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**ON THE HAMILTONIANS INDUCED FROM
A FUCHSIAN SYSTEM**

Abstract. Fuchsian systems on a complex manifold with nontrivial topology are investigated and Hamiltonians, whose dynamic equations reduce to a Fuchs type differential equation, are given. These Hamiltonians and equations correspond to realistic physical models encountered in the literature.

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1. SOLUTION SPACES OF REGULAR AND FUCHSIAN SYSTEMS

Let X be an m -dimensional complex analytic manifold and suppose $D = \cup_{i=1}^n D_i$ is a divisor such that D_j are generic 1-codimensional submanifolds of X . It means that for any point $x \in X$ and for any holomorphic functions s_{pi} which are local equations in a neighborhood U_x of x for those submanifolds D_{pi} of D which contain x , the forms ds_{pi} are linearly independents at x . Let

$$df = \omega f \quad (1)$$

be a completely integrable Pfaffian system on X , where ω is an $d \times d$ -matrix-valued holomorphic 1-form on $X \setminus D$. The complete integrability condition means that ω satisfies $d\omega - \omega \wedge \omega = 0$. From the complete integrability of ω it follows that the solution space of (1) is an d -dimensional vector space. In this section we describe this space following the papers [1], [2], [3].

Let $\mathcal{D} \subset \mathbb{C}$ be the unit disk with center 0. Denote $\mathcal{D}^m = \prod_{i=1}^n \mathcal{D}_i$ and $\mathcal{D}_n^m = \mathcal{D}^m \setminus \bigcup_{i=1}^n \{x_i = 0\}$. Let $p : \tilde{\mathcal{D}}_n^m \rightarrow \mathcal{D}_n^m$ be the universal covering. Let $y = (y_1, \dots, y_m)$ and $x = (x_1, \dots, x_m)$ be points from $\tilde{\mathcal{D}}_n^m$ and \mathcal{D}_n^m , respectively.

Denote by $L_{d \times s}(\tilde{\mathcal{D}}_n^m)$ the space of holomorphic maps from $\tilde{\mathcal{D}}_n^m$ to the space of constant $d \times s$ -rectangular complex matrices $M_{d \times s}$.

Let $f \in L_{d \times s}(\tilde{\mathcal{D}}_n^m)$. We will say that f has *polynomial growth* at 0 if there exist integers $k_1, \dots, k_m \in \mathbb{Z}$, such that $\lim_{p(y) \in U, p(y) \rightarrow 0} f(y_1, \dots, y_m) \times \prod_{i=1}^n y_i^{k_i} = 0$, where $U \subset \tilde{\mathcal{D}}_n^m$. Denote by $L_{d \times s}^0(\tilde{\mathcal{D}}_n^m)$ the subspace of $L_{d \times s}(\tilde{\mathcal{D}}_n^m)$ which consists of the functions of polynomial growth at 0.

The fundamental group $\pi_1(\tilde{\mathcal{D}}_n^m) = \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$ acts on the space $L_{d \times s}(\tilde{\mathcal{D}}_n^m)$ as $\gamma^*(f(y)) = f(\gamma^{-1}y)$, where $\gamma \in \pi_1(\tilde{\mathcal{D}}_n^m)$ and $f(y) \in L_{d \times s}(\tilde{\mathcal{D}}_n^m)$.

Let \mathcal{F} be a subspace of $L_{d \times 1}^0(\tilde{\mathcal{D}}_n^m)$ with properties:

1. $\dim \mathcal{F} = d$

and

2. the space \mathcal{F} is invariant under the action of the fundamental group $\pi_1(\tilde{\mathcal{D}}_n^m)$.

Proposition 1 ([1]). *Let $f \in \mathcal{F}$. Then any coordinate functions $f^{(i)}(y)$ of $f(y)$ are the following logarithmic sums:*

$$f^{(i)}(y) = \sum_{\bar{j}, \bar{l} \in \sigma} f_{\bar{j}\bar{l}}^{(i)}(x) y^{\rho_{\bar{j}}} \log^{b_{\bar{l}}} y,$$

where $\bar{j} = (j_1, \dots, j_n)$ and $\bar{l} = (l_1, \dots, l_n)$ are multiindices; $f_{\bar{j}\bar{l}}^{(i)}(x)$ are convergent Laurent series with finitely many principal parts; $0 \leq \operatorname{Re} \rho_{j_i} < 1$; b_{l_i} are nonnegative integers; the sum is finite and similar terms are collected.

Denote by $\varphi_k(f_{\tilde{j}l}^{(i)})$ the order of zero (or of pole, with the minus sign) of the series $f_{\tilde{j}l}^{(i)}$ with respect to the k -th coordinate, where $k = 1, \dots, n$. We introduce the norm of the element of $f \in \mathcal{F}$ with respect to the k -th coordinate as $\varphi_k(f) = \min_{j \in \sigma; i=1, \dots, d} \varphi_k(f_{\tilde{j}l}^{(i)})$.

Let γ_i denote the generator of $\pi_1(\tilde{\mathcal{D}})$ corresponding to going around the hypersurface $x_i = 0$. The fundamental group $\pi_1(\tilde{\mathcal{D}})$ is abelian and from this it follows that \mathcal{F} splits into a sum $\mathcal{F} = \bigoplus_{i=1}^q \mathcal{F}_i$, where every \mathcal{F}_i is the eigenspace for the operators $\gamma_1, \dots, \gamma_n$ with eigenvalues $(v_1^i, \dots, v_n^i) \neq (v_1^j, \dots, v_n^j)$ if $i \neq j$.

The functions φ_k have finite many values on every \mathcal{F}_l , which we denote by ${}^l\varphi_k^1, \dots, {}^l\varphi_k^{l_k}$, with multiplicities d_1, \dots, d_{l_k} equal to the dimensions of those subspaces of \mathcal{F}_k on which the φ_k are constant. Denote $\rho_k^r = \frac{1}{2\pi i} \log v_k^r$, $0 \leq \operatorname{Re} \rho_k^r < 1$. The numbers ${}^r\beta_k^l = {}^r\varphi_k^l + \rho_k^r$, $k = 1, \dots, n$, $l = 1, \dots, q$, $l = 1, \dots, l_k$ are called the *exponents* of the space \mathcal{F} at zero. A matrix function Ψ whose columns are elements of some basis of the space \mathcal{F} is called a *fundamental matrix* of \mathcal{F} . Let Ψ be a fundamental matrix of the space \mathcal{F} . Then $\det \Psi(y) = \prod_{i=1}^n y_i^{\beta_i + \alpha_i} \varphi(x)$, where β_i is a sum of the i -exponents of \mathcal{F} with multiplicities, the α_i are nonnegative integers and $\varphi(x) \neq 0$.

The space \mathcal{F} is called *weakly singular* at zero, if $\det \Psi(y) = \prod_{i=1}^n y_i^{\beta_i} \varphi(x)$ and $\varphi(x) \neq 0$.

The space \mathcal{F} is the space of solutions of a Fuchs type Pfaffian system on $(\mathcal{D}^\varepsilon)^m$, where ε is the radius of the polydisc, iff \mathcal{F} is weakly singular. In this case the exponents ${}^r\beta_k^l$ are eigenvalues $\omega_k(0)$ at zero of the matrix function

$$\omega(x) = \sum_{i=1}^n \omega_i(x) \frac{dx_i}{x_i} + \sum_{j=n+1}^m \omega_j(x) dx_j.$$

We consider the particular case $n = 2$. Then the vector space \mathcal{F} is the space of solutions of a Fuchs type Pfaffian system on $(\mathcal{D}^\varepsilon)^m$, iff for \mathcal{F} there exists a fundamental matrix $\Psi(y)$ of the form

$$\Phi(y) = U(x) y_1^{\tilde{A}_1} y_2^{\tilde{A}_2} y_1^{E_1} y_2^{E_2},$$

where \tilde{A}_1, \tilde{A}_2 , are diagonal matrices with integer entries and their columns are φ_1 and φ_2 , $E_i = \frac{1}{2\pi i} \log \gamma_i^*$ and $U(x)$ is holomorphic and invertible in $(\mathcal{D}^\varepsilon)^m$.

Let the Fuchs system (1) be completely integrable in \mathcal{D}^m . Then there exist diagonal matrices E_i , $i = 1, \dots, n$, with integer entries and a holomorphic invertible matrix function $U(x)$ such that under the substitution

$f = U(x) \prod_{i=1}^n x_i^{E_i} g$, the system (1) acquires the form

$$dg = \left(\sum_{i=1}^m B_i \frac{dx_i}{x_i} \right) g,$$

where B_i are constant matrices.

A space \mathcal{F} is called *weakly singular* in D , if it is weakly singular at every point $x \in D$. The space \mathcal{F} is called *weakly singular* in the manifold $X_m = X \setminus D$ if $\det \Phi(y) \neq 0$ for every $y \in \tilde{X}_m \setminus \tilde{D}$.

The space \mathcal{F} is the space of solutions of a Fuchs type Pfaffian system in X iff \mathcal{F} is weakly singular in X . In this case the form $\omega = d \log \det \Phi(y)$ in the neighborhood U_x , $x \in D$ has the form

$$\omega = \sum_i \beta_{ji} \frac{ds_{ji}}{s_{ji}} + \phi_x,$$

where β_i is the sum of i -exponents of \mathcal{F} , and ϕ_x is a holomorphic function in U_x .

Theorem 1 ([1]). 1. *The space \mathcal{F} is the space of solutions of a Fuchs type Pfaffian system in X iff the cocycle $\sum_{i=1}^n \beta_i D_i$ is homological to zero.*

2. *Let $D = \bigcup_{i=1}^n D_i$ be a generic divisor and suppose that equations for D_i are given by homogeneous polynomials $f_i(x_1, \dots, x_m)$. Let the space \mathcal{F} be weakly singular in $\mathbb{C}\mathbb{P}^m \setminus D$ and suppose that the sum of i -exponents of \mathcal{F} satisfies the conditions $\sum_{i=1}^n \beta_i \deg(f_i) \leq 0$. The space \mathcal{F} is the solution space of a Fuchs type Pfaffian system on $\mathbb{C}\mathbb{P}^m$ iff $\sum_{i=1}^n \beta_i \deg f_i = 0$.*

From the results of this section it follows that a Fuchs type Pfaffian system defines a finite n -dimensional functional vector space \mathcal{F} whose elements have polynomial growth on the branched submanifolds D of X and define a monodromy representation

$$\rho : \pi_1(X \setminus D, z_o) \rightarrow GL_n(\mathbb{C}).$$

Monodromy matrices act on \mathcal{F} as linear operators. The integer valued function φ has finitely many values $\infty > n^1 > \dots > n^l$ on \mathcal{F} and defines a filtration

$$0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \dots \subset \mathcal{F}^l = \mathcal{F},$$

where $\mathcal{F}^j = \{f \in \mathcal{F} \mid \varphi_i(f) \geq n^j\}$. The monodromy operators preserve this filtration.

2. HAMILTONIANS OF QUANTUM SYSTEMS AND THE HYPERGEOMETRIC EQUATION

Theorem 2. *The hypergeometric equation*

$$z(z-1)\frac{d^2g_1}{dz^2} + (\gamma - (1 + \alpha + \beta))\frac{dg_1}{dz} - \alpha\beta g_1(z) = 0 \quad (2)$$

is a Schrödinger type equation

$$i\frac{\partial f(t)}{\partial t} = H(t)f(t), \quad (3)$$

where $f(t) = (f_1(t), f_2(t))$ and the time dependent Hamiltonian $H(t)$ has the form

$$H(t) = \begin{pmatrix} \varepsilon(t) & V(t) \\ V(t) & -\varepsilon(t) \end{pmatrix}, \quad (4)$$

where $\varepsilon(t) = E_0 \sec h(t/T) + E_1 \tanh(t/T)$, $V(t) = V_0$ and E_0, E_1, T, V_0 are constants.

Proof. First we consider a very well known procedure. Rewrite (3) in the form:

$$if_1'(t) = \varepsilon(t)f_1(t) + V(t)f_2(t), \quad (5)$$

$$if_2'(t) = V(t)f_1(t) - \varepsilon(t)f_2(t). \quad (6)$$

Suppose

$$f_1(t) = g_1(t)e^{-i\int_0^t \varepsilon(\tau)d\tau}, \quad (7)$$

$$f_2(t) = g_2(t)e^{i\int_0^t \varepsilon(\tau)d\tau}, \quad (8)$$

then

$$g_1'(t) = f_1'(t)e^{i\int_0^t \varepsilon(\tau)d\tau} + if_1(t)\varepsilon(t)e^{i\int_0^t \varepsilon(\tau)d\tau}, \quad (9)$$

$$g_2'(t) = f_2'(t)e^{-i\int_0^t \varepsilon(\tau)d\tau} - if_2(t)\varepsilon(t)e^{i\int_0^t \varepsilon(\tau)d\tau}. \quad (10)$$

Substituting in the expression (9) $f_1'(t)$ from (5), one obtains

$$g_1'(t) = -if_1(t)\varepsilon(t)e^{i\int_0^t \varepsilon(\tau)d\tau} - iV(t)f_2(t)e^{i\int_0^t \varepsilon(\tau)d\tau} + if_1(t)\varepsilon(t)e^{i\int_0^t \varepsilon(\tau)d\tau} \Rightarrow g_1'(t) = -iV(t)f_2(t)e^{i\int_0^t \varepsilon(\tau)d\tau}.$$

Changing $f_2(t)$ by (8) one obtains

$$g_1' = -iV(t)g_2(t)e^{2i\int_0^t \varepsilon(\tau)d\tau}. \quad (11)$$

In a similar way we obtain

$$g_2' = -iV(t)g_1(t)e^{-2i\int_0^t \varepsilon(\tau)d\tau}. \quad (12)$$

From (11) we find the second derivative of $g_1(t)$ with respect to t :

$$g_1''(t) = -iV'(t)g_2(t)e^{2i\int_0^t \varepsilon(\tau)d\tau} - iV(t)g_2'(t)e^{2i\int_0^t \varepsilon(\tau)d\tau} + 2V(t)g_2(t)\varepsilon(t)e^{2i\int_0^t \varepsilon(\tau)d\tau}.$$

Substituting in this expression $g_2(t)$ from (12), we obtain

$$g_1''(t) = -iV(t)g_2(t)e^{2i\int_0^t \varepsilon(\tau)d\tau} - V^2(t)g_1(t) + 2V(t)g_2(t)\varepsilon(t)e^{2i\int_0^t \varepsilon(\tau)d\tau}.$$

In order to eliminate $g_2(t)$ from the latter, we will use (11). Finally we obtain the following second order differential equation with respect to $g_1(t)$:

$$g_1''(t) - \left(2i\varepsilon(t) + \frac{V'(t)}{V(t)}\right)g_1'(t) + V^2(t)g_1(t) = 0. \quad (13)$$

In a similar way we obtain a second order equation for $g_2(t)$:

$$g_2''(t) - \left(\frac{V'(t)}{V(t)} - 2i\varepsilon(t)\right)g_2'(t) + V^2(t)g_2(t) = 0. \quad (14)$$

Now we use the specification of matrix entries of $H(t)$. By substituting ε and $V(t)$ into (13) and adopting the change of variable as

$$z(t) = \frac{\sin h(t/T) + i}{2i},$$

the equation (13) can be reduced to the hypergeometric equation (2), where

$$\begin{aligned} \alpha &= iT \left(-E_1 + \sqrt{E_1^2 + V_0^2} \right), \\ \beta &= iT \left(-E_1 - \sqrt{E_1^2 + V_0^2} \right), \\ \gamma &= \frac{1}{2} + E_0T - iE_1T. \end{aligned}$$

Analogously from (14) we obtain the hypergeometric equation with respect to $g_2(z)$:

$$z(z-1)\frac{d^2g_2}{dz^2} + (\gamma' - (1 + \alpha' + \beta'))\frac{dg_2}{dz} - \alpha'\beta'g_2(z) = 0,$$

where

$$\begin{aligned} \alpha' &= iT \left(E_1 + \sqrt{E_1^2 + V_0^2} \right), \\ \beta' &= iT \left(E_1 - \sqrt{E_1^2 + V_0^2} \right), \\ \gamma' &= \frac{1}{2} - E_0T + iE_1T. \quad \square \end{aligned}$$

Remark 1. This theorem is true in more general cases. We consider one from the so called analytical solvable model of quantum dynamics. First to consider such approach were Landau, Rosen and Zener which has been subsequently generalized by several authors (see [5] and references there in). Using the methods of analytic differential equations, in [6] an analytic calculation of nonadiabatic transition probabilities for a two level quantum system is given.

Theorem 3. *The Fuchs type Pfaffian system*

$$(z\mathbb{I} - C) \frac{d\Phi(z)}{dz} = A\Phi(z), \quad (15)$$

where \mathbb{I} is the identity matrix, C and A are respectively a diagonal and arbitrary matrix, is a Schrödinger type equation

$$i \frac{\partial \Psi(t)}{\partial t} = H(t)\Psi(t) \quad (16)$$

with time depending Hamiltonian $H(t) = (H_{ij}(t))$, $i, j = 1, \dots, N$, where

$$\begin{aligned} H_{11} &= \varepsilon(t), H_{12} = V_2, H_{13} = V_3, \dots, H_{1N} = V_N, \\ H_{21} &= V_2, H_{31} = V_3, \dots, H_{2N} = V_N \quad \text{and} \quad H_{ij} = 0 \quad \text{otherwise,} \end{aligned}$$

and $\Psi(t) = (\psi_1(t), \dots, \psi_N(t))$ is a wave function. Here the time dependent part ε is given as $\varepsilon(t) = E_1 \tan h(t/T)$ and V_j are constant.

Proof. Consider the following transformation of the vector function $\Psi(t) = (\psi_1(t), \dots, \psi_N(t))$:

$$g_1(t) = \psi_1(t) e^{i \int_0^t \varepsilon(\tau) d\tau}, \quad g_j(t) = \psi_j(t), \quad j = 2, \dots, N.$$

From this and the identity $i \int_0^t \varepsilon(\tau) d\tau = iE_1 T \log(\cos h(t/T))$ follows the following system of equations:

$$\begin{aligned} g_1'(t) &= T^{-1} \sum_{j=2}^N v_j (\cos ht/T)^{2\varepsilon_1} g_j, \\ g_j'(t) &= (T)^{-1} v_j (\cos ht/T)^{-2\varepsilon_1} g_1, \quad \text{if } 2 \leq j \leq N, \end{aligned}$$

where $\varepsilon_1 = iE_1 T/2$, $v_j = -iV_j T$. After the change of the time variable $z(t) = \sin h(t/T)$, the above system becomes

$$\begin{cases} \frac{dg_1(z)}{dz} = \sum_{j=2}^N v_j (1+z^2)^{\varepsilon_1-1/2} g_j(z), \\ \frac{dg_j(z)}{dz} = v_j (1+z^2)^{-\varepsilon_1-1/2} g_1(z), \quad 2 \leq j \leq N. \end{cases}$$

Let us take arbitrary numbers $\lambda_2, \dots, \lambda_N$ satisfying the equality $\sum_{j=2}^N \lambda_j = 1$ and change the variable once more:

$$\begin{cases} \phi_1(z) = (1+z^2)^{-\varepsilon_1-1/2} g_1(z), \\ \phi_j(z) = \frac{v_j g_j}{z+i} - \lambda \left(\varepsilon_1 + \frac{1}{2} \right) \frac{z-i}{z+i} \phi_1(z), \quad 2 \leq j \leq N. \end{cases}$$

Finally we obtain the following system:

$$\begin{cases} (z-i)\frac{d\phi_1(z)}{dz} = -\left(\varepsilon_1 + \frac{1}{2}\right)\phi_1(z) + \sum_{j=2}^N \phi_j(z), \\ (z+i)\frac{d\phi_j(z)}{dz} = \lambda_j\left(\varepsilon_1^2 + v_i^2 - \frac{1}{4}\right)\phi_1(z) - \phi_j(z) - \lambda_j\left(\varepsilon_1 + \frac{1}{2}\right)\sum_{k=2}^N \phi_k(z), \\ 2 \leq j \leq N, \end{cases}$$

which after writing in a matrix form will give (15). \square

Remark. The Fuchsian system (15) is known as the *Okubo equation* [7].

Theorem 4. *Let \mathcal{F} be a four-dimensional weakly singular vector space in $CP^1 \setminus \{s_1, s_2, \infty\}$ with exponents at these points $(a_{11}, a_{22}, 0, 0)$, $(0, 0, a_{33}, a_{44})$, $(\beta_1, \beta_1, \beta_3, \beta_4)$. Then \mathcal{F} is a solution space of a fourth order Fuchsian differential equation of Okubo type.*

Remark. This theorem is a modification of the main result from [8] in spirit of Section 1 of this paper.

Sketch of proof. Take the matrix C from (15) as $C = \text{diag}(s_1, s_1, s_2, s_2)$, and let A be diagonalizable and have nonresonant nonnegative eigenvalues $\beta_1 = \beta_2, \beta_3, \beta_4$. From this it follows that A has a block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11}, A_{22} are diagonalizable 2×2 -matrices with nonresonant eigenvalues. This system is Fuchsian and has three singular points at s_1, s_2, ∞ with exponents prescribed by the theorem. \square

Remark. In [8] the monodromy group of the so obtained fourth order Fuchsian system is calculated in terms of exponents. In that paper also necessary and sufficient conditions of irreducibility for the monodromy group in terms of the exponents are obtained. From this and a result of A. Bolobruh (see [4]) it follows that in this case a condition of solvability of Riemann-Hilbert monodromy problem [9], [10] in terms of the exponents can be obtained. Moreover, the obtained system of equations will be an equation of Okubo type and therefore has interpretation as an equation describing dynamics of a quantum system (for example, quantum manipulation of qubits [11], [12]).

3. OTHER HAMILTONIANS

A deformation of a Fuchsian system is a family of Fuchsian systems depending on parameters:

$$\frac{d\Phi(z)}{dz} = \left(\sum_j^n \frac{A_j(s)}{z - s_j} \right) \Phi(z). \quad (17)$$

This means that the coefficients A_j , $j = 1, \dots, n$, depend on the parameters $s = (s_1, \dots, s_n)$. Let s belong to some open set of the space \mathbb{C}^n and suppose that the coefficients $A_j(s_1, \dots, s_n)$ depend on s_1, \dots, s_n holomorphically. Such a deformation is said to be a holomorphic deformation of the Fuchsian system.

The Schlesinger system (see [13]) is an overdetermined Pfaffian system of differential equations of the form

$$\frac{\partial A_i}{\partial s_j} = \frac{[A_i, A_j]}{s_i - s_j}, \quad 1 \leq i, j \leq n, \quad i \neq j, \quad (18)$$

$$\frac{\partial A_i}{\partial s_i} = - \sum_{1 \leq j \leq n, i \neq j} \frac{[A_i, A_j]}{s_i - s_j}, \quad 1 \leq i, j \leq n, \quad i \neq j, \quad 1 \leq i \leq n, \quad (19)$$

where A_1, \dots, A_n are $k \times k$ -matrix functions of $s = (s_1, \dots, s_n) \in \mathbb{C}_*^n = \mathbb{C}^n \setminus \text{diagonals}$. The system of equations (18)–(19) can be rewritten as

$$dA_i = \sum_{j=1, j \neq i}^n [A_j, A_i] d \log(s_j - s_i), \quad i = 1, \dots, n. \quad (20)$$

Here d is the exterior differential. The integrability condition for the Schlesinger system is

$$d \left(\sum_{j=1, j \neq i}^n [A_j, A_i] d \log(s_j - s_i) \right) = 0, \quad i = 1, \dots, n. \quad (21)$$

This condition must be fulfilled if $A_1(s), \dots, A_n(s)$ satisfy the equations (20).

Denote by $gl_n(C)^N = gl_n(C) \oplus \dots \oplus gl_n(C)$ the direct sum of N copies of $gl_n(C)$. This is the space of N -tuples A_1, \dots, A_N of $n \times n$ -matrices. The group $GL_n(C)$ acts on this space by the diagonal coadjoint action: $A_j \mapsto gA_jg^{-1}$. Each coadjoint orbit \mathcal{O}_i is left invariant under the t -flows. Thus the Schlesinger equation is actually a family of non-autonomous dynamical systems on a direct product $\mathcal{O}_1 \times \dots \times \mathcal{O}_N$ of coadjoint orbits in $gl_n(C)$. The coadjoint structure leads to a Hamiltonian formalism of the Schlesinger equation (see [14], [15], [16]). Let us introduce a Poisson structure on the vector space $gl_n(C)^N$ by defining the Poisson bracket of the matrix elements of $A_i = (A_i^{pq})$ as:

$$\{A_i^{pq}, A_j^{rs}\} = \delta_{ij} (-\delta_{qr} A_i^{ps} + \delta_{sp} A_i^{rq}).$$

In each component of the direct sum $gl_n(C) \oplus \dots \oplus gl_n(C)$ this Poisson bracket is just the ordinary Kostant-Kirrilov Poisson bracket. The Schlesinger equation can be written in the Hamiltonian form

$$\frac{\partial A_j}{\partial t_i} = \{A_j, H_i\}, \quad (22)$$

where the Hamiltonians are given by

$$H_i = \sum_{j \neq i} \text{Tr} \left(\frac{A_i A_j}{s_i - s_j} \right), \quad (23)$$

and the involution

$$\{H_i, H_j\} = 0. \quad (24)$$

Theorem 5 (see [13]). *Let $A_1(s), \dots, A_n(s)$ be holomorphic with respect to $s_1, \dots, s_n \in \mathcal{U} \subset C_*^n$ $k \times k$ -matrix functions. Assume the square $k \times k$ -matrix function $\Psi(z, s)$, which is a) holomorphic for $z \in C$ and for $s \in \mathcal{U}, z \neq s_1, \dots, s_n$, b) is nondegenerate: $\det \Psi(z, s) \neq 0$, for $s \in \mathcal{U}, z \neq s_1, \dots, s_n$ and c) satisfies the following system of differential equations*

$$\frac{\partial \Psi(z, s)}{\partial z} = \left(\sum_{1 \leq j \leq n} \frac{A_j}{z - s_j} \right) \Psi(z, s),$$

$$\frac{\partial \Psi(z, s)}{\partial s_k} = \frac{A_k}{z - s_k} \Psi(z, s), \quad k = 1, \dots, n.$$

Then the matrix functions $A_1(s), \dots, A_n(s)$ satisfy the Schlesinger system for $s \in \mathcal{U}$.

From the results of Section 1 it follows that the fundamental matrices of a Fuchsian system in the neighborhood of a nonsingular point have the form:

$$\Psi(z) = U(z) z_1^{D_i} z_1^{E_i},$$

where $U(z)$ are holomorphic matrix function on the considered neighborhood, $D = \text{diag}(\varphi_1^1, \dots, \varphi_1^n)$, and E_i a logarithm of the monodromy matrix. Every isomonodromic deformation preserves the eigenvalues of the coefficient matrices $A_j(s)$ and entries of the matrices D_j .

Consider a particular case of the Schlesinger theorem. Let $n = 2$ and let the corresponding Fuchsian system have four regular singular points: fixed singular points $0, 1, \infty$ and one removable singular point. Then the entries of the matrices A_j can be expressed as functions of this removable singularity.

Theorem 6. *The Painlevé transcendents*

- I. $f''(z) = 6f^2(z) + z$;
- II. $f''(z) = 2f^3(z) + zf(z) + a$;
- III. $f''(z) = \frac{2f'^2(z)}{f(z)} - \frac{f'(z)}{z} + \frac{1}{z}(af^2 + b) + cf^3(z) + \frac{d}{f(z)}$;
- IV. $f''(z) = \frac{f'^2}{2f(z)} + \frac{3}{2}f^3(z) + 4zf^2(z) + 2(z^2 - a)f(z) + \frac{b}{f(z)}$;
- V. $f''(z) = \frac{3f(z)-1}{2f(z)(f(z)-1)}f'^2(z) - \frac{1}{z}f'(z) +$
 $+ \frac{1}{z^2}(f(z) - 1)^2 \left(af(z) + \frac{b}{f(z)} \right) + \frac{c}{z} + \frac{df(z)(f(z)+1)}{f(z)-1}$;

$$\text{VI. } f''(z) = \frac{1}{2} \left(\frac{1}{f(z)} + \frac{1}{f(z)-1} + \frac{1}{f(z)-1} f'^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{f(z)-z} \right) f' + \right. \\ \left. + \frac{f(z)(f(z)-1)(f(z)-z)}{z^2(z-1)^2} \left(z + \frac{bz}{f^2(z)} + \frac{c(z-1)}{(f(z)-1)^2} + \frac{dz(z-1)}{(f(z)-z)^2} \right) \right),$$

where a, b, c, d are complex numbers, are Hamilton type equations that describe the dynamics of the removable singular point.

Proof of this theorem follows from the following results: 1) these are isomonodromic deformations of second order Fuchsian differential equations and therefore are Schlesinger equations and 2) Schlesinger equations are Hamilton equations.

Remark. The Hamiltonian structure of Painlevé equations has been studied by many authors. For the first time this problem has been posed, in the our opinion, in the papers [18], [19]. In the papers [15], [16] the methods of moment maps are applied to loop algebras to give explicit construction of Painlevé equations, including analytic expression of Hamiltonian, from regular linear systems.

Let V_1, \dots, V_m be \mathfrak{sl}_2 -modules. Put $V = V_1 \otimes \dots \otimes V_m$. The linear operators $\Omega_{ij} : V \rightarrow V$, $i < j$, act as $\Omega \otimes 1 + 1 \otimes \Omega$ on $V_i \otimes V_j$ and trivially on all of the other factors, where $\Omega \in \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$ is the tensor corresponding to an invariant scalar product. The Fuchs type Pfaff system

$$\frac{\partial \Psi(z_1, \dots, z_n)}{\partial z_i} = \frac{1}{\lambda} \sum_{j=1, i \neq j}^n \frac{\Omega_{ij}}{z_i - z_j} \Psi, \quad i = 1, \dots, n, \quad (25)$$

where $\Psi(z_1, \dots, z_n)$ is a V -valued function on $X_n = \mathbb{CP}^N \setminus \bigcup_{i,j=1, i \neq j}^m \{z_i - z_j = 0\}$, is called the Knizhnik–Zamolodchikov equation (see [17]). Here λ is a complex parameter. Solutions of (25) are covariant constant sections of the trivial bundle $X_n \times V \rightarrow X_n$ with the flat connection

$$\sum_{j=1, i \neq j}^n \frac{\Omega_{ij}}{z_i - z_j} d(z_i - z_j).$$

The solution space has the form described in Section 1. Monodromy representation of this system is a representation of the Artin braid group.

The Hamiltonian connected to this system has the form (see [20]):

$$H = \sum_{i,j=1, i \neq j}^N \frac{\Omega_{ij}}{(z - z_i)(z_i - z_j)} + \sum_{i=1}^N \frac{\mathcal{C}_i}{(z - z_i)^2}.$$

This Hamiltonian describes a chain of N metal atoms with each atom having just one electron. The first term in H describes a jump from position i to position j . The second term is an internal energy that depends on the element of the chain.

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