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ON AN EXTREMUM PROBLEM

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The result presented was announced in [3]. We provide here a detailed proof and discussion. We believe that, in view of its generality, it could be helpful in many applications. In [3] it was used for estimation of the characteristic function of a quadratic form in normally distributed random variables, see also [1]. Similar ideas were used also in [2].

1. Notation and Results. For a natural number n let $x = (x_1, \dots, x_n)$, $x_j \geq 0$, $j = 1, \dots, n$, and denote

$$\Psi(x) = \prod_{j=1}^{n}(1 + x_j),$$

$$A_n = A_n(D, E) = \Big\{x : \sum_{j=1}^{n} x_j = D,\ \sum_{j=1}^{n} x_j^2 = E,\ x_1 \geq x_2 \geq \dots \geq x_n \geq 0\Big\},$$

where D and E are such that $D > 0$, and

$$D^2/n \leq E \leq D^2. \tag{1}$$

(By the Cauchy inequality A_n is non-empty if (1) holds and the point $(D/m, \dots, D/m, 0, \dots, 0)$ belongs to A_n for some m, $1 \leq m \leq n$, if $D^2/E = m$; it is the only point of A_n for $m = 1, n$.) Let

$$\Psi_* = \Psi_*(D, E) = \min\Big\{\Psi(x) : x \in A_n\Big\}, \quad \Psi^* = \Psi^*(D, E) = \max\Big\{\Psi(x) : x \in A_n\Big\}.$$

Similar to [1] it is easy to obtain a lower bound for Ψ_*. Indeed, according to the Method of Lagrange's multipliers (say, λ_1, λ_2 in our case) the coordinates $l_1, \dots, l_n$ of the point at which the minimum is attained, satisfy the equations

$$\Psi_* - \lambda_1(1 + l_j) - 2\lambda_2 l_j(1 + l_j) = 0, j = 1, \dots, n.$$

Thus l_j's can take at most two nonzero values and without loss of generality there exists a positive real α such that $l_j = \alpha$ for $j = 1, \dots, r$ with $r \geq m/2$ and $D_1 = r\alpha \geq D/2$ where m stands for the number of positive l_j's. Since $r = D_1^2/(r\alpha^2) \geq D^2/(4E) := r_0$, we have

$$\Psi_* \geq (1 + \alpha)^r \geq (1 + D/(2r))^r \geq (1 + D/(2r_0))^{r_0}.$$

Detailed analysis leads to precise extreme values for $\Psi(x)$ and lower and upper bounds for minimum and maximum, respectively, which are more accurate than the last lower bound for Ψ_*. Our approach consists in using classical methods for the case $n \leq 3$ and extending the result to the case $n > 3$.

Let $m(h) = h$ for an integer h and $m(h) = [h] + 1$ otherwise; denote

$$b(1) = 0, \quad b(h) = \Big\{(m(h)/h - 1)/(m(h) - 1)\Big\}^{1/2} \quad \text{as} \quad h > 1. \tag{2}$$

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Note that $0 = b(h) = b(h-0) \neq b(h+0) = 1/h$ for an integer h; at non-integer points $b(h)$ is continuous. Below we consider $h = D^2/E$ which varies in $[1, n]$. We have $m(1) = 1$ and $m(h) = j$ for $h \in (j-1, j]$ where $j = 2, \ldots, n$. We will also use the notation

$$b_j = b_j(D, E) = b_j(h) = [((j/h) - 1)/(j-1)]^{1/2}, \quad h \in [1, j], \quad j = 2, \ldots, n, \quad b_1 = 0;$$

b_j coincides with $b(h)$ when $h \in (j-1, j]$. Observe that $b_j(D, D^2/(j-1)) = 1/(j-1)$.

Proposition. 1°. *The function* $\Psi(x)$ *attains its minimum on* A_n *at a unique point* $l = (l_1, \ldots, l_n)$ *with coordinates*

$$l_1 = \cdots = l_{m-1} = D(1+b)/m, \quad l_m = D(1-(m-1)b)/m, \quad l_{m+1} = \cdots = l_n = 0,$$

where $m = m(h)$, $b = b(h)$ *and* $h = D^2/E$. *The function* $\Psi(l) = \Psi_* = \Psi_*(D; h)$ *is increasing in* h, *and*

$$\Psi_* = \Psi_*(D; h) = \left(1 + \frac{D(1+b)}{m}\right)^{m-1} \left(1 + \frac{D(1-(m-1)b)}{m}\right) \geq \left(1 + \frac{D}{[h]}\right)^{[h]}. \quad (3)$$

2°. *The function* $\Psi(x)$ *attains its maximum on* A_n *at a unique point* $u = (u_1, \ldots, u_n)$ *with coordinates*

$$u_1 = D(1+(n-1)b_n)/n, \quad u_2 = \cdots = u_n = D(1-b_n)/n,$$

$\Psi(u) = \Psi^* = \Psi^*(D, n; h)$ *is the increasing function in* h, *and*

$$\Psi^* = \Psi^*(D, n; h) = \left(1 + \frac{D(1+(n-1)b_n}{n}\right)\left(1 + \frac{D(1-b_n)}{n}\right)^{n-1} \leq \left(1 + \frac{D}{n}\right)^n. \quad (4)$$

Remark. We have

$$\Psi(x) = 1 + s_1(x) + s_2(x) + \cdots + s_n(x),$$

where

$$s_k(x) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}, k = 1, 2, ..., n,$$

are the elementary symmetric polynomials. Since $s_1(x) = D$ and $s_2(x) = (D^2 - E)/2$ on $A_n(D, E)$, the proposition provides extreme values for the sum of all elementary symmetric polynomials in n nonnegative real variables when $s_1(x)$ and $s_2(x)$ are fixed.

Below we will give the proof of the proposition (compare it with that of Lemma 1 from [2]). In what follows (1) is supposed to hold.

2. Proof of the Case $n \leq 3$. Since $m(1) = 1$ and $b(1) = b_1 = 0$, the case $n = 1$ is trivial. Clearly, A_2 consists of only one point $x = l = u$ with coordinates

$$x_1 = (D/2)(1+b_2), \quad x_2 = (D/2)(1-b_2)$$

and

$$\Psi(x_1, x_2) = \Psi_* = \Psi^* = 1 + D + (D^2 - E)/2.$$

So, (3) and (4) hold for $n = 2$.

We turn to the case $n = 3$. Note that A_3 is an arch of the circumference obtained by the intersection of the sphere $S_{M_0,R}$ centered at M_0 and having the radius $R, R^2 = E - D^2/3$, with the plane domain surrounded by the triangle $M_0M_1M_2$ (see Fig. 1), where

$$M_0 = (D/3, D/3, D/3), \quad M_1 = (D/2, D/2, 0), \quad M_2 = (D, 0, 0).$$

For the end point M of the arch A_3, which lies on M_0M_2, we have

$$M = \Big((D/3)(1+2b_3)\Big), (D/3)(1-b_3), (D/3)(1-b_3)\Big).$$

As for N, the other end point of A_3, note first that it lies on M_0M_1 if $E \le D^2$ and since $N = (x_1, x_1, D - 2x_1) \in S_{M_0,R}$, we obtain

$$N = \big((D/3)(1 + b_3), (D/3)(1 + b_3), (D/3)(1 - 2b_3)\big) \quad \text{if} \quad D^2/3 \le E \le D^2/2. \qquad (5)$$

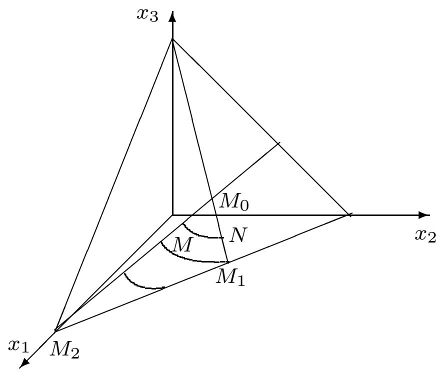

Fig. 1

If $D^2 \ge E \ge D^2/2$ the end point N of A_3 lies on M_1M_2 and since $N = (x_1, D - x_1, 0) \in S_{M_0,R}$ we have

$$N = \big((D/2)(1 + b_2), (D/2)(1 - b_2), 0\big) \quad \text{for} \quad D^2 \ge E \ge D^2/2. \qquad (6)$$

Note that two expressions (5) and (6) for N coincide at the boundary $E = D^2/2$.

Furthermore, for $x \in A_3$

$$\Psi(x) = 1 + D + (D^2 - E)/2 + x_1x_2x_3.$$

So the extrema are to be found for $x_1x_2x_3$. Clearly

$$x_1x_2x_3 = x_3[x_3^2 - Dx_3 + (D^2 - E)/2] := f(x_3).$$

Solving the equation $f(x_3) = 0$, we obtain the following roots $x_3^{(1)} = 0$, $x_3^{(2)} = (D/2)(1 - b_2)$, $x_3^{(3)} = (D/2)(1 + b_2)$, where the two last roots become complex when $E \le D^2/2$.

If $D^2/2 \le E \le D$, the third coordinate x_3 varies in the interval $[0, (D/3)(1 - b_3)]$. Calculating the derivative we obtain $f'(x_3) = 3x_3^2 - 2Dx_3 + (D^2 - E)/2$, which gives that possible extreme points are $x_3^* = (D/3)(1 - b_3)$ and $x_3^{**} = (D/3)(1 + b_3)$; the latter one lies outside the interval considered and $f''(x_3^*) > 0$. Thus the minimum is attained at $x_3 = 0$ and the maximum at $x_3 = (D/3)(1 - b_3)$.

If $E \le D^2/2$, x_3 varies in the interval $[(D/3)(1 - 2b_3), (D/3)(1 - b_3)]$ and since the derivative is positive on it, we have the minimum at $x_3 = (D/3)(1 - 2b_3)$ and the maximum at $x_3 = (D/3)(1 - b_3)$.

Summing up we conclude that for $n = 3$ the function Ψ attains its maximum at the unique point

$$u = \big((D/3)(1 + 2b_3), (D/3)(1 - b_3), (D/3)(1 - b_3)\big) \qquad (7)$$

and its minimum at the unique point

$$l = \big((D/3)(1 + b_3), (D/3)(1 + b_3), (D/3)(1 - 2b_3)\big) \quad \text{if} \quad D^2/3 \le E \le D^2/2 \qquad (8)$$

and at the unique point

$$l = \big((D/2)(1 + b_2), (D/2)(1 - b_2), 0\big) \quad \text{if} \quad D^2/2 \le E \le D^2. \qquad (9)$$

Formulas (7) and (8) give the same answers on the boundary $E = D^2/2$.

Now we again use the notation $m = m(D, E) = m(D^2/E)$ introduced above for the least integer which is greater than or equals to D^2/E and $b = b(D, E) = b(D^2/E)$ given by formula (2). Two formulas (8) and (9) presenting Ψ_* for $n \le 3$ are unified by the formula

$$\Psi_* = \Big[1 + (D/m)(1 + b)\Big]^{m-1}\Big[1 + (D/m)(1 - (m-1)b)\Big], \quad n \le 3. \tag{10}$$

The expression for Ψ^* which covers the case $n = 2$ too is simpler since it does not contain m:

$$\Psi^* = \Big[1 + (D/n)(1 + (n-1)b_n)\Big]\Big[1 + (D/n)(1 - b_n)\Big]^{n-1}, \quad n \le 3. \tag{11}$$

Let us now show that (10) and (11) are valid for the case $n > 3$ as well.

3. Proof for the Case $n > 3$. Minimum. We should show that the minimum is attained at $l \in A_n$ which has the form

$$l = (\underbrace{\alpha, \ldots, \alpha}_{m-1}, \beta, \underbrace{0, \ldots, 0}_{n-m}), \tag{12}$$

where $\alpha \ge \beta > 0$.

Let us equip Ψ and Ψ_* with an additional subscript n, i.e.,

$$\Psi_{*n} = \Psi_{*n}(D, E) = \Psi_n(l), \quad l = (l_1, \ldots, l_n), \quad l_1 \ge \cdots \ge l_n \ge 0. \tag{13}$$

For $n \le 3$ the relation (12) has been proved. Let us now consider the case $n > 3$ and take the last three positive coordinates l_{m-2}, l_{m-1}, l_m of l. Denote

$$l_{m-2} + l_{m-1} + l_m = D', \quad l_{m-2}^2 + l_{m-1}^2 + l_m^2 = E'$$

and find the minimum $\Psi_{*3}(D', E')$ of $\Psi_3(x_1, x_2, x_3)$. If this minimum has been less than $\Psi_3(l_{m-2}, l_{m-1}, l_m)$ this would have contradicted to our assumption that at l minimum is attained by Ψ_n. According to (8) $l_{m-2} = l_{m-1} \ge l_m$. Arguing similarly we can show that $l_{m-3} = l_{m-2}$, etc. Denote now

$$l_1 = \cdots = l_{m-1} = \alpha, \quad l_m = \beta, \quad m - 1 = k.$$

We have the following conditions

$$k\alpha + \beta = D, k\alpha^2 + \beta^2 = E, \alpha \ge \beta > 0. \tag{14}$$

Having in mind that

$$E/D^2 - 1/(k+1) \ge 0, \tag{15}$$

which is the case since $(k+1) \le n$ we obtain

$$\alpha = \frac{D}{k+1}\left(1 + \sqrt{\frac{(k+1)E/D^2 - 1}{k}}\right), \quad \beta = \frac{D}{k+1}\left(1 - k\sqrt{\frac{(k+1)E/D^2 - 1}{k}}\right),$$

and $\beta > 0$, if

$$E/D^2 < 1/k. \tag{16}$$

Now we are ready to define k. Inequalities (15) and (16) lead to $k < D^2/E \le k + 1$ which implies that $k + 1 = m = m(D, E) = m(D^2/E)$ and this solution is unique. We conclude that the conditions (14) determine l in the form (12) where there are $n - m$ zeros, first $m - 1$ positive coordinates are equal to α and the mth one to β, where m, α and β are expressed in terms of D and E in the way stated in the part 1° of Proposition.

Next we study $\Psi_*(D, E)$ as a function of $h = D^2/E$, which varies in $[1, n]$. According to the properties of $m(h)$ and $b(h)$ described above this function is continuous in each interval $(j, j+1], j = 1, \ldots, n-1$, and since $\Psi_*(D; h)$ equals to $(1 + D/j)^j$ for $h = j$ and it has the same limit as $h \to j+0$ for each $j = 1, \dots, n-1$, $\Psi_*(D;h)$ is continuous on the whole interval $[1,n]$. The derivative of $\Psi_*(D;h)$ w.r.t. h is positive for $h \in (j, j+1]$, whence $\Psi_*(D;h)$ increases in this interval and

$$\Psi_*(D;h) \geq \lim_{h\to j+0} \Psi_*(D;h) = \big(1+D/j\big)^j = \big(1+D/[h]\big)^{[h]}, \quad h \in (j, j+1], \quad j = 1, \dots, n-1.$$

As $\Psi_*(D;j) = (1+D/j)^j$ in the integer points, we obtain the following lower estimate

$$\Psi_*(D,E) = \Psi_*(D;h) \geq \left(1+\frac{D}{[h]}\right)^{[h]} = \left(1+\frac{D}{[D^2/E]}\right)^{[D^2/E]}.$$

Of course, one can take $\Psi_*(D;h_0)$ with any h_0, $1 \leq h_0 < h$, as a lower estimate for $\Psi_*(D;h)$.

4. Proof for the Case $n > 3$**. Maximum.** It is easy to show that for $E < D^2$ the point of maximum of Ψ looks like

$$u = (\alpha, \beta, \dots, \beta), \quad \alpha \geq \beta > 0. \tag{17}$$

Indeed, let $u = (u_1, \dots, u_n)$ and consider the first three coordinates of u. Denote $u_1 + u_2 + u_3 = D'$, $u_1^2 + u_2^2 + u_3^2 = E'$. As in (13) we equip Ψ^* with an additional subscript n. It is evident that the problem of finding maximum for $\Psi_3(x_1, x_2, x_3)$ in $A_3(E', D')$ has the unique solution of the form (u_1, u_2, u_2). According to (7) it means that $u_1 \geq u_2 = u_3$. Arguing similarly, we obtain that $u_3 = u_4$, and so on until we arrive at (u_{n-2}, u_{n-1}, u_n). Thus we see that our hypothesis (17) is true.

Introduce the notation

$$u_1 = \alpha, \quad u_2 = \cdots = u_n = \beta, \quad \alpha \geq \beta > 0.$$

From the conditions

$$\alpha + (n-1)\beta = D, \quad \alpha^2 + (n-1)\beta^2,$$

we obtain

$$\alpha = (D/n)\big(1 + (n-1)b_n\big), \quad \beta = (D/n)(1-b_n),$$

and hence the validity of (11) for any natural n.

If $E = D^2$, then $b_n = 1$, $\alpha = D$ and $\beta = 0$, which corresponds to the singular case $A_n(D, D^2) = \{(D, 0, \dots, 0)\}$, when u has the same form (17) where we set $\beta = 0$.

Let us now study (11) as the function of h. We need no calculations to claim that

$$\Psi^*(D;E) = \Psi^*(D;h) \leq \big(1 + D/n\big)^n \tag{18}$$

since $(1+D/n)^n$ is a maximum of $\Psi(x)$ with the only constraint $\sum x_i = D$, $x_i > 0$, $i = 1, \dots, n$. But as in the case of minimum, we can prove that $\Psi^*(D;h)$ increases (since its derivative w.r.t. h is positive). This will lead to (18) after substituting $h = n$ in the expression of $\Psi^*(D;h)$.

As an upper estimate of $\Psi^*(D;h)$ we can take $\Psi^*(D;h_0)$ with any h_0 such that $h < h_0 < n$.

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