha}{4k} - \frac{i}{2}\right)\left(\frac{1}{2k^2} - i\frac{\beta}{4k}\right)} e^{-2iky}.$$

We note that the limit $\beta \to 0$ (respectively $\alpha \to 0$) in the equation (28) gives the unitary on-shell scattering matrix $\mathcal{S}_{\alpha,y}(k)$ ($\mathcal{S}_{\beta,y}(k)$) associated with the pair $(-\Delta_{\alpha,y}, -\Delta)$ and $(-\Delta_{\beta,y}, -\Delta)$, respectively [11]. One obtains

$$\mathcal{S}_{\alpha,\beta,y}(k) \xrightarrow[\beta\to 0]{} (2k+i\alpha)^{-1} \begin{bmatrix} 2k & -i\alpha e^{-2iky} \\ -i\alpha e^{2iky} & 2k \end{bmatrix} = \\ = \mathcal{S}_{\alpha,y}(k), \quad k \geq 0, \quad -\infty < \alpha \leq \infty, \quad y \in \mathbb{R},$$

$$\mathcal{S}_{\alpha,\beta,y}(k) \xrightarrow[\alpha\to 0]{} (2-i\beta k)^{-1} \begin{bmatrix} 2 & -i\beta k e^{-2iky} \\ -i\beta k e^{2iky} & 2 \end{bmatrix} = \\ = \mathcal{S}_{\beta,y}(k), \quad k \geq 0, \quad -\infty < \beta \leq \infty, \quad y \in \mathbb{R}.$$

In the low-energy limit $k \to 0$ we get

$$S_{\alpha,\beta,y}(k) \xrightarrow[k\to 0]{} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

and in the high energy limit we obtain

$$S_{\alpha,\beta,y}(k) \xrightarrow[k\to\infty]{} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We note that the pole of $S_{\alpha,\beta,y}(k)$ coincides with the bound state $(\alpha, \beta < 0)$ or resonance $(\alpha, \beta > 0)$ of $\Delta_{\alpha,\beta,y}$.

Acknowledgement

The second author would like to thank Prof. Blanchard, Universität Bielefeld, for his invitation to the Research Institute BIBOS and for interesting discussions. We are both thankful to Prof. Exner for useful comments on a preliminary draft of this paper and for sharing with us a copy of an unpublished preprint.

References

1. P. Exner and H. Grosse, Some properties of the one dimensional generalized point interactions. *Preprint* 1999.
2. S. G. Davison and M. Steslicka, Relativistic theory of Tamm surface states. *J. Phys. C* **2** (1969), 1802–1807.
3. S. G. Davison and M. Steslicka, Relativistic treatment of localized states. *Inter J. Quantum Chem.* **4** (1971), 445–453.
4. M. Steślicka and S. G. Davison, Boundary conditions for the relativistic Kronig–Penney model. *Phys. Rev. B (3)* **1** (1970), 1858–1860.
5. R. Subramanian and K. V. Bhagwat, The relativistic Tamm model. *J. Phys. C* **5** (1972), 798–806.
6. R. Subramanian and K. V. Bhagwat, Relativistic generalization of the Saxon-Hutner theorem. *Phys. Status Solidi (B)* **48** (1971), No. 1, 399–406.
7. W. M. Fairbairn, M. L. Glasser and M. Steślicka, Relativistic theory of surface states. *Surf. Sci* **36** (1973), 462–471.
8. G. Gumbs, Relativistic scattering states for a finite chain with δ-function potentials of arbitrary position and strength. *Phys. Rev. A* **32** (1985), 1208–1210.
9. I. R. Lapidus, Relativistic one-dimensional hydrogen atom. *Amer. J. Phys.* **51** (1983), 1036–1038.

10. F. Gesztesy and P. Seba, New analytically solvable models of relativistic point interactions. *Lett. Math. Phys.* **13** (1987), 345–358.
11. S. Albeverio, F. Gesztesy, R. Hoegh-Krohn and H. Holden, Solvable models in quantum mechanics. *Texts and Monographs in Physics. Springer-Verlag, New York,* 1988.
12. J. Dittrich, P. Exner and P. Seba, Dirac operators with a spherically symmetric δ-shell interaction. *J. Math. Phys.* **30** (1989), 2875–2882.
13. J. Shabani and A. Vyabandi, Exactly solvable models of relativistic δ-sphere interactions in quantum mechanics. *J. Math. Phys.* **43** (2002), No. 12, 6064–6084.
14. J. Shabani and A. Vyabandi, A note on a series of papers on relativistic δ-sphere interactions in quantum mechanics published by M. N. Hounkonnou and G. Y. H. Avossevou in the Journal of Mathematical Physics. *J. Math. Phys.* **43** (2002), No. 12, 6380–6384.
15. J. Shabani and A. Vyabandi, In preparation.
16. N. I. Akhiezer and I. M. Glazman, Theory of linear operators in Hilbert space. Vol. II. *Monographs and Studies in Mathematics,* 10. *Pitman* (*Advanced Publishing Program*), *Boston, Mass.-London,* 1981.
17. R. G. Newton, Inverse scattering. I. One dimension. *J. Math. Phys.* **21** (1980), No. 3, 493–505.
18. N. Dunford and J. T. Schwartz, Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space. *Interscience Publishers John Wiley & Sons New York-London,* 1963.

(Received 7.03.2005)

Authors' addresses:

J. Shabani*
UNESCO Harare Cluster office
Harare, Zimbabwe
E-mail: j.shabani@unesco.org

A. Vyabandi
Université du Burundi
Institut de Pédagogie Appliquée(IPA)
BP 5223 MutangaI Bujumbura, Burundi
E-mail: vyabandi@hotmail.com

*On leave of absence from University of Burundi, Faculty of Science, BP 2700 Bujumbura, Burundi