Bogusław Broda, Grzegorz Duniec, and Giorgi Khimshiashvili

THE NON-ABELIAN STOKES THEOREM
IN LOW DIMENSIONS

Abstract. We discuss some aspects of the non-Abelian version of Stokes theorem in two and three dimensions. In particular, the existence of isotopy for practical implementation of the non-Abelian Stokes theorem for topologically nontrivial knots, possibly with self-intersections, in three-dimensional Euclidean space is proved. A generalization to the case of a more general 3-manifold $\mathcal{M}^3$ is proposed and some results concerned with the issues of links and stability are presented. A 2-dimensional version of the non-Abelian Stokes theorem is also established and its generalization to the case of a real analytic surface with isolated singularities is formulated in the setting of square-integrable forms.

რეზიუმე. ნაშრომში განხილულია ორ და სამ განზომილებიანი სტოქსის თეორემის არააბელური ვერსიის ზოგიერთი ასპექტი. კერძოდ, დამტკიცებულია სამგანზომილებიან ევკლიდურ სივრცეში ტოპოლოგიურად არატრივიალური მარყუჟებისათვის სტოქსის არაბელური თეორემის პრაქტიკული განხორციელებისათვის საჭირო იზოტოპიის არსებობა. მოყვანილია განზოგადება უფრო ზოგადი, 3-განზომილებიანი $\mathcal{M}^3$ მრავალსახეობის შემთხვევისათვის, აგრეთვე კავშირების და მდგრადობის პრობლემებთან დაკავშირებული შედეგები. აგრეთვე დადგენილია სტოქსის არააბელური თეორემის 2-განზომილებიანი ვერსია და კვადრატით ინტეგრებადი ფორმების ტერმინებში ჩამოყალიბებულია მისი განზოგადება ნამდვილი ანალიზური ზედაპირისათვის იზოლირებული განსაკუთრებულობით.

1. Introduction

The (Abelian, i.e. classical) Stokes theorem is one of the central points of analysis on manifolds. The formula

$$\int_{\partial M} \omega = \int_{M} d\omega,$$

where ω is a differential $(d-1)$-form on d-dimensional compact oriented manifold M with boundary ∂M, is well-known. The lowest-dimensional (non-trivial) version of the Stokes theorem reads

$$\oint_{C} A = \int_{S} F, \tag{1}$$

where S is a two-dimensional surface, A is a connection 1-form, $F = dA$ is its curvature 2-form, and $C = \partial S$ is a closed contour (loop). It is sometimes called the (proper) Stokes theorem and appears extremely useful in applications (e.g. in Abelian gauge theory).

Formula (1), which may be considered as the Abelian Stokes theorem can be generalized to the non-Abelian case [1], [2]. There are several practical approaches to the non-Abelian Stokes theorem (NAST) [1], [2], [3], [4], and a lot of various aspects of the NAST have been already discussed [5]. One of them, initiated in [6], concerns NAST for a knot C (and also possibly for links). In particular, in the 3-dimensional case (i.e. when the ambient manifold is three dimensional), it may happen that C is knotted (or linked, for a multi-component C), and a direct application of NAST is impossible.

A typical situation of such kind arises when one wishes to calculate or estimate some physical quantities associated with a knotted or linked object, e.g., a loop of wire C in three-dimensional Euclidean space. To do that, it is often natural to consider a two-dimensional surface which spans C (in the sense that $\partial S = C$) and satisfies certain physically meaningful conditions (equations). For example, one may wish to take the soap film S spanning C (i.e., the solution of Plateau problem with boundary C) and estimate the tension in it or the force acting upon the wire on the side of the film. Notice that for a knotted C the corresponding soap film is necessarily singular and one has to invent a version of Stokes theorem appropriate for such situation.

Another sort of obstacle for using the classical Stokes theorem arises in the case when one deals with analytic or algebraic surfaces with isolated singularities, i.e., when the integration contour bounds a singular two-dimensional surface. This, as usual, means that the tangent plane is not well-defined at all points of the surface (like, e.g., in the case of a usual quadratic cone in three-dimensional space). Differential geometry of such surfaces suggests several issues in the spirit of the Stokes theorem. In such instances one has to look for more general formulations and procedures some of which are proposed in the sequel.

Symbolically, e.g., in the language of product integration [1], we can write the NAST for a disk S as

$$P \exp\left(i \oint_{C=\partial S} A\right) = \mathbb{P} \exp\left(i \int_S \mathcal{F}\right), \qquad (2)$$

where P and $\mathbb{P}$ are appropriately defined orderings, and $\mathcal{F}$ is the "twisted" non-Abelian curvature F of the Lie algebra valued connection A, $F = dA + \frac{1}{2}A \wedge A$ (see [1], [2], [3], [4], for details). If C bounds a disk S, equation (2) is directly applicable, but if C is, e.g., a non-trivial knot, one should resort to [6], where a version of the NAST for knots and links has been formulated. Another challenging situation arises if one considers a two-dimensional surface with an isolated singularity and looks for a version of NAST for the integral over the link of the singularity.

The aim of this paper is to clarify some of these issues, namely: to reformulate the procedure of [6] more explicitly, to allow intersecting knots, to generalize NAST to a more general 3-manifold $\mathcal{M}^3$, to work out the issues of links and of stability, and to present a version of NAST for 2-manifolds $\mathcal{M}^2$, including the case of a real-analytic surface with an isolated singularity.

2. NAST for Knots in $\mathbf{R}^3$

The essential step of the standard NAST is a decomposition of the initial $C = \partial S$ into "lassos" bounding disks of infinitesimal areas. If S is a disk, the procedure is straightforward and well-known but for a knotted C the decomposition is non-trivial. An elegant solution of the problem has been proposed in [6], where the authors have found a general decomposition of C suitable for a direct application of the NAST. The starting point of their analysis is an arbitrary compact connected oriented 2-dimensional surface S_c given in a canonical form (see, [7] page 209). Since a knot is always a boundary of a 2-dimensional surface, the Seifert surface S_s (a connected orientable surface), the problem is solved once an appropriate decomposition for this surface is found. But for practical purposes and for implementation of the theorem in [6] it is not obvious how one can actually relate the decomposition of [6] for the surface S_c given in a canonical form and decomposition of the actual Seifert surface S_s. To fill the gap, we propose a theorem establishing an appropriate isotopy explicitly (see [8], [9], [10], for a related material).

To begin with, following [7] we are recalling the construction of the Seifert surface S_s for a knot C. Let us assign C an orientation, and examine its regular projection. Near each crossing point, let us delete the over- and under-crossings, and replace them by "short-cut" arcs. We now have a disjoint collection of closed curves bounding disks, possibly nested. These disks can be made disjoint by pushing their interiors slightly off the plane. Then, let us connect them together at the old crossings with half-twisted strips to

form S_s. In the case of a multi-component C (link), we join components by tubes, if necessary.

Theorem 1. *There exists, in an explicit form, an (ambient) isotopy connecting the Seifert surface S_s for a knot C (for a link, in fact) and the 2-dimensional surface given in the canonical form S_c. Actually, the isotopy is valid for any two compact connected orientable homeomorphic 2-dimensional surfaces.*

To prove Theorem 1, we will explicitly present all components of the isotopy. The starting point are disks (0-handles) connected with strips (1-handles). Shortening a strip, and bringing any two connected by them 0-handles together we join them reducing their number by one. Let us repeat this procedure until we end up with a single disk with a bunch of strips. Next, let us concentrate on the first two strips. They can be "crossed" or "nested". In the case of nested strips, we can "decouple" them sliding the first one over the second one. In the case of crossed strips, we slide together the whole two bunches of all interior strips ("counterclockwise") over the two crossed strips separating the bunches from them. Let us keep repeating this procedure until we end up with a sequence of "decoupled" single strips and single pairs of crossed strips. Of course, the decoupling takes place only on the boundary of the disk, and the strips can be intertwined in a very complicated way. For a link, it may happen that the initial Seifert surface is disconnected. In such a case, after obtaining a single disk for each component, let us join the disks by tubes. Before we engage in ordering of strips we should cancel the tubes. Reducing the "size" of the first disk and shortening the first tube, and next bringing the two first disks together we join them decreasing their number by one. Each such an operation creates a hole with strips inside the second disk. Pushing the hole out of the interior of the disk we obtain a standard disk with a larger number of strips. Let us keep on repeating the procedure until all the tubes disappear and return to disentangling strips described earlier for a (single) knot C.

The method used for proving the theorem can also be used to extend the NAST to the case of self-intersecting knots (links). Let us return to the construction of the Seifert surface S_s. Now, some of the crossing points of a regular projection are "true" crossing points (intersection points). Splitting the intersection points arbitrarily (in one of the two ways) we get rid off the true crossing points, and the procedure of the previous section becomes fully applicable. However the memory about the intersections should remain encoded in the form of "pinching" lines identifying the intersection points. These lines lie on strips, and in the course of all necessary rearrangements they persist in lying on the Seifert surface. After joining all the disks according to the procedure described in Theorem 1 all the lines fall in the final disk. The ordering procedure consisting in sliding the strips drags the lines inside the strips. Therefore, the Seifert surface brought to the canonical form S_c is covered by two independent systems of curves. The first, very

regular one follows from the procedure of [6] and is responsible for cutting S_c into simply connected surfaces (disks). The second system of curves, possibly complicated, generated by the present recipe, indicates necessary pinching and identification of points. Each disk created by the first system of curves should now be pinched by curves of the second system becoming a finer bunch of disks, and the standard NAST becomes applicable.

3. Multi-Component Links and Stability

It is possible to easily generalize the NAST to the case of a more general ambient 3-dimensional manifold $\mathcal{M}^3$ in the framework of surgery. Both Dehn surgery and handle surgery [7] are relevant in this context. In the following theorem we assume that the surgery used to construct a 3-manifold considered, is defined by combinatorial data called surgery coefficients [7].

Theorem 2. *Let $\{C, C_1, C_2, \ldots, C_n\}$ be a link, where C is a knot, and C_i define surgery prescription for a 3-manifold $\mathcal{M}^3$ together with their surgery coefficients r_i. For a homologically trivial knot C, i.e., when C is a boundary, there exists a corresponding (generalized) Seifert surface S_s.*

In the first step we construct the Seifert surface S_s for C ignoring C_i. Thus, C_i can pierce S_s in several points. Thickened C_i (linked tori) cut small disks D_{ia} ($a = 1, 2, \ldots, n_i$) out of S_s. What to do with these holes in S_s depends crucially on C_i and its r_i. Only $r_i = \infty$ means triviality of the surgery operation along C_i (regluing back the same torus), and no disk is actually cut out. For $r_i \neq \infty$ we should mark all the circles B_{ia} (boundaries B_{ia} of the removed disks D_{ia}, i.e. $B_{ia} = \partial D_{ia}$) corresponding to C_i. In the identified (glued back) solid torus each B_{ia} lies on its surface, homologically non-trivially. Cutting the torus along a meridian we can see all B_{ia} on its section as points P_{ia}. The points P_{ia} should be connected in pairs with intervals which should be prolonged to bands in the interior of the solid tori and finally closed forming short tubes. The assignment of the pairs to the points enters as additional and arbitrary information in our construction. Now we return back to the original Seifert surface S_s with holes, and cancel the holes by simply connecting them with the tubes. In case the number n_i is odd for some i one B_{ia} leaves unpaired, and this simply means that the knot can be homologically non-trivial. The Seifert surface S_s modified by the surgery, i.e. with several tubes added, is also a Seifert surface, possibly non-minimal, and the general decomposition procedure applies.

As for the issue of multiple loops and stability, our approach corrects and differs from that proposed in [6]. First of all we observe that for a (genuine) link one should treat its components independently, i.e. to consider only one of them at a time forgetting the rest.

It follows from the fact that different components represent different operators, and no mixing them takes place anywhere. Application of the NAST for an N-component link means that we cover appropriate Seifert surface(s) with N independent nets of small lassos. They can intersect or may not,

they can overlap or may not, but they are always independent. In other words, if we consider a common surface for several loops we must decree areas for particular components anyway, and the approach effectively decays onto independent treatment of single components. Obviously, in general case, Seifert surfaces corresponding to different components can intersect.

Another closely issue is concerned with the NAST for stably equivalent surfaces, i.e. for Seifert surfaces differing by a handle, which of course changes the topology of an initial Seifert surface.

Proposition. *Any two connected Seifert surfaces of a knot C are stably equivalent to each other.*

In fact, this exactly corresponds to addition of two crossed strips, as it could be represented by addition of (connected sum with) a torus (see e.g. [10], Chapter 5 A). For this new surface the NAST dictates another decomposition which is still equivalent to the former one. This follows from the fact that different families of the lassos in 3-dimensional $\mathcal{M}^3$ are (and must be), by the very construction of the NAST, still isotopic to the initial knot.

4. NAST in Two Dimensions

In two dimensions the NAST assumes a non-trivial form. Therefore we have to invent a new formulation of the main result.

Theorem 3. *Let $\mathcal{M}^2$ be a compact connected orientable 2-manifold with boundary, and C be a homologically trivial cycle on $\mathcal{M}^2$. In this case the NAST can be written in the form*

$$\mathbf{C} = \mathbf{S}\, \mathbf{c_k B_k^{-1} c_k^{-1}} \ldots \mathbf{c_1 B_1 c_1^{-1} b_n a_n} \ldots \mathbf{b_1 a_1}, \qquad (3)$$

where $\mathbf{a_i}$, $\mathbf{b_i}$, $\mathbf{B_j}$, $\mathbf{c_j}$, $\mathbf{C}$ denote operators of parallel transport (or global connection) in an irreducible representation $\mathcal{R}$ of the Lie group G, and $\mathbf{S}$ is given by the right hand side of (2).

To prove the theorem we use a general classification of compact connected 2-manifolds with boundary. According to [8] (Ch. 1, Sect. 10), the normal form for $\mathcal{M}^2$ is given by a polygon with $4n + 3k$ sides (n – the genus, k – number of holes) which are identified as follows:

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} c_1 B_1 c_1^{-1} \cdots c_k B_k c_k^{-1}, \qquad (4)$$

where a_i, b_i correspond to the properly identified sides of the polygon representing a genus n 2-manifold in a standard fashion, and c_i are cuts to the boundary components B_i corresponding to the k holes. C is an additional hole which can be placed, for example, in front of (4). Now the standard NAST (2) can be applied. After necessary rearrangements we arrive at the formula (3).

Another non-trivial aspect of NAST in two-dimensional case is related with consideration of surfaces with isolated singular points. The geometric structure of a real analytic surface near its isolated singular points is known

in great detail (see, e.g., [11]), which makes it possible to analyze the behavior of both sides of NAST near the singular point and find a formulation appropriate to this case.

Let X be a real analytic surface with isolated singular points p_j, $j = 1, \ldots, n$. As was shown in [11], all these singularities are of the so-called *horn type*, which means that locally the induced metric is quasi-isometric to the metric of the form $(dr)^2 + r^{2\gamma}(d\phi)^2$ on $S^1 \times [0, \delta)$. If $\gamma = 1$, then a singular point is said to be of a *cone type*. With each of those singular points p_j on can associate a real number r_j, which describes the behavior of the induced metric on X near point p_j. Namely, the number r_j is defined as the leading coefficient in the asymptotic expansion of the arc length function $l(\delta)$ on a small link $X \cap B_\delta^2$ [11]. Let the singular points p_j be numbered in such way that all singular points of cone type are the points $p_1, \ldots, p_s$.

Consider now a circle bundle E over X which is locally trivial over the regular part of X. Then, as is well known, there exists a connection ω on E and one can consider its curvature form Ω. In order to formulate a version of Stokes theorem in this situation, one considers a surface with boundary Y which is obtained by intersecting X with the union of small balls B_j around singular points, and compares the integral of ω over the boundary $C = X \cap (\cup B_j)$ of Y, with the integral of Ω over Y. It turns out that there appears a correction term as compared with the usual Stokes theorem and this correction term is completely determined by the geometric structure of singular points.

Theorem 4. *With the assumptions and notation as above, the curvature form Ω is integrable on Y and one has the equality:*

$$\int_C \omega = \int_Y \Omega + \frac{1}{2\pi} \sum_{j=1}^s r_j - n. \tag{5}$$

The proof of this theorem can be easily obtained by passing to the vector bundle associated with E and applying the Gauss-Bonnet theorem from [11]. The result itself is not surprising in view of [11], but it seems useful since it suggests what can be form of NAST for surfaces with isolated singularities. Namely, it becomes clear that one should introduce a correction term of the above type. However, in non-abelian case one has to properly modify the definition of path integral of curvature form and establish its basic properties. We were not able to find such discussion in the existing literature so we plan to consider it in full detail in a forthcoming publication and present a rigorous version of NAST for singular surfaces.

Notice that the dimensions 2 and 3 are the only dimensions of real interest in the context of NAST as in higher dimensions there are no topologically non-trivial configurations of closed curves. Thus the settings and results presented above seem to adequately match some non-trivial physical phenomena of essentially topological nature which one encounters in gauge theory, like the Aharonov-Bohm effect and its various manifestations. It

seems plausible that further research in this direction may lead to better understanding of those phenomena.

Summing up, in this article we have shown how the theorem proposed in [6] for the application of the NAST to topologically nontrivial knots can be effectively implemented. As a by-product of our construction, we have extended the main result to knots with intersections. Next, we have proposed a generalization of the NAST to a more general 3-manifold defined by surgery, and settled the issue of links and of stability under the addition of a handle. Finally, the NAST for a 2-manifold $\mathcal{M}^2$ has been formulated in such a form which permits generalization to the case of surface with isolated singularities. In forthcoming publications we intend to discuss some straightforward applications of our results to certain situations arising in non-abelian gauge theory and differential geometry.

Acknowledgments

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Authors' addresses:

B. Broda and G. Duniec
Department of Theoretical Physics
(Section of Mathematical Physics)
University of Łódź
Pomorska 149/153, PL-90-236 Łódź
Poland
E-mail: bobroda@uni.lodz.pl
          gduniec@merlin.fic.uni.lodz.pl

G. Khimshiashvili
Department of Geometry of Topology
A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: khimsh@rmi.acnet.ge