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A SURVEY OF RESULTS IN THE PLAIN THEORY OF ELASTICITY OBTAINED BY GEORGIAN SCIENTISTS DURING THE LAST TEN YEARS

Abstract. The paper gives a survey of the results obtained for the last decade by the Georgian scientists on the problems of elasticity with the use of the theory of analytic functions of a complex variable. A description of basic stages of the development in this area is presented.

In the present work we mention more or less noteworthy results obtained by the Georgian mathematicians in the last decade of the past century.

The basis for application of the theory of analytic functions to the plane theory of elasticity is formed by the well-known Kolosov-Muskhelishvili formulas which express stress and displacement components through two functions analytic in the domain occupied by a body. Using these formulas, boundary value problems of the plane theory of elasticity are reduced to the boundary value problems of the theory of analytic functions. This enables one to employ various methods of the theory of analytic functions for solving boundary value problems of the plane theory of elasticity. Using Cauchy type integrals, the first and the second boundary value problems for domans mapped onto a disk by means of a rational function are explicitly solved in a simple way. However, in the case of multiply connected domans with a smooth enough boundary the first and the second boundary value problems are reduced to Fredholm integral equations, and existence of solutions is proved.

Investigation of the basic mixed and contact problems turned out to be more complicated, since they are reduced to boundary value problems with discontinuous coefficients. When the conformal mapping function onto the disk is rational, the problems are reduced to those of linear conjugation with discontinuous coefficients. These problems have been solved by N. Muskhelishvili and his pupils early in the 40s of the last century. The basic mixed problem in the general case has been reduced by D.I. Sherman to singular integral equations with discontinuous coefficients and studied subsequently by G.F. Manjavidze.

Thus the classical theory has in a certain sense got a completed form. Early in the 60s, for the investigation of mixed problems of elasticity the use was made of integral transformations. Applying these transformations, certain classes of problems of the theory of elasticity, such as plane, spatially statistical and dynamical, are reduced to the boundary value problems of the theory of analytic functions. The obtained problem is the problem of linear conjugation whose solution is given in quadratures.

Thus the use of the integral transformation has extended a class of elasticity problems, solvable by the method of the theory of analytic functions.

Starting from the 60s, one of the most important classes of contact problems of elasticity dealing with the interaction of thin-shelled elements (stringers) and inclusions with massive bodies received primary attention. The fundamental work in this direction belongs to E. Melan. Unlike classical problems of elasticity, boundary conditions in this case involve a combination of boundary values of an unknown function and of its derivative.

These problems in the general case are reduced either to an integral differential equation or to a system of integral differential equations. Different methods of approximate and effective solutions have been developed. A general enough method of effective construction of a solution is based on the application of integral transformations and of the apparatus of the the-

ory of analytic functions. Using the Fourier or Melin transform, a wide class of problems is reduced to the following boundary value problem of the theory of analytic functions: find in the strip $0 < \operatorname{Im} z < \beta$ an analytic function $\phi(z)$, satisfying the boundary condition $\phi(x+a) = -G(x)\phi(x) + F(x)$, where $G(x) \neq 0$, $G(\infty) = G(-\infty) = 1$, $\operatorname{Ind} G(x) = 0$, $F(x)$ and $G(x)$ are given functions from the class H,

$$\phi(z)e^{-\mu|z|} \to 0, \quad z \to \infty, \quad \mu < \frac{2\pi\beta}{\alpha^2 + \beta^2}, \quad a = \alpha + i\beta.$$

The solution of the problem has the form

$$\phi(z) = \frac{X(z)}{2a} \int_{-\infty}^{\infty} \frac{F(t)dt}{X(t)\operatorname{sh} p(t-z)},$$

where

$$X(z) = \exp\left[\frac{\operatorname{ch} pz}{2a} \int_{-\infty}^{\infty} \frac{\ln G(t)dt}{\operatorname{ch} pt \operatorname{sh} p(t-z)}\right], \quad p = \frac{\pi i}{a}.$$

This problem is called the Carleman type problem.

The same period is noticeable by investigation of the so-called inverse problem in the theory of elasticity and plate bending.

In such kind of problems shape of holes in the infinite plane and their location at infinity are unknown, when the hole boundary is free or under the action of constant normal pressure, and at infinity stretching or contracting forces are assumed to act. To find unknown holes the tangential normal stress is required to take constant values (the problems of plate bending are formulated analogously). Such holes are called equistrengthened. It is proved that Kolosov-Muskhelishvili's potential $\phi(z)$ is constant. If we map the given domain into a multiply connected region, then the problem is reduced for the given domain to two Dirichlet problems from which one can determine the second Kolosov-Muskhelishvili potential ψ and the conformal mapping function. If the domain is finite with exterior boundary prescribed, while interior contours are equistrengthened, the study of the problem becomes complicated and is reduced to a non-linear boundary value problem which fails to be studied. It has been shown in the works of R. D. Bantsuri and R. S. Isakhanov that if the exterior boundary of the doubly connected domain is a convex broken line, the interior boundary is equistrengthened, and the normal displacement on the exterior boundary is constant, while tangential stresses are equal to zero, then the Kolosov–Muskhelishvili potential ϕ is constant. This problem is reduced to the Riemann–Hilbert problem for an annulus. This makes it possible to define the conformal mapping function of the unknown region and the complex potential ψ.

1. In contact problems of the plane theory of elasticity the elastic thin-shelled elements are represented in terms of thin elastic supports or inclusions of constant rigidity. The contact problems of the plane theory of elasticity as well as of the theory of plate bending have been considered. In the plane theory of elasticity, the contact tangential stresses at the ends of

elastic elements have integrable singularities (of the order $1/2$), i.e., there takes place concentration of stresses. In the theory of plate bending, a jump of cutting forces in the neighborhood of the inclusion ends has a nonintegrable singularity (of the order $3/2$). Therefore there naturally arises the problem on reducing or eliminating of these singularities.

The works of N. Shavlakadze deal mainly with the contact problems of interaction between elastic, isotropic or anisotropic plates and other elements of variable rigidity. Under conditions of continuity of contact deformations with respect to unknown contact stresses or their jumps integral differential equations with continuous coefficients are obtained. When the rigidity of elastic supports changes qualitatively and correspondingly the coefficient of the singular operator in the integral differential equation turns at the ends of integration interval to zero of higher order, then the equation varies qualitatively. Specifically, it reduces equivalently to a singular integral equation of the third kind.

Works [17, 18] deal with the contact problems of interaction between semi-infinite stringers or inclusions and another plane or a half-plane. Using the methods of the theory of analytic functions and integral transformations, these problems are solved explicitly. Paper [22] investigates the contact problem of interaction of an elastic plate and a finite inclusion when the inclusion rigidity varies according to the law $E(x) = x^\alpha b(x)$, $(b(x) > 0$, $\alpha \geq 0$, $0 < x < 1)$. The plate is under stretching forces at infinity.

The basic equations are of the form

$$E(x)\frac{d^2 u_0(x)}{dx^2} = \tau_-(x) - \tau_+(x), \quad q_-(x) = q_+(x) \equiv q(x),$$

$$\int_0^1 (\tau_+(t) - \tau_-(t))dt = p_1 - p_2, \quad \varepsilon_x(x) = \varepsilon_x^{(0)}(x), \quad 0 < x < 1,$$

where $q_\pm(x)$ and $\tau_\pm(x)$ are, respectively, unknown normal and tangential contact stresses on the upper (with the index "+") and on the lower (with the index the "−") contours of the inclusion, $u_0(x)$ is horizontal displacement of its points, $\varepsilon_x(x)$ and $\varepsilon_x^{(0)}(x)$ are, horizontal deformations of the points of ox-axis and the points of the inclusion, respectively, $E(x) = \frac{E_0(x)h_0(x)}{1-v_0}$, $(E_0(x), h_0(x)$ and v_0 are, respectively, elasticity modulus, thickness and Poisson coefficient for the inclusion material), p_1 and p_2 are unknown axial forces at the ends $x = 0$ and $x = 1$, respectively.

As for the jump $\tau(x) = \tau_+(x) - \tau_-(x)$ and axial forces p_1 and p_2, we have the following Prandtl type integral differential equation:

$$\int_{-\frac{h_0(x)}{2}}^{\frac{h_0(x)}{2}} \sigma_x(0, y)dy = p_1, \quad \int_{-\frac{h_0(x)}{2}}^{\frac{h_0(x)}{2}} \sigma_x(1, y)dy = p_2.$$

Investigation of the above-obtained equations shows that the jump of tangential contact stresses in the vicinity of the inclusion end $x = 0$ is of

the form

$$\tau(x) = \begin{cases} O(x^{-\frac{1}{2}}), & 0 \le \alpha < 1, \\ O(x^{-1+\delta_0}), & \alpha = 1, \quad \delta_0 > \frac{1}{2}, \\ O(1), & 1 < \alpha \le 2, \\ O(x^{\alpha-2}), & \alpha > 2. \end{cases}$$

The contact problem of bending of a plate, fastened across a finite portion with inclusion of varying bending rigidity $D(x)$, has been studied in the same paper. The plate bending $\omega(x, y)$, satisfying the biharmonic equation in the given region, cut along the segment $(0, 1)$, satisfies on that segment the following boundary conditions:

$$\langle \omega \rangle = \langle \omega'_y \rangle = \langle M_y \rangle = 0, \quad \langle N_y \rangle = \mu(x),$$
$$\omega(x, 0) = \omega_0(x),$$
$$\frac{d^2}{dx^2} D(x) \frac{d^2 \omega_0(x)}{dx^2} = -\mu(x), \quad 0 < x < 1,$$
$$D(x)\omega''_0(x)|_{x=0} = M_1, \quad D(x)\omega''_0(x)|_{x=1} = M_2,$$
$$[D(x)\omega''_0(x)]'_{x=0;1} = 0, \quad \int_0^1 \mu(t)dt = 0, \quad \int_0^1 t\mu(t)dt = -M_1 - M_2.$$

Here the use is made of the notation $\langle f \rangle = f(x, -0) - f(x, +0)$, M_y and N_y are, respectively, the bending moment and force, $D(x) = \frac{E_0(x)h_0^3(x)}{12}$, M_1 and M_2 are unknown moments at the ends $x = 0$ and $x = 1$, respectively.

The behavior of the unknown transversal force in the neighborhood of the point $x = 0$ is given by the formula

$$\langle N_y \rangle = \begin{cases} O(x^{-\frac{3}{2}}), & 0 \le \alpha < 1, \\ O(x^{-2+\delta_0}), & \alpha = 1, \quad \delta_0 > \frac{1}{2}, \\ O(x^{-1}), & 1 < \alpha \le 2, \\ O(x^{\alpha-3}), & \alpha > 2. \end{cases}$$

The contact problem of interaction between a piecewise homogeneous plate and an elastic inclusion across the line of contact of two elastic materials is investigated in [20]. The inclusion undergoes tangential and normal loads. For unknown tangential and normal stresses the following system of integral differential equations is obtained

$$-A\psi''(x) + \frac{B}{\pi} \int_0^\infty \frac{\varphi'(t)dt}{t-x} = \frac{\varphi(x)}{E(x)} - \frac{f_1(x)}{E(x)},$$
$$A\varphi'(x) + \frac{B}{\pi} \int_0^\infty \frac{\psi'''(t)}{t-x} = \frac{\psi(x)}{D(x)} - \frac{f_2(x)}{D(x)}, \quad x > 0,$$
$$\varphi(0) = 0, \quad \varphi(\infty) = T_0, \quad \psi(0) = 0, \quad \psi(\infty) = M_0, \quad \psi'(0) = 0, \quad \psi'(\infty) = p_0.$$

The problem of interaction between a piecewise homogeneous plate and a semi-infinite inclusion under the normal load is solved in [21]. In the

same work, for a jump of transversal force an integral differential equation is obtained which is solved explicitly, as well as asymptotic estimates.

In [24] one can find investigation of the contact problem of plate bending, fastened across a finite portion with an inclusion of varying rigidity: $D(x) = (a^2 - x^2)^{n+1/2}p(x)$ (n is a positive integer, $p(x)$ is a polynomial, $|x| < a$). The Prandtl integral differential equation is obtained, which for $n = 0$ is investigated by many authors. Effective and approximate solutions are obtained therein. The methods applied earlier for any positive integer n turned out to be useless, therefore it became necessary to carry out more complex investigations based on the modification of already available methods of the theory of analytic functions. The behavior of an unknown transversal force in the neighborhood of the points $x = \pm a$ has the form $\langle N_y \rangle = O((a^2 - x^2)^{n-3/2})$, $x \to \pm a$.

The paper [25] is devoted to the investigation of contact problems of finite, isotropic and infinite anisotropic plate, as well as of the problems with circular holes. The obtained integral differential equation has in the general case the form

$$\varphi(x) - D(x) \int_{-a}^{a} \frac{\varphi'(t)dt}{t - x} + D(x) \int_{-a}^{a} K(x,t)\varphi(t)dt = D(x)f(x), \quad |x| < a,$$

where $K(x,t)$ is a function infinitely differentiable in the square, whose presence is specified by the plate finiteness, while for an infinite plate only characteristic part of that equation remains. In the general case this equation is reduced to a Fredholm integral equation of the second kind, which in its turn is reduced to the equivalent infinite system of linear algebraic equations by using the methods of orthogonal polynomals. The question on the regularity and solvability of such systems is studied.

The paper [23] considers the contact problem of interaction between an elastic plate and elastic, located periodically, inclusions of variable rigidity. The effective solutions are obtained.

The contact problem on discrete interaction between an infinite wedge-shaped plate and an elastic strengthening is investigated in [19]. The interaction is realized discretely by clamps located according to the exponential step law. For unknown concentrated forces in the paper an infinite system of algebraic equations of special type is obtained:

$$\sum_{j=-\infty}^{\infty} \Gamma_{k-j}b_j = a^k b_k + f_k,$$

where $\Gamma = \{\Gamma_k\}_{-\infty}^{\infty}$, $f = \{f_k\}_{-\infty}^{\infty}$ are the known and $b = \{b_k\}_{-\infty}^{\infty}$ are the unknown vectors from the space l_1. Using the Fourier discrete transformation, the above system is reduced to the Carleman boundary value problem for an annulus.

2. In studying boundary value problems, the formulas obtained by Lekhnitskiĭ for anisotropic bodies turned out to be less effective. When

the characteristic equation has simple roots, the components of the stress and displacement vectors are defined by means of two analytic functions $\phi_1(z_1)$ and $\phi_2(z_2)$, where $z_k = x_k + iy_k$; $x_k = x - \alpha_k$, $y_k = \beta_k y$ and (x, y) are the points of a physical domain. Therefore, to solve boundary value problems there arises the necessity in considering three domains.Then the investigation of these problems is reduced to the problems of displacement. However, investigation of such problems is connected with great difficulties. Effective solution of the problems can be obtained only in the half-plane, interior and exterior domain of the ellipse and in the infinite plane with cuts located along one straight line. Constants appearing in Lekhnitskiĭ's formulas, which are expressed by roots of the characteristic equation of fourth order and by Hook's coefficients, are of special difficulty for the investigation of boundary value problems. Therefore one fails to study them completely, in particular, an exact asymptotics of solutions in the vicinity of the points, at which the boundary conditions are changed, is not studied completely for the mixed problems.

M. Basheleishvili introduced four new constants which depend both on the roots of the characteristic equation and on the Hook's coefficient. Using these constants, it became possible to represent components of the stress and displacement vectors for any values of the root of the characteristic equation in a more simple way characteristic equation, than Lekhnitskiĭ's formulas. The above-mentioned constants enable one to get effective solutions of simple boundary value problems for specific domains. A new relation between the roots of the characteristic equation and Hook's constants is written out. This dependence is very important in defining pseudo-energies which are most significant for investigation of boundary value problems. The obtained results can be found in the monograph due to M. Basheleishvili [4].

In their work, M. Bashaleishvili and Sh. Zazashvili [8] have studied the basic mixed problem of anisotropic bodies for finite simply connected domains.

The solution of the problem is given in terms of the double layer potential of second kind with a real-valued density. The problem is reduced to a Sherman type singular integral equation with the regular part containing an unknown function and its complex conjugate.

Such type of equations for an isotropic body have been obtained by D. Sherman. A complete investigation of Sherman's equations is given in the works by G. Manjavidze and treated in N. Muskhelishvili's monograph "Singular Integral Equations".

On the basis of the results obtained by G. Manjavidze and M. Basheleishvili, the above-mentioned singular integral equation has been studied for anisotropic bodies.

In the work of M. Basheleishvili and Sh. Zazashvili [7] it is presented an effective solution of the basic mixed problem of anisotropic body for an exterior domain of an ellipse. The exact asymptotics of solutions is obtained in the neighborhood of the point at which the boundary conditions change.

Sh. Zazashvili [32, 33] investigates boundary contact problems for a piecewise homogeneous plane composed of different anisotropic half-planes, when across the contact line there is a finite number of cuts. By this is meant that the differences of limiting values of the displacement and stress vectors are given outside of the cuts, while at the cut ends are given either limiting values of the displacement vector (the first boundary value problem) or the limiting values of the stress vector (the second boundary value problem), or the limiting values of the displacement vector are given at one end of the cut and those of the stress vector at the other end (the mixed problem).

D. Natroshvili and Sh. Zazashvili [14] have studied the boundary contact problem formulated by M. Komninu for a piecewise homogeneous anisotropic plane composed of two different anisotropic half-planes with cuts along the contact line. It is assumed that in a sufficiently small neighborhood of cut ends the cut is not open, and between the cut ends there is a contact free from friction; the remaining part of the cut is open. It turns out that under such a statement of the problem the cut ends have no stress oscillations.

L. Gogolauri [9] investigated the contact problem of the plane theory of elasticity for an elastic orthotropic half-plane fastened by (an infinite number of) periodically situated stringers of equal resistance. Using the methods of the theory of functions of a complex variable, the problem is reduced to the M. Keldysh and L. Sedov problem for a circle and the solution of the problem is constructed.

The problem of a semi-infinite crack distribution with constant velocity along the interface of a piecewise homogeneous orthotropic plane is studied by R. Bantsuri [1]. The obtained dynamical equation, written in the moving coordinate system which is immovably connected with the crack end, is elliptic, if the velocity V satisfies the following conditions:

$$V^2 < G/\rho,$$

where $\frac{G}{\rho} = \min\left(\frac{G_1}{\rho_1}; \frac{G_2}{\rho_2}\right)$, G_1 and G_2 are displacement moduli of different half-planes, ρ_1 and ρ_2 are densities of the corresponding half-planes.

The effective solutions are constructed with the help of the methods of analytic functions and the behavior of stresses in the vicinity of the crack end is established.

In her monograph E. Obolashvili [16] considered problems of the plane theory of elasticity for domains with cuts. Using the Riemann-Schwarz symmetry principle proven in the theory of elasticity, the following problem is solved effectively: the half-plane $x > 0$ has cuts $[a_k, b_k]$, $k = 1, \ldots, n$, along the ox-axis or cuts along arcs of the circumference $|z| = 1$, $x > 0$. Normal displacements and tangential stresses on the axis $x = 0$ are equal to zero, and along the cuts one of the following conditions

1. $2\mu[u^+(x,0) + iv^+(x,0)] = f^+(x)$, $v^-(x,0) = g(x)$, $X_n^- = g_1(x)$, $x \in L$.

2. $2\mu[u^\pm(x,0) + iv^\pm(x,0)] = f^\pm(x)$, $x \in L_1$, $v^\pm(x,0) = g^\pm(x)$, $X_n^\pm = g_1(x)$, $x \in L_2$ is fulfilled, where $L_1 \cup L_2 = L$, L_1, L_2 are segments of the

ox-axis.

The conditions prescribed on the *oy*-axis allow one to apply the Riemann-Schwarz symmetry principle, and the problem is reduced to the problem on the infinite plane which is cut along segments of the *ox*-axis or along the arcs of the circumference $|z| = 1$. The effective solutions are constructed.

For the wave equation in the theory of elasticity, a solution is given on the half-plane, cut along a half-line. The solution is constructed in quadratures.

Using the method of analytic continuation, G. Kutateladze [13] solved the problem of torsion for a prismatic bar having the semi-circular cross-section with cuts.

3. R. Bantsuri [2] investigated the problem on plate bending for a doubly connected region whose exterior boundary is the union of a given broken line and unknown arcs. The interior boundary is an unknown smooth contour.

The angle of rotation on the boundary segments is required to be piecewise constant, the cross-cutting force to be equal to zero, and the acting bending moments on the unknown parts of the boundary to be constant. It is required to define unknown parts of the boundary, if tangential normal moments take constant values. In this case it is proved that the complex potential ϕ is constant. If the region occupied by the body is mapped conformally onto a circular ring, and segments of the broken line are mutually perpendicular, then the solution of the problem is reduced to the Carleman type problem for a circular ring:

$$\varphi(R^2\sigma) = G(\sigma)\varphi(\sigma) + g(\sigma), \quad \sigma \in \gamma, \quad \gamma = \{\sigma : |\sigma| = 1\},$$

where

$$G(\sigma) = -1, \quad g(\sigma) = 1/2g_0(\sigma), \quad \sigma \in \gamma_1, \quad \gamma_1 \subset \gamma,$$
$$G(\sigma) = 1, \quad g(\sigma) = 0, \quad \sigma \in \gamma/\gamma_1, \quad \gamma\backslash\gamma_1.$$

Here γ_1 is the part of γ which corresponds to parallel segments of the *ox*-axis. The solution of the problem is given in quadratures. This allows one to determine the unknown parts of the boundary and the complex potentials.

G. Kapanadze [10, 11] investigated the problem of bending for finite and infinite doubly connected plates, when the plate boundary is hinged and represents convex broken lines. By means of a conformal mapping the problem on a circular ring is reduced to the Riemann-Hilbert problem for a circular ring. The effective solutions are constructed.

4. Nonlocal boundary value problems for the plane theory of elasticity and polyharmonic functions have been investigated by E. Obolashvili [15]. Let D be a half-plane, $y > 0$, $x \in R$, D_1 be the quarter-plane $x > 0$, $y > 0$ and $u(x,y)$, $v(x,y)$ are the components of elastic displacement. An infinitely decreasing displacement on the half-plane $y > 0$ is defined from the following problems:

a) $u(x,0) = g_1(x) \quad x < 0; \quad u(x,0) = \lambda u(x,h) + g(x), \quad x > 0,$
   $v(x,0) = g_2(x), \quad x \in R, \quad 0 < \lambda \leq 1;$

$b)$ $u(x,0) = \lambda_1 u(x,h_1) + g_1(x)$, $x>0$; $u(x,0) = \lambda_2 u(x,h_2) + g_2(x)$, $x<0$,
$v(x,0) = g_3(x)$, $x \in R$, $0 < \lambda_1 \neq \lambda_2 \leq 1$, $h_1 \neq h_2$;

$c)$ $u(x,0) = u(x,h_1) + mu(-x,h_1) + g_1(x)$, $x > 0$,
$u(x,0) = u(x,h_2) + mu(-x,h_2) + g_2(x)$, $x < 0$,
$v(x,0) = g(x)$, $x \in R$, $h_1 \neq h_2$.

These problems are reduced to the Wiener-Hopf integral equation, to dual integral equation and also to such dually-integral equations, whose kernels require the sum $k_j(x-t) + k_j(x+t)$, $j = 1,2$.

5. For a nonhomogeneous, isotropic body of special type, O. Shinjikashvili obtained a complex representation of stress and displacement vector components by means of two analytic functions, when

$a)$ $\mu = \mathrm{const}$, $\frac{1}{\varkappa+1} = A_0(1 + \beta e^{\alpha y})$,

$b)$ $\mu = \mathrm{const}$, $\frac{1}{\varkappa+1} = \sum_{k=0}^{n}(A_k \cos k\alpha y + B_k \sin k\alpha y)$,

where A_k, B_k, α, β are well-defined real constants, μ is the displacement module, and $\frac{1}{\varkappa+1} = \frac{1+\sigma}{4}$ for a generalized plane stressed state, $\frac{1}{1+\varkappa} = \frac{1}{4(1-\sigma)}$ for plane deformation and σ is the Poisson coefficient.

On the basis of the above representations, the following problems have been investigated: the first and the second boundary value problems for the half-plane [26]; the problem of rigid punch pressure on the half-plane boundary [27, 28]; the problem of interaction of the half-plane and a stringer [29, 30]; the problem of linear cut for the infinite plane [31].

All the above-mentioned problems are reduced to the problem of linear conjugation and the solution is given explicitly.

G. Kutateladze [12] investigated for an annulus composed of three heterogeneous concentric rings inserted successively one into another, and for unbounded elastic isotropic plane with a circular hole and a supporting ring soldered in it , consisting of two concentric elastic rings with different elastic characteristics. Using the method of the theory of analytic functions and applying analytic continuation of the unknown function, the solution of the problem is obtained in the form of series.

6. M. Basheleishvili [5] generalized the known common representations of Kolosov-Muskhelishvili in the elastic mixture theory of statics of an isotropic elastic body. While in Kolosov-Muskhelishvili's representations there appear two analytic functions, in the theory of mixture statics the vectors of the partial displacement u and of the stress T_u are expressed in terms of four functions.

Basic equations of isotropic, elastic mixture statics have the form

$$a_1 \Delta u' + b_1 \operatorname{grad} \operatorname{div} u' + c\Delta u'' + d \operatorname{grad} \operatorname{div} u'' = 0,$$
$$c\Delta u' + d \operatorname{grad} \operatorname{div} u' + a_2 \Delta u'' + b_2 \operatorname{grad} u'' = 0,$$

where $u' = (u'_1, u'_2)$ and $u'' = (u''_1, u''_2)$ are the vectors of partial displacement, a_1, b_1, c, d, a_2, b_2 are constants characterizing physical properties of

the mixture.

Let $(u', u'') \equiv (u_1, u_2, u_3, u_4)$. Introducing the vector $u = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix}$, we can write

$$u = m\varphi(z) + \frac{l}{2} z \overline{\varphi}'(z) + \overline{\psi}(z),$$

where

$$m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \ l = \begin{bmatrix} l_4 & l_5 \\ l_5 & l_6 \end{bmatrix}, \ z = x_1 + ix_2, \ \varphi(z) = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \ \psi(z) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

and

$$T_u = \frac{\partial}{\partial s(x)} [(A - 2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}];$$

E is the unit matrix, $A = 2\mu m$, $B = 2\mu m$, $\mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}$, $\frac{\partial}{\partial s(x)} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1}$, $n = (n_1, n_2)$ is an arbitrary unit vector.

Using these representations, in the works [3, 5], the basic plane boundary value problems are reduced to the first and the second problems of statics of an isotropic elastic body. Thus it is shown that the results obtained for equations of statics of isotropic body can be obtained in the theory of elasticity of mixtures as well.

In [6], the basic boundary value problems for equations of the theory of elastic mixtures are effectively solved both in the interior and in the exterior domain of an ellipse (by series inside, and by integrals outside of the ellipse).

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