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QUALITATIVE AND QUANTITATIVE CHARACTERISTICS OF LIMIT CYCLES

Abstract. In this article there are presented the criteria for existence, and there are also considered the questions of number, multiplicity and stability of limit cycles of the two-dimensional dynamic systems associated with a specific inversion of the Bendixson-Dulac criterion about the absence of closed trajectories in dynamic systems, and with the proposed by the author classification of limit cycles, based on the properties of the divergence of the vector field and the regularity conditions of cycles..

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Key words and phrases: Dynamic system, trajectory (of a dynamic system), limit cycle (of a dynamic system), divergence (of a vector field), stable (unstable, semistable) limit cycle, divergent limit cycle, rough (unrough) limit cycle, strict (nonstrict, generalized-strict) limit cycle, multiple (tuple) limit cycle, regular (non-regular) limit cycle, Poincaré function, topographical system of the curves, universal curvilinear-coordinate system.

რეზიუმე. ნაშრომში მოყვანილია ორგანზომილებიანი დინამიური სისტემების ზღვრული ციკლების არსებობის კრიტერიუმები, რომლებიც დაკავშირებულია დინამიური სისტემების შეკრული ტრაექტორიების არარსებობის ბენდიქსონ–დიულაკის კრიტერუმის ერთ სპეციფიკურ შექცეულ დებულებასთან. გარდა ამისა, განხილულია ზღვრული ციკლების რაოდენობის, ჯერადობის და მდგრადობის საკითხები. მოყვანილია ზღვრული ციკლების ავტორისეული კლასიფიკაცია, რომელიც ემყარება ვექტორული ველის დივერგენციის თვისებებს და ციკლების რეგულარობის პირობებს.

Introduction

As it is known, in the initial period of the development of the theory of differential equations the main task was to develop methods and techniques for integrating motion equations of dynamic systems in so called closed form, when for the description of the motions it was necessary to obtain an analytic formula whose application supposed the action of a finite number of known operations over the known elementary functions.

However, after the French mathematician J. Liouville proved in 1841 that motion equations can be integrated in the closed form only in seldom cases, the efforts of mathematicians and mechanical scientists were directed for research of various properties of motions according to the known properties of motion equations themselves.

The fact is that even if during the process of integrating motion equations one uses infinite series of this or that form, then, even in this case, in spite of the possibility to solve significantly more equations than if we had solved them in closed form, it very often happens so that most essential and interesting properties of motions can be by no means detected from the form of the series obtained.

Moreover, if one chances to have integrated motion equations in closed form, then, it is far from always that such a motion can have been analyzed because the obtained dependence between different parameters often appears to be greatly and greatly complicated.

Thus, the necessity for techniques and methods that could enable one, without integrating motion equations themselves, however, to get necessary information about these or those properties of motions according to the known properties of an initial dynamic system has become evident.

One of the most important problems, in this respect, is the development of the effective methods of solving the problems of existence, the number and stability of periodic motions of dynamic systems.

As H. Poincaré [1, p. 75] noted "...the specific importance of these periodic solutions is the fact that they are the only bridge through which we could attempt to penetrate into the domain that hasn't been considered to be obtainable".

Herewith, one should take into account that already while considering the dynamic systems with the two-dimensional phase space there is a significant difference between periodic motions, to which on the phase plane there corresponds the continuum of closed trajectories completely filling some domain, and that it is possible both for the case of linear motion equations and for the case of nonlinear ones; and periodic motions, to which on the phase plane there corresponds an isolated closed trajectory (a limit cycle), and that it is possible only for the case of nonlinear motion equations.

Considering the case of non-constant periodic motions of two-dimensional dynamic systems to which on the phase plane there correspond limit cycles, note, that in spite of the very variable nature of such motions they are united by one common property, with respect to which A.A. Andronov [2, p. 32] wrote: "There exists a number of arrangements, being able to generate non-damped oscillations for the account of non-periodic sources of energy. However, so far there hasn't been a sufficiently strict and general theory of such auto-oscillations. Herewith, there exists an adequate mathematical construction created without any connection with the theory of oscillations, making it possible to establish general viewpoint for all similar processes, for the case of one degree of freedom. This construction is the Poincaré "limit cycles" theory".

In 1929 A.A. Andronov published an important article concerned with the investigations of Van der Pol on the radio theory, which concerned the description of the established oscillations with a constant amplitude in a triode vacuum tube.

In this article it was, for the first time, shown that a stable periodic solution of the differential equation obtained by Van der Pol, corresponds on the phase plane to a stable limit cycle. It should be noted that in the radio theory this limit cycle corresponds to those transferring radio signals that appear in electronic generators.

The oscillations of the above mentioned type often appear both in mechanical and acoustical systems, and, besides, in the systems that are, generally speaking, the subject of research of that part of non-linear mechanics that is associated with one of its sections, namely, the theory of oscillations.

The significance of studying limit cycles is confirmed, as well, by the fact that the problem about the maximum number of limit cycles and their location was included by D. Hilbert, as the 16th point, into the number of 23, mentioned by him in 1900, most important problems, requiring their solution. This problem hasn't been solved yet despite it having been attacked by many well-known scientists.

Discussing auto-oscillations it is necessary also to note that circumstance that limit cycles corresponding to them belong to the so called singular trajectories of dynamic systems. And, as the investigations of A.A. Andronov and E.A. Leontovich showed, singular trajectories of the dynamic systems of the second order, i.e., the equilibrium states, separatricies and limit cycles, determine that "skeleton" that enables one to build the qualitative picture of the behaviour of the phase trajectories over all the phase space. Herewith, if for the studying of the separatricies and the behaviour of the trajectories in the neighbourhood of the equilibrium states of the dynamic systems there are available reliable methods, then for the studying of limit cycles there aren't reliable available any regular methods.

The fact is that the theory of limit cycles belongs to that domain of the qualitative theory of differential equations that is associated with the brightly expressed global problem, when solving this problem there appear difficulties of the principal character due to the fact that it is not possible *to directly* use mathematical apparatus of differential calculus according to its very idea provided for solving local questions.

In this article there are presented the criteria for existence, and there are also considered the questions of number, multiplicity and stability of limit cycles of the two-dimensional dynamic systems associated with a specific inversion of the Bendixson-Dulac criterion about the absence of closed trajectories in dynamic systems, and, with the proposed by the author classification of limit cycles, based on the properties of the divergence of the vector field and the regularity conditions of cycles.

1. Divergent Closed Trajectories

Consider differential systems

$$\frac{dx}{dt} = X(x,y), \quad \frac{dy}{dt} = Y(x,y) \tag{1.1}$$

and

$$\frac{dx}{dt} = X(x,y)B(x,y), \quad \frac{dy}{dt} = Y(x,y)B(x,y), \tag{1.2}$$

where x, y and t are scalar real variables, and X, Y and B are real functions, having continuous partial derivatives in some domain G (we shall further call such differential systems dynamic systems [3]). Remind that if Z is a vector field determined by the system (1.1), that is, if

$$Z = X(x,y)\frac{\partial}{\partial x} + Y(x,y)\frac{\partial}{\partial y},$$

then **div** Z is the divergence of the vector field Z giving the volume - dilation rate for the corresponding flow which is given by the following equality

$$\mathbf{div}\ Z = \frac{\partial X(x,y)}{\partial x} + \frac{\partial Y(x,y)}{\partial y}$$

Definition 1.1 ([4])**.** The closed trajectory $\Gamma : \gamma(x,y) = 0$ of the dynamic system (1.1) is said to be a divergent closed trajectory of the system (1.1), if along this trajectory the divergence of the vector field, determined by the system (1.1), preserves its constant meaning, that is, if there holds the equality $\mathbf{div}\ Z\big|_{\gamma(x,y)=0} = \lambda$, where the constant $\lambda \in \mathbb{R}$. A divergent closed trajectory may be both an isolated closed trajectory (a limit cycle) and may be a curve belonging to the set of curves forming some continuum.

For example, the dynamic system

$$\frac{dx}{dt} = -y + x(x^2+y^2-1)(x^2+y^2-2)(x^2+y^2-3)^2,$$
$$\frac{dy}{dt} = x + y(x^2+y^2-1)(x^2+y^2-2)(x^2+y^2-3)^2$$

has divergent limit cycles defined by the equations $\gamma_1(x,y) \equiv x^2+y^2-1 = 0$, $\gamma_2(x,y) \equiv x^2+y^2-2 = 0$, $\gamma_3(x,y) \equiv x^2+y^2-3 = 0$. That fact that the relations $\gamma_i(x,y) = x^2+y^2-i = 0$, $i = 1,2,3$, define limit cycles, may be easily shown by the transition in the system considered to the polar coordinates. The divergency of these cycles may be just directly verified. In this case $\mathbf{div}\ Z = 2(x^2+y^2-1)[5(x^2+y^2)^3-31(x^2+y^2)^2+56(x^2+y^2)-22]-8$

The latter equality may be rewritten in the form $\mathbf{div}\ Z = 2(x^2+y^2-2)[5(x^2+y^2)^3-26(x^2+y^2)^2+35(x^2+y^2)-8]+4$ or in the form $\mathbf{div}\ Z = 2(x^2+y^2-3)[5(x^2+y^2)^3-21(x^2+y^2)^2+24(x^2+y^2)-6]$. And then

$$\mathbf{div}\ Z\big|_{\gamma_1(x,y)=0} = -8, \quad \mathbf{div}\ Z\big|_{\gamma_2(x,y)=0} = 4, \quad \mathbf{div}\ Z\big|_{\gamma_3(x,y)=0} = 0.$$

As another example, consider the dynamic system

$$dx/dt = y, \quad dy/dt = -x + xy(x^2+y^2-1)^2.$$

The vector field determined by this system is symmetric with respect to the axis of the ordinates of the phase plane. The system has in the finite part of the phase plane the only critical point $O(0,0)$, being the centre.

Here $\mathbf{div}\ Z = x(x^2+y^2-1)(x^2+5y^2-1)$, and the divergent closed trajectory $x^2+y^2-1=0$ is a curve belonging to the set of the centre trajectories.

Theorem 1.1 ([5]). *Let the functions $X(x,y)$ and $Y(x,y)$ belong to the class C^k, $k=1,2,3,\ldots$, in any finite domain of the phase plane $\mathbb{R}^2$. Then, if the dynamic system* (1.1) *has a closed trajectory Γ, then there exist a constant $\lambda \in \mathbb{R}$ and a nonzero function $B : \mathbb{R}^2 \to \mathbb{R}^+$ of the class C^k such that the equation* $\mathbf{div}\ BZ = \lambda^*$ *defines a curve*† *having a finite real branch coinciding with the trajectory Γ.*

The proof of this theorem, in fact, coincides with the proof of the lemma from [6].

Remark 1.1. The function B, and, hence, the constant λ, being present in the Theorem 1.1, generally speaking, are not the only ones.

Thus, for example, consider the dynamic system

$$dx/dt = y(x^2-xy+y^2) - 2x(x^2+y^2-1), \quad dy/dt = -x(x^2-xy+y^2),$$

having the closed trajectory (a stable limit cycle), the equation for which is $x^2+y^2=1$. For the latter system the divergence of the vector field is defined by the relation $\mathbf{div}\ Z = 2-5x^2-3y^2$.

It can be easily verified that with either real λ the equation $2-5x^2-3y^2=\lambda$ does not define the trajectory of the initial dynamic system.

Herewith, if the function B is determined by the equality $B(x,y) = 3x^2-4xy+7y^2+3$ then $\mathbf{div}\ BZ = (x^2+y^2-1)(-23x^2+16xy-25y^2-20)-14$ and the equation $\mathbf{div}\ BZ = -14$ defines the curve whose real branch $x^2+y^2=1$ turns out the divergent closed trajectory.

Besides the above mentioned function B, one may also consider, for example, the polynomial $B(x,y) = \frac{1}{7}(23x^2-20xy+43y^2+7)$.

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*$\mathbf{div}\ BZ$ is the divergence of the vector field, determined by the dynamic system (1.2)

†The fact that the equation $\mathbf{div}\ BZ = \lambda$ defines a curve imposes on the right-hand side of the considered dynamic system definite restrictions. We shall assume them to be valid always when considering those or some other curves

In this case **div** $BZ = (x^2+y^2-1)(-187x^2+80xy-149y^2-84)/7-10$. Here, the desired divergent branch $x^2+y^2-1=0$ is already defined by the equation **div** $BZ=-10$. Note, further, that, taking into account the definition 1.1, the theorem 1.1 may be reformulated as follows.

Theorem 1.2 ([5]). *Let functions $X(x,y)$ and $Y(x,y)$ belong to the class C^k, $k=1,2,3,\dots$, in any finite domain of the phase plane $\mathbb{R}^2$. Then, if the dynamic system* (1.1) *has a closed trajectory Γ, then there exists a nonzero function $B:\mathbb{R}^2\to\mathbb{R}$ of the class C^k such that for the dynamic system* (1.2) *the curve Γ will be a divergent closed trajectory.*

Thus, with the exactness to the transformation of time, any closed trajectory of dynamic system (1.1) may be always considered as a divergent closed trajectory. This result is substantial for the theory of periodic motions of monodromic dynamic systems of the second order.

Taking into account the properties of the Poincaré function and coincidence of the phase portraits of the dynamic system (1.1) and (1.2) (with a nonzero function B) one can prove that there holds

Theorem 1.3 ([5]). *Let the functions $X(x,y)$ and $Y(x,y)$ belong to the class C^k, $k=1,2,3,\dots$, in any finite domain of the phase plane $\mathbb{R}^2$. Then, for the dynamic system* (1.1) *to have a rough (unrough) limit cycle, the existence of a nonzero function $B:\mathbb{R}^2\to\mathbb{R}^+$ of the class C^k, such that the dynamic system* (1.2) *have a rough (unrough) divergent limit cycle, is necessary and sufficient.*

Consider now along with the dynamic system (1.1), the real dynamic system of the form

$$\frac{dx}{dt}=P(x,y),\quad \frac{dy}{dt}=Q(x,y) \tag{1.3}$$

supposing that the functions P, Q, as well as the functions X, Y, are the functions of the class C^k, $k=1,2,3,\dots$, in any finite domain of the phase plane $\mathbb{R}^2$.

Denote as E the function defined by the equality $E(x,y)=X(x,y)\times$ $\times Q(x,y)-Y(x,y)P(x,y)$ and satisfying the condition ${E'_x}^2+{E'_y}^2\neq 0$ on the set

$$\{(x,y)\big|E(x,y)=0\}. \tag{1.4}$$

The condition (1.4) means, in particular, that the equation

$$E(x,y)=0 \tag{1.5}$$

defines on the phase plane $\mathbb{R}^2$ some curve.

With the above mentioned assumptions we shall formulate a simple, however useful for the qualitative analysis of the dynamic systems, statement.

Theorem 1.4 ([4]). *For the trajectory $\Gamma:\gamma(x,y)=0$ of the dynamic system* (1.1) *to be the trajectory of the dynamic system* (1.3)*, it is necessary and sufficient that the curve, defined by the equation* (1.5)*, have a coinciding* *with the curve* Γ *real branch, not containing critical points of the system* (1.3).

Corollary 1.1. *For the trajectory* $\Gamma : \gamma(x, y) = 0$ *of the dynamic system* (1.1) *to be the trajectory of the dynamic system* (1.3), *it is necessary and sufficient that along the curve* Γ *the equalities*

$$P(x,y)\big|_{\gamma=0} = X(x,y)\big|_{\gamma=0} \cdot \mathcal{B}(x,y),$$
$$Q(x,y)\big|_{\gamma=0} = Y(x,y)\big|_{\gamma=0} \cdot \mathcal{B}(x,y)$$

hold, where $\mathcal{B}$ *is a nonzero on the curve* Γ *function of the class* C^k.

Remark 1.2. The condition of the absence of critical points of the dynamic system (1.3) on the closed branch of the curve, defined by the equation (1.5), being present in the theorem 1.4, is substantial. Indeed, consider the dynamic systems

$$\frac{dx}{dt} = -y + x(x^2 + y^2 - 1), \quad \frac{dy}{dt} = x + y(x^2 + y^2 - 1) \tag{1.6}$$

and

$$\frac{dx}{dt} = x^2 + y^2 - 1, \quad \frac{dy}{dt} = x^2 + y^2 - 1. \tag{1.7}$$

The equation (1.5) is, in this case, of the form

$$(x^2 + y^2 - 1)[2x + (y - x)(x^2 + y^2 - 1)] = 0. \tag{1.8}$$

The branch $x^2 + y^2 - 1 = 0$ of the curve, defined by the equation (1.8), is for the system (1.6) a trajectory (a limit cycle).

It should be noted that for the system (1.7) the above mentioned branch has already been a singular closed trajectory fully consisting of critical points of the system.

Theorem 1.5 ([4]). *If the dynamic system* (1.1) *has the closed trajectory* $\Gamma : \gamma(x, y) = 0$, *then there exist the functions* P *and* Q *of the class* C^k $k = 1, 2, 3, \ldots,$ *in* $\mathbb{R}^2$ *such that for the dynamic system* (1.3) *the curve* Γ *will be a divergent closed trajectory.*

Remark 1.3. From the theorem 1.5 follows, in particular, that there exist infinitely great deals of the dynamic systems of the form (1.1) that have the same closed trajectories (in whose neighbourhood, however, the behaviour of the trajectories may be different).

Theorem 1.6 ([4]). *If the dynamic system* (1.1) *has the closed trajectories* $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$, *then there exist the functions* P *and* Q *of the class* C^k, $k = 1, 2, 3, \ldots,$ *in* $\mathbb{R}^2$ *such that for the dynamic system* (1.3) *the curves* Γ_1, $\Gamma_2, \ldots, \Gamma_m$ *will be divergent closed trajectories.*

2. Divergent and Generalized–Strict Limit Cycles

Turning to the investigation of divergent limit cycles note that if in the definition (1.1) the closed trajectory Γ is isolated, and if a constant λ is a nonzero constant, then Γ is a limit cycle, which will, evidently, be a rough (simple, one-multiple, hyperbolic) divergent limit cycle. It should be taken into consideration, that it will be stable, if $\lambda < 0$, and unstable, if $\lambda > 0$. If, however, $\lambda = 0$, then a cycle Γ will be an unrough (multiple, compound) divergent limit cycle.

Theorem 2.1 ([5]). *Let the functions $X(x, y)$ and $Y(x, y)$ belong to the class C^k, $k = 1, 2, 3, \dots$, in any finite domain of the phase plane $\mathbb{R}^2$. Then, for the dynamic system* (1.1) *to have a rough divergent limit cycle, the existence of a nonzero constant $\lambda \in \mathbb{R}^-(\mathbb{R}^+)$ such that the curve defined by the equation $\mathbf{div} Z = \lambda$ have a finite real branch, being a closed trajectory of the system* (1.1)*, is necessary and sufficient.*

From the theorems 1.3 and 2.1 follows

Theorem 2.2 ([5]). *Let the functions $X(x, y)$ and $Y(x, y)$ belong to the class C^k, $k = 1, 2, 3, \dots$, in any finite domain of the phase plane $\mathbb{R}^2$. Then, for the dynamic system* (1.1) *to have a rough limit cycle, the existence of a nonzero constant $\lambda \in \mathbb{R}^-(\mathbb{R}^+)$ and a nonzero function $B : \mathbb{R}^2 \to \mathbb{R}^+$ of the class C^k such, that the equation $\mathbf{div}\ BZ = \lambda$ define the curve, having a finite real branch, being a closed trajectory of the system* (1.2)*, is necessary and sufficient.*

Theorem 2.3 ([5]). *Let the functions $X(x, y)$ and $Y(x, y)$ belong to the class C^k, $k = 2, 3, \dots$, in any finite domain of the phase plane $\mathbb{R}^2$. Then, for the dynamic system* (1.1) *to have a rough limit cycle, the existence of a nonzero constant $\lambda \in \mathbb{R}^-(\mathbb{R}^+)$ and a nonzero function $B : \mathbb{R}^2 \to \mathbb{R}^+$ of the class C^k such, that the the curve, defined by the equation $\mathbf{div}\ BZ = \lambda$, have a finite real branch Γ_p, being a p-multiple, without critical points of the system* (1.1)*, simple closed curve of the class C^{k+1}, that would coincide with q-multiple $(p \le q)$*[‡] *branch of Γ_q of the curve defined by the equation*

$$D_t^{(1.2)}\, \mathbf{div}\ BZ = 0 \tag{2.1}$$

where the symbol $D_t^{(1.2)}$ denotes the differentiation operator with respect to t due to the system (1.2)*, is necessary and sufficient.*

Proof. Let the dynamic system (1.1) have a rough limit cycle. Then, according to the theorem 2.2, there exist a nonzero constant $\lambda \in \mathbb{R}^-(\mathbb{R}^+)$ and a nonzero function $B : \mathbb{R}^2 \to \mathbb{R}^+$ of the class C^k such that the equation $\mathbf{div}\ BZ = \lambda$ will define a curve having a finite real p-multiple branch

[‡]The coinciding p-multiple Γ_p and the q-multiple Γ_q branches of the corresponding curves are defined by the equations $\varphi^p(x, y) = 0$ and $\varphi^q(x, y) = 0$, where $\varphi(x, y)$ is a non-factorable function, and the natural numbers p and q are the indicies of the φ^p and φ^q.

$\Gamma_p : \gamma(x, y) = 0$ being a divergent closed trajectory of the system (1.2). Owing to the fact that the curve Γ_p is a trajectory of the system (1.2), we come to the conclusion that $D_t^{(1.2)}\gamma(x, y) = F(\gamma(x, y), x, y)$, where $F(o, x, y) = 0$. The latter fact, taking into account the divergency of Γ_p, just means that the curve, defined by the equation (2.1), has a q-multiple ($p \leq q$) branch, coinciding with the branch Γ_p without critical points of the system (1.1). And, vice versa, since from the conditions of the theorem it follows that the dynamic system (1.2) has a divergent closed trajectory, then the reference for the preceding theorem just completes the proof. □

One of the examples, illustrating the theorem 2.3, is the dynamic system

$$\frac{dx}{dt} = -y + x(x^2 + y^2 - 1), \quad \frac{dy}{dt} = x + y(x^2 + y^2 - 1). \qquad (2.2)$$

For this system $\mathbf{div}\ Z = 4(x^2+y^2-1)+2$, $D_t^{(2.2)}\,\mathbf{div}\ Z = (x^2+y^2)(x^2+y^2-1)$ and, thus, the system (2.2) has a rough (unstable) limit cycle, defined by the equation $x^2 + y^2 = 1$ (here $\lambda = 2$).

The geometric interpretation of the theorem 2.3 is the following.

Theorem 2.4. *Let the functions $X(x, y)$ and $Y(x, y)$ belong to the class C^k, $k = 2, 3, \ldots$, in any finite domain of the phase plane $\mathbb{R}^2$. Then, for the dynamic system* (1.1) *to have a rough limit cycle, the existence of a nonzero function $B : \mathbb{R}^2 \to \mathbb{R}^+$ of the class C^k such, that the equation* $\mathbf{div}\ BZ = \Lambda$ *where Λ is a real parameter, changing on some interval $I \subset \subset \mathbb{R}^-(\mathbb{R}^+)$, define a topographical system, whose contact curve would have a q-multiple finite real branch, coinciding with one of the p-multiple ($p \leq q$) curves of the same topographical system, is necessary and sufficient.*

Here, saying about a topographical system of the curves, defined by the equation of the form $\Phi(x, y) = C$, where C is a real parameter, we shall understand a family of non-intersecting and embracing each other of the class C^{k+1} simple closed curves, surrounding one or some critical points of the system (1.1), and completely filling some two-connected domain without critical points of the system (1.1).

From the theorem 2.3 follows

Theorem 2.5 ([5])**.** *Let the functions $X(x, y)$ and $Y(x, y)$ belong to the class C^k, $k = 2, 3, \ldots$, in any finite domain of the phase plane $\mathbb{R}^2$. Then, for the dynamic system* (1.1) *to have a rough limit cycle, the existence of a nonzero constant $\lambda \in \mathbb{R}^-(\mathbb{R}^+)$ and a nonzero function $B : \mathbb{R}^2 \to \mathbb{R}^+$ of the class C^k such, that the equation*

$$D_t^{(1.2)}\,\mathbf{div}\ BZ + \mathbf{div}\ BZ = \lambda \qquad (2.3)$$

define a curve, having a finite real branch, being a simple without critical points of the system (1.1) *closed curve of the class C^{k+1}, whose multiplicity is equal to its multiplicity, when regarded as a divergent curve of the system* (1.2)*, is necessary and sufficient.*

Remark 2.1. The theorem 2.5 shows not only the qualitative aspect of the problem but also defines the technique for finding the required function B that not obligatorily must be the solution of the equation (2.3). But if we are able to find the function B as a solution of the equation (2.3) in some domain D, then, by this, we are be able to find all rough limit cycles of the system (1.1)(if such ones exist), located in D. In particular, if polynomial systems are considered, then for to find the function B it is possible to use the known results of the theory of ovals of planar algebraic curves.

Thus, the introduction of divergent limit cycles allowed to deduce the necessary and sufficient conditions for the existence of rough limit cycles belonging to the dynamic systems of the general form (1.1).

What other useful possibilities does the introduction of divergent limit cycles provide? First of all, it is the possibility to get the upper estimate for the number of rough divergent limit cycles of polynomial dynamic systems (such cycles are always strict limit cycles, i.e., the cycles, along which the divergence of the vector field, given by the dynamic system, is strictly negative or strictly positive).

Theorem 2.6 ([5]). *Let $X(x, y)$ and $Y(x, y)$ be polynomials of the degree not higher than $n \geq 3$. Then:*

1) *the general number of strict limit cycles of the dynamic system* (1.1) *does not exceed the number $N = \frac{1}{2}\sum\limits_{\nu=1}^{k}(m_\nu - 1)(m_\nu - 2) + k$, where $\sum\limits_{\nu=1}^{k} m_\nu =$ $= n - s - p - 1$, m_ν is the order of the non-reducible branch $f_\nu(x, y) = 0$ of the reducible curve, defined by the equation*

$$\mathbf{div}\ Z = f_1(x, y) \cdots f_\nu(x, y) \cdots f_k(x, y) \cdot f_s(x) \cdot f_p(y) = 0$$

and s and p are the degrees of the polynomials $f_s(x)$ and $f_p(y)$ (*generally speaking, reducible*) *correspondingly;*

2) *the general number of strict limit cycles of the dynamic system* (1.1), *forming a nest* (*a set of embedded, successively, into each other, ovals*) *and surrounding one critical point of the system* (1.1), *does not exceed the number $N = \left[\frac{n-s-p-k}{2}\right]$, where $[\cdot]$ is a whole part of a number.*

The idea of the proof of the theorem 2.6 is to show that to each strict limit cycle of the polynomial system (1.1) there corresponds, at least, one "its" closed finite real branch of the algebraic curve, defined by the equation $\mathbf{div}\ Z = 0$ (theorem 7 [8]). And then the general number of strict limit cycles of the system considered will not exceed the general number of the closed finite real branches of the curve $\mathbf{div}\ Z = 0$ or, what is the same, the general number of the domains of the constance of the sign of the divergence. But, in this case, taking into account all the non- reducible branches of the curve $\mathbf{div}\ Z = 0$, which may define closed curves, and, basing on the Harnack theorem [9], stating that a real non-reducible non-singular algebraic curve of the order p cannot have more than $(p-1)(p-2)/2 + 1$ real cycles, we, indeed, obtain the estimate of the maximum number of the domains of the constance of the sign of the divergence, and, hence, the upper estimate of the number of strict limit cycles of the polynomial dynamic systems of the form (1.1).

Theorem 2.6 generalizes and strengthens the S.P. Diliberto theorem 7 [8]. The generalization and the strengthening of the S.P. Diliberto result is the fact that, first of all, the theorem 2.6 considers rather polynomial dynamic systems of the general form than only those, which have only strict limit cycles as it is assumed by S.P. Diliberto.

Secondly, the theorem 2.6 gives, in the general case, the upper estimate of the number of limit cycles substantially more exact than the S.P. Diliberto theorem 7. These estimates coincide in the case when the curve, defined by the equation $\mathbf{div}\ Z = 0$, turns out to be a non-reducible curve.

Thus, consider, for example, the dynamic system

$$\frac{dx}{dt} = -y+x(1+y^2)^m(x^2+y^2-1), \quad \frac{dy}{dt} = x+y(1+y^2)^m(x^2+y^2-1) \quad (2.4)$$

with the only critical point $O(0,0)$ and $m \in \mathbb{N}$. For this system

$$\mathbf{div}\ Z = 2(1+y^2)^{m-1}\{(x^2+y^2-1)[1+(m+1)y^2]+(x^2+y^2)(1+y^2)\}$$

and thus, $n = 2m+3$, $s = 0$, $p = 2m-2$, $k = 1$.

According to S.P. Diliberto theorem the general number of strict limit cycles of the system (2.4) does not exceed the number $m(2m+1)+1$ for an arbitrary m and, according to the theorem 2.6, the number of the indicated cycles, for an arbitrary m, does not exceed 4. If, however, one considers strict limit cycles of the system (2.4), forming a nest, then, according to S.P. Diliberto theorem, their number does not exceed the number $m+1$, and, as we see, again the upper estimate of the number of strict limit cycles depends upon the number m.

However, according to the theorem 2.6, the number of strict limit cycles, forming a nest, does not exceed, in this case, 2 (and does not depend upon the number m). When $m = 1$, i.e., when the curve, defined by the equation $\mathbf{div}\ Z = 0$, turns out to be non-reducible, the upper estimate of the number of strict limit cycles, obtained in the theorem 2.6, coincides with the S.P. Diliberto estimate.

Let us turn now to the consideration of unrough divergent limit cycles.

Here, let us note, first of all, the following result.

Theorem 2.7 ([5]). *Let the functions $X(x,y)$ and $Y(x,y)$ belong to the class C^k, $k = 1,2,3,\ldots$, in any finite domain of the phase plane $\mathbb{R}^2$. Then, if the dynamic system* (1.1) *has an unrough limit cycle Γ and, if there exists the function $B(x,y)$ of the class C^k such that in some neighbourhood of Γ the function* $\mathbf{div}\ BZ$ *is the function of the sign-constant, then Γ is an unrough divergent limit cycle of the system* (1.2).

The theorem 2.7 is, in fact, a reformulation of the theorem 2 of the work [10]. The pointed out result may be made more exact. That is, the evident discussions allow to come to the following conclusion: an unrough divergent limit cycle of the system (1.2), discussed in the theorem 2.7, is odd-multiple, it being stable or unstable, in dependence of the fact, whether **div** BZ is non-positive or non-negative in some neighbourhood of Γ. Thus, consider, for example, the dynamic system

$$\frac{dx}{dt} = -y + x(x^2 + y^2 - 1)^3, \quad \frac{dy}{dt} = x + y(x^2 + y^2 - 1)^3. \tag{2.5}$$

This system has a three-multiple unstable divergent limit cycle, defined by the equation $x^2 + y^2 = 1$, in whose sufficiently small neighbourhood

$$\mathbf{div}\, Z = 2(x^2 + y^2 - 1)^2[4(x^2 + y^2) - 1] \tag{2.6}$$

is a sign-positive function.

As for the even-multiple divergent limit cycles (and that means, semi-stable ones), it should be noted that in the neighbourhood of such cycles the divergence of the vector field of the corresponding dynamic system will necessarily change sign, changing it in the very arbitrary way.

Indeed, consider the dynamic system

$$\frac{dx}{dt} = y + xy^2(x^2 + y^2 - 1)^2, \quad \frac{dy}{dt} = -x$$

with an unrough divergent two-multiple limit cycle, defined by the equation $x^2 + y^2 = 1$. In this case $\mathbf{div}\, Z = y^2(x^2 + y^2 - 1)[5x^2 + y^2 - 1]$ and, as we see, in the outer half-neighbourhood of the limit cycle, the divergence is non-negative, and in its inner half-neighbourhood it changes sign.

Turn now to the dynamic system

$$\frac{dx}{dt} = -y + x(x^2 + y^2 - 1)^2, \quad \frac{dy}{dt} = x + y(x^2 + y^2 - 1)^2$$

which also has an unrough divergent two-multiple cycle defined by the equation $x^2 + y^2 = 1$. Here,

$$\mathbf{div}\, Z = 2(x^2 + y^2 - 1)[3(x^2 + y^2) - 1],$$

and in the outer half-neighbourhood of the limit cycle the divergence is positive, and in the inner one it is negative.

Without further considering all possible cases of the divergence changing sign in the neighbourhood of a limit cycle, we note only that there also exist such dynamic systems with unrough divergent limit cycles in whose outer and inner half-neighbourhoods the divergence of the vector field changes sign.

As an example of such a dynamic system is the system

$$\frac{dx}{dt} = y + x(y - 1/2)(y + 1/2)(x^2 + y^2 - 1)^2, \quad \frac{dy}{dt} = -x$$

with the same limit cycle as in the preceding two cases, and

$$\mathbf{div}\, Z = (y - 1/2)(y + 1/2)(x^2 + y^2 - 1)(5x^2 + y^2 - 1).$$

Theorem 2.8 ([5]). *Let the functions $X(x,y)$ and $Y(x,y)$ be holomorphic in any finite domain of the phase plane $\mathbb{R}^2$, and let, with some nonzero function $B:\mathbb{R}^2\to\mathbb{R}^+$ of the class C^∞, the curve, defined by the equation* $\mathbf{div}\ BZ=0$, *have a finite real branch Γ, being a closed trajectory of the system* (1.2)*. Then, if in an outer or an inner half-neighbourhood of Γ the function* $\mathbf{div}\ BZ$ *is the sign-constant one, then Γ is an unrough divergent limit cycle of the system* (1.2)*.*

The validity of the theorem 2.8 follows from the Dulac criterion for a two-connected domain , from the holomorphy of the functions $X(x,y)$ and $Y(x,y)$, and from the properties of the Poincaré function.

Theorem 2.9 ([5]). *Let the functions $X(x,y)$ and $Y(x,y)$ be holomorphic, and the nonzero function $B:\mathbb{R}^2\to\mathbb{R}^+$ belong to the class C^∞ in any finite domain of the phase plane $\mathbb{R}^2$. Then, if the curve, defined by the equation* $\mathbf{div}\ BZ=0$, *has a finite real branch Γ_p, being a p-multiple simple without critical points of the system* (1.2) *analytic closed curve, coinciding with a q-multiple ($p\le q$) branch of the curve, defined by the equation* (2.1)*, and if in an outer or an inner half-neighbourhood of this branch the function* $\mathbf{div}\ BZ$ *is the sign-constant one, then the dynamic system* (1.1) *has an unrough limit cycle.*

The validity of the theorem 2.9 follows from the coincidence of the branches Γ_p and Γ_q, turning out the divergent closed trajectories of the dynamic system (1.2), from the positivity of the function B, and from the theorem 2.8.

Remark 2.2. Pay attention to the fact that the multiplicities of the coinciding branches of the curves , discussed in the theorems 2.3 and 2.9, generally speaking, are not necessarily the same, as well as they are not necessarily to coincide with the multiplicity of the limit cycle itself either.

Thus, for example, consider the dynamic system (2.5) with a triple limit cycle. For this system $D_t^{(2.5)}\,\mathbf{div}\,Z=24(x^2+y^2-1)^4[2(x^2+y^2-1)^2+3(x^2+$ $+y^2-1)+1]$ and, thus, the coinciding branch $x^2+y^2=1$ of the curves defined by the equations $\mathbf{div}\ Z=0$, where $\mathbf{div}\ Z$ is of the form (2.6), and $D_t^{(2.5)}\,\mathbf{div}\ Z=0$, has the multiplicities 2 and 4, correspondingly.

The dynamic system

$$\begin{aligned}\frac{dx}{dt}&=4y+x(x^2+y^2-2)(x^2+y^2-1),\\ \frac{dy}{dt}&=-4x+y(x^2+y^2-2)(x^2+y^2-1),\end{aligned}\tag{2.7}$$

as well as in the previous case, has a limit cycle, defined by the equation $x^2+y^2=1$. This limit cycle is a rough divergent stable limit cycle. Here $\mathbf{div}\ Z=6(x^2+y^2-1)^2$, $D_t^{(2.7)}\,\mathbf{div}\ Z=12(x^2+y^2)(x^2+y^2-2)(x^2+y^2-1)^2$.

As we see, the double branch $x^2 + y^2 = 1$ of the curves, defined by the equations $\mathbf{div}\ Z = 0$ and $D_t^{(2.7)}\,\mathbf{div}\ Z = 0$ is, in this case, a limit cycle of multiplicity one.

Consider, now, the dynamic system

$$\frac{dx}{dt} = y + x(x^2 + y^2 - 1)^2, \quad \frac{dy}{dt} = -x, \tag{2.8}$$

having an unrough double divergent limit cycle, namely, a unit circle with the centre in the origin of coordinates of the phase plane. Here $\mathbf{div}\,Z = (x^2 + y^2 - 1)(5x^2 + y^2 - 1)$ and

$$D_t^{(2.8)}\,\mathbf{div}\,Z = 4x(x^2 + y^2 - 1)[2y + x(x^2 + y^2 - 1)(5x^2 + 3y^2 - 3)]$$

and, thus, the coinciding one-multiple branch $x^2 + y^2 = 1$ of the curves, defined by the equations $\mathbf{div}\,Z = 0$ and $D_t^{(2.8)}\,\mathbf{div}\ Z = 0$, is a limit cycle of multiplicity two.

Remark 2.3. The above obtained results, concerning the existence of limit cycles, are a specific inversion of the Bendixon-Dulac criterion about the absence of closed trajectories in the dynamic system (1.1).

Definition 2.1 ([11])**.** A limit cycle Γ of the dynamic system (1.1) is said to be a generalized-strictly stable (unstable) limit cycle if the divergence of the vector field, determined by the system (1.1), not vanishing identically on Γ, satisfies, on this cycle, the condition $\mathbf{div}\ Z \leq 0$ (≥ 0).

As follows from the definition 2.1, the concept of a generalized-strict limit cycle, generalizes the concept of a strict limit cycle, introduced by the S.P. Diliberto. According to this contents this generalization is essential, since in contrast to the case of strict limit cycles in the case of generalized-strict limit cycles, their transversal intersection and tangency by various branches of a reducible curve, defined by the equation $\mathbf{div}\ Z = 0$, are allowed.

Thus, consider the dynamic system

$$\frac{dx}{dt} = -y + xy^2(1 - x^2 - y^2), \quad \frac{dy}{dt} = x,$$

having in the origin of coordinates of the phase plane an unstable focus. For this system, the curve, defined by the equation $x^2 + y^2 = 1$, is a generalized-strictly stable limit cycle. In this situation, $\mathbf{div}\,/, Z = y^2(1 - 3x^2 - y^2)$, and, as it is easily seen, at the points $M_1(0;1)$, $M_2(0;-1)$, $M_3(1;0)$, $M_4(-1;0)$ of a limit cycle the divergence vanishes, and is negative at its other points. Herewith, at the points M_1, M_2 there takes place tangency of a limit cycle with one of the branches of the curve, defined by the equation $\mathbf{div}, Z = 0$, and at the points M_3, M_4 there takes place transversal intersection of a cycle with other branch of the above mentioned curve.

The generalized-strict limit cycles play an important role in the qualitative theory of rough limit cycles. The foundation for this statement is

Theorem 2.10 ([11]). *Let the functions $X(x,y)$ and $Y(x,y)$ belong to the class C^k, $k = 1,2,3,\dots$, in any finite domain of the phase plane $\mathbb{R}^2$. Then, if the dynamic system* (1.1) *has a rough stable* (*unstable*) *limit cycle* Γ, *then there exists a nonzero function $B : \mathbb{R}^2 \to \mathbb{R}^+$ of the class C^k such that for the dynamic system* (1.2) *the cycle Γ will be a generalized-strictly stable* (*unstable*) *limit cycle.*

The proof of this theorem is similar to the proof of the lemma from the article [6].

3. Classification of Limit Cycles

The further proposed classification of limit cycles is based on the concepts of a strict and regular (monotonic) limit cycles.

Remind that the concept of a strict limit cycle is connected with the properties of the divergence of the vector field of the dynamic system (1.1). As for the concept of a regular or monotonic limit cycle introduced in the work [12], it is formally connected with the properties of the operator D^k, such that

$$D^k S(x,y) = \left[-Y(x,y)\frac{\partial}{\partial x} + X(x,y)\frac{\partial}{\partial y}\right]^k S(x,y) =$$
$$= \frac{\partial^k S(x,y)}{\partial x^k}(-Y(x,y))^k + C_k^1 \frac{\partial^k S(x,y)}{\partial x^{k-1}\partial y}(-Y(x,y))^{k-1}X(x,y) + \dots +$$
$$+ C_k^r \frac{\partial^k S(x,y)}{\partial x^{k-r}\partial y^r}(-Y(x,y))^{k-r}X^r(x,y) + \dots + \frac{\partial^k S(x,y)}{\partial y^k}X^k(x,y), \quad (3.1)$$

gather where $X(x,y)$ and $Y(x,y)$ are the right-hand sides of the dynamic system (1.1), C_k^r is the number of r-combinations of k elements, and the foundation for this being the further stated theorem 3.1.

The geometrical interpretation of a regular limit cycle is that, beginning at some time, $t = t_0$, the distance from the representative point, moving on a spiral in the neighbourhood of a regular limit cycle, to the limit cycle, is strictly monotonic for increasing or decreasing t.

Herewith, substantial is the fact that for the calculation of the multiplicity of a regular limit cycle it is not necessary to calculate the multiplicity of the root of the Poincaré function (in general this is a transcendental problem), and it is sufficient only to verify satisfiability of a number of conditions expressed in the terms of the right-hand sides of the dynamic system (1.1) (see the theorem 3.1).

But before turning to the theorem 3.1 let us pay attention to some considerations concerned with the definition of a regular cycle.

So, let

$$x = f(t), \quad y = g(t), \quad (3.2)$$

where f and g are periodic functions of t with the period ω, be the parametric equations of a negatively oriented limit cycle Γ of the dynamic system

(1.1). As for the functions X and Y, we suppose them to have continuous partial derivatives of any required order.

Introduce now a new, so called universal curvilinear-coordinate system, with the parameter t in equations Γ replaced by the length s of the cycle measured from a fixed point on Γ, and with positive direction clockwise. Then a change in the direction of s to the positive direction corresponds to a change of $t = \tau(s)$ also in the positive direction.

Let l be the arc length of one passage around the limit cycle, and let

$$x = f(\tau(s)) = \varphi(s), \quad y = g(\tau(s)) = \psi(s) \tag{3.3}$$

where $0 \leq s \leq l$, be the parametric equations of the limit cycle (3.2).

Then, in the universal curvilinear-coordinate system $(s, \mathbf{n})$, the following relations hold [12]:

$$x = \varphi(s) - \mathbf{n}\psi'(s), \quad y = \psi(s) + \mathbf{n}\varphi(s), \tag{3.4}$$

where

$$\varphi'(s) = \frac{X(\varphi(s), \psi(s))}{\sqrt{X^2(\varphi(s), \psi(s)) + Y^2(\varphi(s), \psi(s))}},$$

$$\psi'(s) = \frac{Y(\varphi(s), \psi(s))}{\sqrt{X^2(\varphi(s), \psi(s)) + Y^2(\varphi(s), \psi(s))}}.$$

We note that, if a point B of the limit cycle has rectangular coordinates $(\varphi(s), \psi(s))$, then the formulae (3.4) uniquely relate the rectangular coordinates (x, y) of a point A on the directed normal in through B to its curvilinear coordinates.

By virtue of the formulae (3.4) we can write the differential equation of trajectories of the system (1.1) in curvilinear coordinates, in the neighbourhood of a limit cycle:

$$\frac{d\mathbf{n}}{ds} = \frac{Y\varphi' - X\psi' - \mathbf{n}(X\varphi'' + Y\psi'')}{X\varphi' + Y\psi'} = F(s, \mathbf{n}). \tag{3.5}$$

With the use of the equation (3.5), in the work [12] the following definition of a regular cycle is introduced: *a limit cycle* Γ *of the dynamic system (1.1) is said to be a regular limit cycle, if the function* $F(s, \mathbf{n})$ *has constant sign in each of the semicircles* $\mathbf{n} > 0$ *and* $\mathbf{n} < 0$.

Herewith, as it follows from the further context of the work [12], the function $F(s, \mathbf{n})$ cannot vanish.

Such definition became generally accepted, and was usually applied in the works on limit cycles.

Along with this, it should be noted that, if to take into consideration geometrical sense of regular limit cycles, then to such cycles there also belong limit cycles, in whose neighbourhood the function $F(s, \mathbf{n})$ does not change sign, but can vanish, for example, at the isolated points of each of the curves $\mathbf{n} = \text{const}$, and that completely agrees with the criterion of a strict monotony of a function (see, e.g., [13, p. 102]).

Taking into account the above mentioned considerations, the P.N. Papush definition [12] does not cover all the class of regular limit cycles, formulating only the sufficient regularity condition. On this account a number of results on the theory of regular limit cycles have to be made more exact. For instance, the V.A. Krasnogorov theorems about the fact that:

1) a holomorphic dynamic system

$$\frac{dx}{dt} = X(y), \quad \frac{dy}{dt} = Y(x,y)$$

cannot have regular limit cycles [14];

2) regular limit cycles of the holomorphic dynamic system (1.1) cannot have common points with an isocline of the differential equation

$$\frac{dy}{dx} = \frac{Y(x,y)}{X(x,y)},$$

which is ortogonal to its integral curves [15].

Keeping in mind the mentioned above considerations, we use, in the sequel, the following sharpened definition of a regular limit cycle.

Definition 3.1. A limit cycle Γ of the dynamic system (2.1) will be said to be a regular (monotonic) limit cycle, if the transition to universal curvilinear-coordinate system yields a differential equation (3.5) such that in each of certain half-neighbourhoods $0 < \mathbf{n} < \mathbf{n}_0$ and $-\mathbf{n}_0 < \mathbf{n} < 0$ of the cycle Γ, the function $F(s,\mathbf{n})$ has constant sign ($F(s,\mathbf{n}) \leq 0$ (≥ 0)) and does not vanish identically on open arcs of curves $\mathbf{n} = \mathrm{const}$.

Theorem 3.1. *For a limit cycle* Γ *of a holomorphic system* (1.1) *to be k-tuple regular limit cycle, it is necessary and sufficient that along the curve* Γ *there hold the following conditions:*

$$\sigma_i \equiv X(x,y)D^iY(x,y) - Y(x,y)D^iX(x,y) = 0, \tag{3.6}$$

$i = 1, 2, \ldots, k-1,$ *but*

$$\sigma_k \equiv X(x,y)D^kY(x,y) - Y(x,y)D^kX(x,y) \leq 0 \ (\geq 0), \tag{3.7}$$

where the operator D^s *is given by the formula* (3.1)*, and equality is attained in* (3.7) *only at isolated points of the cycle. For odd k the limit cycle is stable* (*unstable*)*, while for even k it is semistable.*

The validity of the theorem practically follows from the reasoning, when the theorem 3[16] is being proved, the definition of the multiplicity of a limit cycle, and a criterion for a function to be strictly monotonic.

Remark 3.1. The functions σ_i, determined in the operator form by the equalities (3.6), are given in an explicit form by the right-hand sides of the dynamic system (2.1) as follows:

$$\sigma_i = \sum_{\nu=0}^{i+1}(-1)^{\nu} X^{i+1-\nu} Y \left(C_i^{\nu} Y_{x^{\nu}y^{i-\nu}} + C_i^{\nu-1} Y_{x^{\nu-1}y^{i-\nu+1}} \right),$$

where C_i^r is the number of r-combinations of i elements, $C_i^{i+1} = 0$, $C_i^{-1} = 0$.

With the use of the theorem 3.1, it may be shown, for instance, that the dynamic system

$$\frac{dx}{dt} = y + x(x^2 + y^2 - 1)^2, \quad \frac{dy}{dt} = -x \tag{3.8}$$

has the semistable double regular limit cycle, defined by the equation $x^2 + +y^2 = 1$.

The fact , that the indicated curve is a trajectory of the system (3.8), follows from the analytical test of a particular integral. And the fact, that the circle $x^2 + y^2 = 1$ is a limit cycle, not belonging to the family of the curves of the centre, results from the Bendixson criterion. Further, the calculations show that in the given case $\sigma_1 = 0$ along a limit cycle. As for as σ_2 is concerned, note that at the points $M_1(0, 1)$, $M_2(0, -1)$ of a limit cycle $\sigma_2 = 0$, and at its other points the inequality $\sigma_2 > 0$ holds.

Turning to the classification of limit cycles, introduce two new concepts.

Definition 3.2. A limit cycle of the dynamic system (2.1) is said to be a nonstrict limit cycle, if along this limit cycle the divergence of the vector field, given by the system (2.1), changes sign.

We use, in the sequel, the symbol $\underset{\sim}{\mathbf{div}}\, z$, meaning that the divergence of the vector field, given by the system (1.1), changes sign along the field considered. If, however, it appears that the divergence has constant sign or constant meaning, or if it vanishes along a limit cycle, then, in these cases, we use, correspondingly, the symbols $\underset{\leq 0(\geq 0)}{\mathbf{div}}\, z$ $(\underset{<0(>0)}{\mathbf{div}}\, z)$, $\underset{\text{const}}{\mathbf{div}}\, z$, $\underset{=0}{\mathbf{div}}\, z$, with the vanishing in the first case being allowed only at isolated points of the cycle.

Definition 3.3. A limit cycle of the dynamic system (1.1) is said to be a non-regular limit cycle, if along this cycle there is k such that the conditions (3.6) hold, but σ_k changes sign.

Herewith, if, already, σ_1 changes sign, then a limit cycle will be said to be a non-regular limit cycle of the first kind; otherwise it will be said to be a non-regular limit cycle of the second kind.

Further, similarly to the symbols $\underset{\sim}{\mathbf{div}}\, z$, $\underset{\leq 0(\geq 0)}{\mathbf{div}}\, z$, $\underset{<0(>0)}{\mathbf{div}}\, z$, $\underset{\text{const}}{\mathbf{div}}\, z$, $\underset{=0}{\mathbf{div}}\, z$ we will use also the symbols $\underset{\sim}{\sigma_k}$, $\underset{\leq 0(\geq 0)}{\sigma_k}$, $\underset{<0(>0)}{\sigma_k}$, $\underset{\text{const}}{\sigma_k}$, $\underset{=0}{\sigma_k}$ in order to indicate that, on Γ, σ_k changes sign, has constant sign, has a constant nonzero value, and vanishes.

Moreover, unless otherwise stated, we consider, for simplicity, generalized-strict $(\underset{\leq 0(\geq 0)}{\mathbf{div}}\, z)$, strict in the sense of S.P. Diliberto $(\underset{<0(>0)}{\mathbf{div}}\, z)$, and rough divergent limit cycles $(\underset{\text{const}}{\mathbf{div}}\, z)$ as strict limit cycles. As for the general symbol, we use, in this case the symbol of a generalized-strict limit cycle.

And then, if to consider, first, a case of rough limit cycles, then, taking into account, that along of a limit cycle there holds the equality

$$\int_0^\omega \frac{\sigma_1}{\sigma_0} dt = \int_0^\omega \mathbf{div}\ z dt,$$

where $\sigma_0 = X^2(x,y)+Y^2(x,y)$, we logically deduce a scheme, represented in the Fig. 3.1. The scheme shows that rough limit cycles may be subdivided into the following four classes:

1) nonstrict non-regular limit cycles;
2) nonstrict regular limit cycles;
3) strict non-regular limit cycles;
4) strict regular limit cycles.

Rough limit cycles

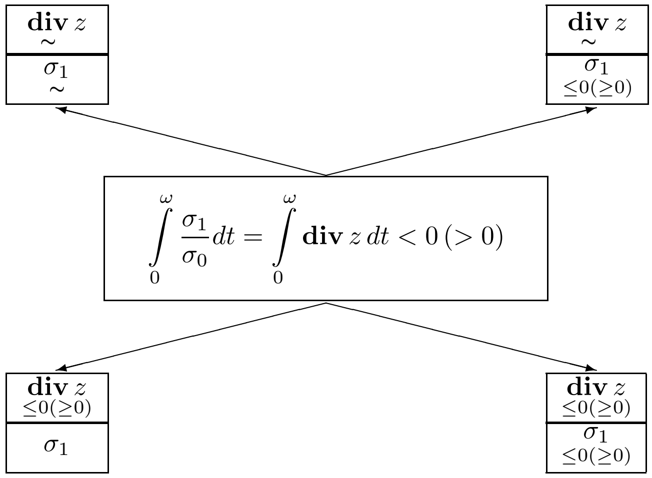

Fig. 3.1

Examples show that each of the four types is realized in practice. In particular, there are types of rough limit cycles that can occur in the division of classes 2)—4) into subclasses, by using the existence of divergence limit cycles ($\underset{\mathrm{const}}{\mathbf{div}\ z}$) and the existence of regular limit cycles on which σ_1 is a nonvanishing constant ($\underset{\mathrm{const}}{\sigma_1}$).

Turn now to consideration of unrough limit cycles. Here we logically deduce a scheme, represented in the Fig. 3.2:

Unrough limit cycles

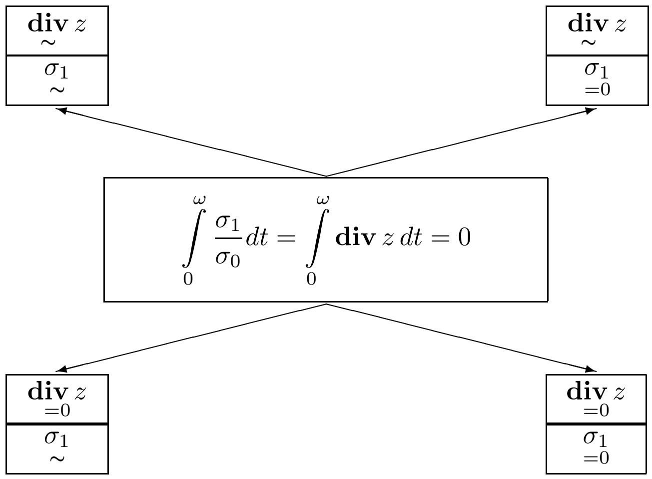

Fig. 3.2

If, however, to separate now regular limit cycles from non-regular limit cycles, then, finally, we obtain the following scheme (Fig. 3.3):

Unrough limit cycles

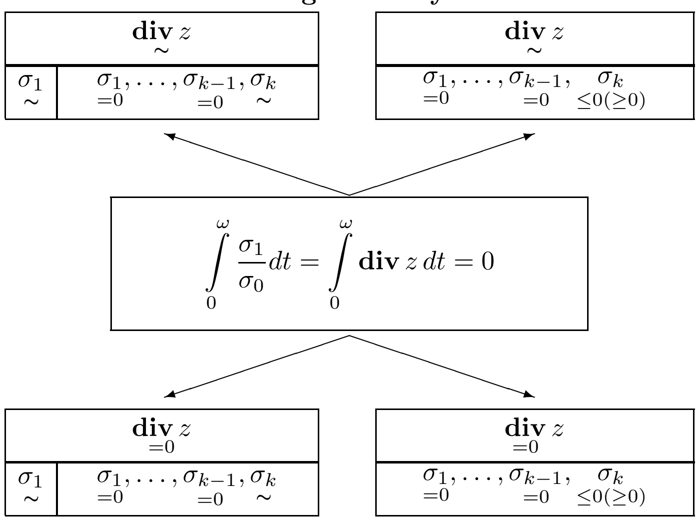

Fig. 3.3

So, we have four classes of unrough limit cycles. They are:
1) nonstrict non-regular limit cycles of the first or second kind;
2) nonstrict regular limit cycles;
3) divergent non-regular limit cycles of the first or second kind;
4) divergent regular limit cycles.

As in the case of rough limit cycles, each of these classes is realized in practice.

Turn now to the proof of the result, principal for the above carried out classification of limit cycles, whose contensive part is the fact that qualitative research of limit cycles in the general case of the holomorphic dynamic systems of the form (1.1) may be always reduced to the investigation of regular divergent limit cycles. Hereby note, that the theorem 3.2, to be further proved, gives, also, mathematical foundation for a new approach to the definition of multiplicity of any limit cycle. This approach is directly concerned with the reducibility problem (regarding this see also [17]).

Theorem 3.2. *Let the functions $X(x, y)$ and $Y(x, y)$ be holomorphic in any finite domain of the phase plane $\mathbb{R}^2$. Then, if the dynamic system* (1.1) *has a limit cycle Γ, whose parametric equations are as in* (3.2)*, then there is a variable change*

$$x = f(t) + \sum_{i=1}^{\infty} \gamma_i(t)\nu^i, \quad y = g(t) + \sum_{i=1}^{\infty} \eta_i(t)\nu^i, \tag{3.9}$$

where γ_i and η_i are ω-periodic functions, transforming the differential equation of trajectories, of the system (1.1)*, in the neighbourhood of Γ, into a differential equation*

$$\frac{d\nu}{dt} = h_1\nu + h_2\nu^2 + \cdots + h_k\nu^k + \cdots \tag{3.10}$$

with constant coefficients, i.e., into a differential equation with a regular trivial solution corresponding to the cycle Γ.

Proof. As it was noted above, the differential equation of trajectories of the dynamic system (1.1), in the neighbourhood of a limit cycle Γ, is, in curvilinear coordinates $(s, \mathbf{n})$, of the form (3.5). In virtue of holomorphicity of the right-hand sides of the system (1.1), the function F in (3.5) is a holomorphic function of $\mathbf{n}$. And then, it being expanded in a series with respect of integer degrees of $\mathbf{n}$ in the neighbourhood of a limit cycle, the equation (3.5) may be rewritten in the form

$$\frac{d\mathbf{n}}{ds} = F'_{\mathbf{n}}(s, 0)\mathbf{n} + \frac{1}{2!}F''_{\mathbf{n}^2}(s, 0)\mathbf{n}^2 + \cdots + \frac{1}{k!}F^{(k)}_{\mathbf{n}^k}(s, 0)\mathbf{n}^k + \cdots . \tag{3.11}$$

Now, if as parametric equations of a cycle Γ to keep in mind rather the equations (3.2) than the equations (3.3), then the coefficients $F^{(k)}_{\mathbf{n}^k}(s, 0)$ of the series in the right-hand side of the equation (3.11) will be ω-periodic functions of t. Hence, taking into consideration, for example, [18, § 26], we come to the conclusion that the differential equation (3.11) for the unknown function $\mathbf{n}$ with coefficients that are ω-periodic functions of t, is reduced by ω-periodic, with respect of t, variable change

$$\mathbf{n} = a_1(t)\nu + a_2(t)\nu^2 + \cdots + a_k(t)\nu^k + \cdots \tag{3.12}$$

to the differential equation with constant coefficients of the form (3.10). Taking into account, however, the fact that universal curvilinear-coordinate system $(s, \mathbf{n})$ is given, in this case, by the equalities

$$x = f(t) - \mathbf{n}g'(t), \quad y = g(t) + \mathbf{n}f'(t),$$

we come to the conclusion that there exists a coordinate change

$$x = f(t) - [a_1(t)\nu + a_2(t)\nu^2 + \cdots + a_k(t)\nu^k + \cdots]g'(t),$$
$$y = g(t) + [a_1(t)\nu + a_2(t)\nu^2 + \cdots + a_k(t)\nu^k + \cdots]f'(t),$$

i.e., a coordinate change of the form (3.9), transforming the differential equation of trajectories of the system (1.1), in the neighbourhood of Γ, into an equation (3.10) with constant coefficients. □

Remark 3.2. From the theorems 1.1 and 3.2 it follows that in the qualitative investigation of the holomorphic dynamic systems of the form (1.1), any limit cycle can be assumed to be a regular divergent limit cycle.

Remark 3.3. The multiplicity of a limit cycle of the dynamic system (1.1) is the same as the multiplicity of the regular trivial solution of the differential equation (3.10); this follows from the form of the Poincaré function and the form of the phase-coordinate transformation, and the multiplicity is determined by the index j of the first nonvanishing coefficient h_j in the differential equation (3.10).

Thus, the theorem 3.2 is the mathematical foundation of the new approach to the determination of multiplicity of a limit cycle Γ of the holomorphic dynamic system (1.1), concerned with reduction of a differential equation of the trajectories of the original dynamic system in the neighbourhood of a limit cycle to a special form.

Remark 3.4. As it is shown above, the given classification of limit cycles is based on the properties of the divergence of the system (1.1) and on the condition of the regularity of a cycle, that is, such characteristics of limit cycles that, from the qualitative point of view, completely characterize any limit cycle of the system (1.1).

4. Multiplicity and Stability of Limit Cycles

As it is known, the multiplicity of limit cycles is determined as the multiplicity of a zero of the Poincaré function. With this in mind, in the work [16] there was undertaken an attempt to obtain the conditions of the multiplicity of a limit cycle of the analytic system of the form (1.1). However, obtained formulae in [16] turned out to be too unwieldy and practically unreadable due to the fact that the connection between each of the conditions deduced hasn't been detected.

Basing on the new approach, proposed in Section 3, we deduce recurrent formulae for the definition of multiplicity of limit cycles, further solving, in particular, simultaneously the problems of stability, unstability and semi-stability of limit cycles.

But before to deduce the pointed out recurrent formulae clear up how the coefficients $F^{(k)}_{\mathbf{n}^k}$ of the Taylor series for the function $F(t,\mathbf{n})$ in the equation (3.5) through the functions X and Y of the right-hand sides of the system (2.1) are expressed.

With this aim in view, we use the formula for presenting dn/dt in (3.5), obtained in the work [19] in the form

$$\frac{dn}{dt} = \sqrt{X^2+Y^2}(1+\tilde{k}\mathbf{n})\Big[(X\mathcal{D}Y - Y\mathcal{D}X) + \frac{1}{2!}(X\mathcal{D}^2Y - Y\mathcal{D}^2X)\mathbf{n}^2 + \\ + \frac{1}{3!}(X\mathcal{D}^3Y - Y\mathcal{D}^3X)\mathbf{n}^3 + \cdots\Big]\Big[(X^2+Y^2) + (X\mathcal{D}X + Y\mathcal{D}Y)\mathbf{n} + \\ + \frac{1}{2!}(X\mathcal{D}^2X + Y\mathcal{D}^2Y)\mathbf{n}^2 + \frac{1}{3!}(X\mathcal{D}^3X + Y\mathcal{D}^3Y)\mathbf{n}^3 + \cdots\Big]^{-1}, \tag{4.1}$$

where $\tilde{k} = \tilde{k}(t)\left(Y\frac{dX}{dt} - X\frac{dY}{dt}\right)(X^2+Y^2)^{-3/2}$, and the operator $\mathcal{D}^k = \left(-\frac{Y}{\sqrt{X^2+Y^2}}\frac{\partial}{\partial x} + \frac{X}{\sqrt{X^2+Y^2}}\frac{\partial}{\partial y}\right)^k$, where the functions X and Y are considered along a cycle Γ with the parametric equations (3.2), is acting as well as the operator $\mathcal{D}^k$ (see the formula (3.1)).

Our further reasoning will be connected with the representation of the expression from the second bracket of (4.1) in the form of the series $\sum\limits_{k=0}^{\infty}\alpha_k\mathbf{n}^k$.

For the sake of abbreviation of the writing we introduce the following notations:

$$X^2+Y^2 = b_0, \quad \sqrt{X^2+Y^2} = \beta_0, \quad \frac{X\mathcal{D}^jX + Y\mathcal{D}^jY}{j!} = b_j, \\ \frac{X\mathcal{D}^iY - Y\mathcal{D}^iX}{i!} = g_i \ \ (i,j=1,2,3,\ldots), \quad \left(Y\frac{dX}{dt} - X\frac{dY}{dt}\right)\Big/ b_0 = \beta_1. \tag{4.2}$$

Then, by virtue of the notations

$$\frac{1}{b_0 + b_1\mathbf{n} + b_2\mathbf{n}^2 + b_3\mathbf{n}^3 + \cdots} = \sum_{k=o}^{\infty}\alpha_k\mathbf{n}^k \tag{4.3}$$

and, thus,

$$\begin{aligned} &b_0\alpha_0 = 1,\\ &b_1\alpha_0 + b_0\alpha_1 = 0,\\ &b_2\alpha_0 + b_1\alpha_1 + b_0\alpha_2 = 0,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &b_k\alpha_0 + b_{k-1}\alpha_1 + \cdots + b_1\alpha_{k-1} + b_0\alpha_k = 0 \quad (k=3,4,5,\ldots). \end{aligned}$$

Taking now into account that in the neighbourhood of a limit cycle $b_0 \neq 0$, we find

$$\alpha_0 = \frac{1}{b_0},$$
$$\alpha_1 = -\frac{1}{b_0^2},$$
$$\alpha_2 = -\frac{1}{b_0}(b_2\alpha_0 + b_1\alpha_1) = \frac{b_1^2}{b_0^3} - \frac{b_2}{b_0^2}, \tag{4.4}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\alpha_k = -\frac{1}{b_0}(b_k\alpha_0 + b_{k-1}\alpha_1 + \cdots + b_1\alpha_{k-1}) \quad (k = 3, 4, 5, \ldots).$$

Hence

$$\sum_{k=0}^{\infty}\alpha_k n^k = \frac{1}{b_0} - \frac{b_1}{b_0^2}\mathbf{n} + \left(\frac{b_1^2}{b_0^3} - \frac{b_2}{b_0^2}\right)\mathbf{n}^2 + \cdots = \frac{1}{X^2+Y^2} -$$
$$-\frac{X\mathcal{D}X+Y\mathcal{D}Y}{(X^2+Y^2)^2}\mathbf{n} + \left[\frac{(X\mathcal{D}X+Y\mathcal{D}Y)^2}{(X^2+Y^2)^3} - \frac{1}{2}\frac{X\mathcal{D}^2X+Y\mathcal{D}^2Y}{X^2+Y^2}\right]\mathbf{n}^2 + \cdots \tag{4.5}$$

Keeping now in view the formulae (4.5) and (4.3), the equality (4.1) may be rewritten in the form

$$\frac{dn}{dt} = \sqrt{X^2+Y^2}\left[1 + \left(Y\frac{dX}{dt} - X\frac{dY}{dt}\right)(X^2+Y^2)^{-3/2}\mathbf{n}\right]\times$$
$$\times\left[(X\mathcal{D}Y - Y\mathcal{D}X)\mathbf{n} + \frac{1}{2!}(X\mathcal{D}^2Y - Y\mathcal{D}^2X)\mathbf{n}^2 + \cdots\right]\times$$
$$\times\left\{\frac{1}{X^2+Y^2} - \frac{X\mathcal{D}X+Y\mathcal{D}Y}{(X^2+Y^2)^2}\mathbf{n} + \left[\frac{(X\mathcal{D}X+Y\mathcal{D}Y)^2}{(X^2+Y^2)^3} -\right.\right.$$
$$\left.\left.-\frac{1}{2!}\frac{X\mathcal{D}^2X+Y\mathcal{D}^2Y}{(X^2+Y^2)^2}\right]\mathbf{n}^2 + \cdots\right\} = (\beta_0 + \beta_1\mathbf{n})(g_1\mathbf{n} + g_2\mathbf{n}^2 +$$
$$+ g_3\mathbf{n}^3 + \cdots)(\alpha_0 + \alpha_1\mathbf{n} + \alpha_2\mathbf{n}^2 + \alpha_3\mathbf{n}^3 + \cdots) = (\beta_0 + \beta_1\mathbf{n})\times$$
$$\times\sum_{i=1}^{\infty}g_i\mathbf{n}^i\sum_{j=0}^{\infty}\alpha_j\mathbf{n}^j = \beta_0\sum_{i=1}^{\infty}g_i\sum_{j=0}^{\infty}\alpha_j\mathbf{n}^{i+j} + \beta_1\sum_{l=1}^{\infty}g_l\sum_{s=0}^{\infty}\alpha_s\mathbf{n}^{l+s+1}. \tag{4.6}$$

Putting now in (3.11) $s = t$ and turning to the representation (4.6) we find finally that

$$F_{\mathbf{n}^k}^{(k)}(t,0) = k!\left[\beta_0\sum_{i=1}^{k}g_i\sum_{j=0}^{k}\alpha_j + \beta_1\sum_{l=1}^{k}g_l\sum_{s=0}^{k}\alpha_s\right], \tag{4.7}$$

where $i + j = l + s + 1 = k$ and b_0, b_j, g_i, β_0, β_1 and α_1 are expressed, correspondingly, by the formulae (4.2) and (4.4).

Turn now to the deduction of recurrent formulae for the definition of multiplicity of a limit cycle. For this, first of all, we differentiate with respect to t the relation (3.12). Then

$$\frac{dn}{dt}=(\dot{a_1}+h_1a_1)\nu+(\dot{a_2}+h_2a_1+2h_1a_2)\nu^2+(\dot{a_3}+h_3a_1+2h_2a_2+3h_1a_3)\nu^3+ \\ +\cdots+(\dot{a_l}+h_la_1+2h_{l-1}a_2+3h_{l-2}a_3+\cdots+lh_1a_l)\nu^l+\cdots \tag{4.8}$$

On the other hand, using (3.12), we find that

$$\frac{dn}{dt}=F'_{\mathbf{n}}(t,0)\mathbf{n}+\frac{1}{2!}F''_{\mathbf{n}^2}(t,0)\mathbf{n}^2+\cdots+\frac{1}{k!}F^{(k)}_{\mathbf{n}^{(k)}}(t,0)\mathbf{n}^k+\cdots=F_1a_1\nu+ \\ +(F_1a_2+F_2a_1^2)\nu^2+(F_1a_3+2F_2a_1a_2+F_3a_1^3)\nu^3+\cdots+\Bigg(\sum_{j=2}^{k}F_j\times \\ \times\sum_{s_1+\cdots+s_{k-1}=j}\frac{j!}{s_1!s_2!\cdots s_{k-1}!}a_1^{s_1}a_2^{s_2}\cdots a_{k-1}^{s_{k-1}}+F_1a_k\Bigg)\nu^k+\cdots, \tag{4.9}$$

where

$$s_1+2s_2+3s_3+\cdots+(k-1)s_{k-1}=k, \tag{4.10}$$

$s_i\geq 0$, $s_i\in\mathbb{Z}$ and

$$F_1=F'_{\mathbf{n}},\quad F_2=\frac{1}{2!}F''_{\mathbf{n}^2},\quad F_3=\frac{1}{3!}F'''_{\mathbf{n}^3},\quad\cdots\quad F_k=\frac{1}{k!}F^{(k)}_{\mathbf{n}^k}.$$

The condition (4.10) results from the considerations that

$$(a_1\nu+a_2\nu^2+\cdots+a_l\nu^l)^m=\sum_{s_1+s_2+\cdots+s_l=m}\frac{m!}{s_1!s_2!\cdots s_l!}(a_1\nu)^{s_1}(a_2\nu^2)^{s_2}\times \\ \times\cdots(a_l\nu^l)^{s_l}=\sum_{s_1+s_2+\cdots+s_l=m}\frac{m!}{s_1!s_2!\cdots s_l!}a_1^{s_1}a_2^{s_2}\cdots a_l^{s_l}\nu^{s_1}\nu^{2s_2}\cdots\nu^{ls_l}= \\ =\sum_{s_1+s_2+\cdots+s_l=m}\frac{m!}{s_1!s_2!\cdots s_l!}a_1^{s_1}a_2^{s_2}\cdots a_l^{s_l}\nu^{s_1+2s_2+\cdots+ls_l}.$$

Equating now the coefficients under the equal degrees of ν in the right-hand sides of the equalities (4.8), (4.9), we obtain the system of ordinary differential equations

$$\begin{aligned}
&\dot{a}_1+(h_1-F_1)a_1=0,\\
&\dot{a}_2+(2h_1-F_1)a_2=F_2a_1^2-h_2a_1,\\
&\dot{a}_3+(3h_1-F_1)a_3=F_3a_1^3+2F_2a_1a_2-h_3a_1-2h_2a_2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\dot{a}_k+(kh_1-F_1)a_k=\sum_{j=2}^{k}F_j\sum_{s_1+\cdots+s_{k-1}=j}\frac{j!}{s_1!s_2!\cdots s_{k-1}!}a_1^{s_1}a_2^{s_2}\cdots a_{k-1}^{s_{k-1}}-\\
&-h_ka_1-2h_{k-1}a_2-3h_{k-2}a_3-\cdots-(k-1)h_2a_{k-1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{4.11}$$

where, remind, the condition (4.10) holds.

Turn, first, to the first equation of the system (4.11). As it is not difficult to see, the solution of this equation is the function a_1, given by the equality

$$a_1(t) = \exp \int_0^t (F_1 - h_1) d\tau.$$

Then, taking into consideration, that the coordinate change (3.12) is ω-periodic with respect to t, we come to the conclusion that $a_1(\omega) = \alpha_1(0)$, and therefore

$$\exp \int_0^\omega (F_1 - h_1) d\tau = 1$$

and, thus,

$$h_1 = \frac{1}{\omega} \int_0^\omega F_1 d\tau = \frac{1}{\omega} \int_0^\omega F_n' d\tau.$$

If a limit cycle is unrough, then $h_1 = 0$, and in this case

$$a_1(t) = \exp \int_0^t F_n' d\tau.$$

And then, if $h_1 = 0$, the solution of the second equation of the system (4.11) may be determined by the equality

$$a_2(t) = a_1(t) \int_0^t (F_2 a_1 - h_2) d\tau.$$

Using ω-periodicity of the function $a_2(t)$, obtain that

$$h_2 = \frac{1}{\omega} \int_0^\omega F_2 \left(\exp \int_0^t F_1 d\tau \right) dt.$$

Now, if $h_1 = h_2 = 0$, then

$$a_2(t) = \frac{1}{2!} \exp \int_0^t F_n' d\tau \left[\int_0^t (F''_{\mathbf{n}^2} \left(\exp \int_0^\tau F_n' ds \right) d\tau \right],$$

or

$$a_2(t) = \frac{1}{2!} a_1 \int_0^t F''_{\mathbf{n}^2} a_1 d\tau.$$

Further, with the account that $h_1 = h_2 = 0$, we find

$$a_3(t) = \exp \int_0^t F_1 d\tau \left[\int_0^t (2F_2 a_2 + F_3 a_1^2 - h_3) d\tau \right].$$

The condition of the ω-periodicity of $a_3(t)$ yields

$$h_3 = \frac{1}{\omega}\int\limits_0^{\omega}\left\{F_2 \exp\int\limits_0^t F_1 d\tau\left[2F_2\left(\exp\int\limits_0^{\tau} F_1 ds\right)d\tau + F_3 \exp 2\int\limits_0^t F_1 d\tau\right]\right\}dt.$$

Simplifying the latter expression we, finally, obtain

$$h_3 = \frac{1}{3!\omega}\int\limits_0^{\omega} F'''_{\mathbf{n}^3} \exp 2\left(\int\limits_0^t F'_n d\tau\right)dt.$$

And, then, if $h_1 = h_2 = h_3 = 0$, then

$$a_3(t) = \frac{1}{3!}a_1\int\limits_0^t F'''_{\mathbf{n}^3} a_1^2 d\tau + a_1\int\limits_0^t F''_{\mathbf{n}^2} a_2 d\tau.$$

Deduce now the formula for the representation of h_k, supposing $a_1, a_2, \dots,$ a_{k-1} to be known, and $h_1 = h_2 = \ldots = h_{k-1} = 0$. With these assumptions, as one can easily see, as the solution of the $k-th$ equation of the system (4.11) one may take

$$a_k(t) = a_1\int\limits_0^t\left(\sum_{j=2}^{k} F^{(j)}_{\mathbf{n}^j}\sum_{s_1+\cdots+s_{k-1}=j}\frac{a_1^{s_1-1}a_2^{s_2}\cdots a_{k-1}^{s_{k-1}}}{s_1!s_2!\cdots s_{k-1}!} - h_k\right)d\tau.$$

And then, keeping in mind the condition of ω-periodicity, we find that

$$h_k = \frac{1}{\omega}\int\limits_0^{\omega}\sum_{j=2}^{k} F^{(j)}_{\mathbf{n}^j}\sum_{s_1+\cdots+s_{k-1}=j}\frac{a_1^{s_1-1}a_2^{s_2}\cdots a_{k-1}^{s_{k-1}}}{s_1!s_2!\cdots s_{k-1}!}dt, \tag{4.12}$$

where $s_1,\ s_2, \dots, s_{k-1}$ satisfy the condition (4.10), and the functions $F^j_{\mathbf{n}^j}$ are determined by the formulae (4.7).

Formula (4.12) is just the recurrent formula for the calculation of h_k.

Thus, we come to the conclusion that if $h_1 = h_2 = \ldots = h_{k-1} = 0$, $h_k \neq 0$, then a limit cycle Γ of the holomorphic dynamic system (1.1) will be a k-tuple limit cycle. Herewith, if k is odd, then for $h_k < 0$ (> 0), a limit cycle will be stable (unstable), and if k is even, a limit cycle will be semistable.

The pointed out conditions are necessary and sufficient conditions of k-tuplicity, stability (unstability) and semistability of a limit cycle Γ, correspondingly.

Further, as it is seen from the analytic representation of constants h_i, the simplification in their calculation may be achieved both on the score of simpler, than (4.7), representation of the functions $F^{(j)}_{\mathbf{n}^j}$ and on the score of decreasing the number of integrals in (4.12) because some number of them may vanish along the trajectory Γ.

So, in the work [20] it was proved that if to introduce into the consideration the functions

$$H_1(s,\mathbf{n}) = X'_x(x(s,\mathbf{n}), y(s,\mathbf{n})) + Y'_y(x(s,\mathbf{n}), y(s,\mathbf{n})),$$
$$H_2(s,\mathbf{n}) = \frac{\partial}{\partial y}\left(\frac{XH_1}{b_0}\right) - \frac{\partial}{\partial x}\left(\frac{YH_1}{b_1}\right),$$
$$H_3(s,\mathbf{n}) = \frac{\partial}{\partial y}\left(\frac{XH_2}{b_0}\right) - \frac{\partial}{\partial x}\left(\frac{YH_2}{b_0}\right),$$
$$H_4(s,\mathbf{n}) = \frac{\partial}{\partial y}\left(\frac{XH_3}{b_0}\right) - \frac{\partial}{\partial x}\left(\frac{YH_3}{b_0}\right),$$

$\delta = \exp\delta_1$, $\delta_2 = \int\limits_0^t H_2(s(\tau),0)\delta d\tau$, where $\delta_1 = \int\limits_0^t H_1(s(\tau),0)d\tau$, then there holds

Theorem 4.1 ([20])**.** *If:* (i) $k=2$ and along the closed trajectory Γ of the system (1.1)

$$\bar h_1 = \int\limits_0^\omega H_1(s(t),0)dt = 0, \quad \bar h_2 = \int\limits_0^\omega H_2(s(t),0)\delta dt \neq 0,$$

then Γ is a double semistable limit cycle of the system (1.1); (ii) $k=3$ and along Γ $\bar h_1 = \bar h_2 = 0$, $\bar h_3 = \int\limits_0^\omega H_3(s(t),0)\delta^2 dt < 0$ (>0), then Γ is a triple stable (unstable) limit cycle of the system (1.1);

(iii) $k=4$ and along Γ $\bar h_1 = \bar h_2 = \bar h_3 = 0$, $\bar h_4 = \int\limits_0^\omega H_4(s(t),0)\delta^3 dt +$ $+2\int\limits_0^\omega H_3(s(t),0)\delta^2\delta_2 dt \neq 0$, then Γ is a fourfold semistable limit cycle of the system (1.1).

In conclusion note that if to introduce into consideration the function

$$H_5(s,\mathbf{n}) = \frac{\partial}{\partial y}\left(\frac{XH_4}{b_0}\right) - \frac{\partial}{\partial x}\left(\frac{YH_4}{b_0}\right),$$

then the following statement may be also proved.

Theorem 4.2 ([21])**.** *If* $k=5$ *and along the closed trajectory* Γ *of the system* (1.1) $\bar h_1 = \bar h_2 = \bar h_3 = \bar h_4 = 0$, $\bar h_5 = \int\limits_0^\omega H_5(s(t),0)\delta^4 dt +$ $+5\int\limits_0^\omega H_4(s(t),0)\delta^3\delta_2 dt + 5\int\limits_0^\omega H_3(s(t),0)\delta^2\delta_2^2 dt < 0$ (>0), *then* Γ *is a five-multiple stable* (*unstable*) *limit cycle of the system* (1.1).

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