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**FACTORIZATION OF MATRIX-FUNCTIONS AND
REGULARIZATION OF SINGULAR INTEGRAL
OPERATORS ON MANIFOLDS WITH BOUNDARY**

Abstract. The problem of expansion of positively homogeneous matrix-functions is studied. Certain algebraic, differential and asymptotic properties of cofactors are established. The results obtained are used to investigate the problem of regularization of singular integral operators on manifolds with boundary in Hölder spaces with weight.

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რეზიუმე. ნაშრომში შესწავლილია დადებითად ერთგვაროვანი მატრიც-ფუნქციების ფაქტორიზაციის საკითხი. დადგენილია ფაქტორიზაციის თანამართლების გარკვეული ალგებრული, დიფერენციალური და ასიმპტოტური თვისებები. ამის საფუძველზე გამოკვლეულია საზღვრიან მრავალსახეობებზე გავრცელებული სინგულარული ინტეგრალური ოპერატორების რეგულარიზაციის საკითხი ჰელდერის წონიან სივრცეებში.

INTRODUCTION

Our consideration involves singular integral operators on manifolds with boundary in weighted Hölder spaces. We further develop the results that were obtained for one-dimensional singular operators in [8].

Let R^m be an m -dimensional Euclidean space, $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$ be points of R^m ,

$$|x| = \left(\sum_{i=1}^m x_i^2 \right)^{\frac{1}{2}}, \quad x' = (x_1, \dots, x_{m-1}).$$

The paper consists of three sections. In §1 we study the question whether the one-dimensional matrix singular operators

$$\mathcal{K}(u)(x) = A(x)u(x) + \frac{1}{\pi i} B(x) \int_R \frac{u(y)}{y-x} dy, \quad x \in R,$$

are Noetherian in spaces of functions having certain differential and asymptotic properties.

$$A(x) = \|a_{ij}(x)\|_{n \times n}, \quad B(x) = \|b_{ij}(x)\|_{n \times n}, \quad u = (u_1, \dots, u_n),$$

where any n -dimensional vector $u = (u_1, \dots, u_n)$ is considered as a one-column matrix $u = \|u_i\|_{n \times 1}$.

The results of §1 are used in §2 to study the representation of a positively homogeneous matrix-function $A(\xi)$ ($\xi \in R^m \setminus \{0\}$, $m \geq 2$) in the form

$$A(\xi) = A_-(\xi', \xi_m) \mathcal{D}(\xi) A_+(\xi', \xi_m),$$

where \mathcal{D} is a certain canonical matrix and the matrix A_+ (A_-) admits an analytic continuation with respect to ξ_m to the upper (lower) complex half-plane. Note that the matrices A_+ , A_- possess certain algebraic, differential and asymptotic properties.

In §3 we use the results of §2 to investigate the question whether the matrix singular integral operator

$$A(u)(x) = C(x)u(x) + \int_{\mathcal{D}} k(x, x-y)u(y) dy, \quad x \in \mathcal{D}, \quad \mathcal{D} \subset R^m,$$

are Noetherian in Hölder spaces with weight.

A partial account of the above results is given in [5].

In the paper we use the following notation:

$$R_+^m = \{x : x \in R^m, x_m > 0\}, \quad R_-^m = \{x : x \in R^m, x_m < 0\}, \\ B(x, a) = \{y : y \in R^m, |y-x| < a\}, \quad S(x, a) = \{y : y \in R^m, |y-x| = a\}.$$

§ 1. SINGULAR INTEGRAL OPERATORS ON R

A function u defined on R belongs to the space $H_\beta^{k,\nu}(R)$ ($0 < \nu < 1$, $\beta \geq 0$, k is a nonnegative integer) iff:

- (i) $\forall x \in R \quad |\partial^p u(x)| \leq c(1 + |x|)^{-p-\beta}$, $p = 0, \dots, k$; $\partial^p \equiv \left(\frac{d}{dx}\right)^p$;
(ii) $\forall x, y \in R \quad |\partial^k u(x) - \partial^k u(y)| \leq c|x - y|^\nu \rho_{xy}^{-k-\beta-\nu}$,

where $\rho_{xy} = \min(1 + |x|, 1 + |y|)$.

The norm in this space is defined by the equality

$$\|u\| = \sum_{p=0}^k \sup_{x \in R} (1 + |x|)^{p+\beta} |\partial^p u(x)| + \sup_{x, y \in R} \rho_{xy}^{k+\beta+\nu} \frac{|\partial^k u(x) - \partial^k u(y)|}{|x - y|^\nu}.$$

It is easy to show that $H_\beta^{k,\nu}(R)$ is a Banach space.

Let a function b defined on R be representable in the form $b(x) = b(\infty) + \overset{\circ}{b}(x)$ where $\overset{\circ}{b} \in H_\lambda^{0,\nu_1}(R)$ ($\lambda > 0$). Then

(i) the operator

$$v(x) \equiv A(u)(x) = \int_R \frac{u(y)}{x - y} dy$$

is bounded in the space $H_\beta^{0,\nu}(r)$ ($0 < \beta < 1$);

(ii) the operator

$$v(x) \equiv B(u)(x) = \int_R \frac{b(x) - b(y)}{x - y} u(y) dy$$

is completely continuous in the space $H_\beta^{0,\nu}(r)$ ($0 < \beta < 1$, $\nu < \nu_1$).

Proof. (i) Setting

$$\mathcal{D}_1 = B(x, \frac{1}{2}(1 + |x|)), \quad \mathcal{D}_2 = B(0, 2|x| + 1) \setminus \mathcal{D}_1, \quad \mathcal{D}_3 = R \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$$

we have

$$v(x) = \int_{\mathcal{D}_1} (x - y)^{-1} [u(y) - u(x)] dy + \sum_{i=2}^3 \int_{\mathcal{D}_i} (x - y)^{-1} u(y) dy.$$

Hence, taking into account that

$$\begin{aligned} \rho_{xy} &\geq c(1 + |x|) \quad (y \in \mathcal{D}_1), \quad |y - x| \geq c(1 + |x|) \quad (y \in \mathcal{D}_2), \\ &|y - x| \geq c(1 + |y|) \quad y \in \mathcal{D}_3, \end{aligned}$$

we obtain

$$|v(x)| \leq c(1 + |x|)^{-\beta} \|u\|. \quad (1.1)$$

Let us estimate the difference $v(x) - v(z)$. Assume that $|x - z| < \frac{1}{8}(1 + |x|)$ (otherwise the required estimate follows from (1.1)). Then $1 + |x| \sim 1 + |z|$ (i.e., $1 + |x| \leq c_1(1 + |z|) \leq c_2(1 + |x|)$). Introduce the sets:

$$\tilde{\mathcal{D}}_1 = B(x, 2|x - z|), \quad \tilde{\mathcal{D}}_2 = B(z, 3|x - z|), \quad \tilde{\mathcal{D}}_3 = B(z, \frac{1}{2}(1 + |x|) \setminus |x - z|).$$

Clearly $\tilde{\mathcal{D}}_1 \subset \tilde{\mathcal{D}}_2 \subset \tilde{\mathcal{D}}_3 \subset \mathcal{D}_1$.

The following representation holds:

$$\begin{aligned} v(x) - v(z) &= \left(\int_{\tilde{\mathcal{D}}_2} + \int_{\mathcal{D}_1 \setminus \tilde{\mathcal{D}}_3} \right) (x - y)^{-1} [u(y) - u(x)] dy - \\ &- \int_{\tilde{\mathcal{D}}_2} (z - y)^{-1} [u(y) - u(z)] dy + \int_{\tilde{\mathcal{D}}_3 \setminus \tilde{\mathcal{D}}_2} [(x - y)^{-1} - (z - y)^{-1}] [u(y) - u(x)] dy - \\ &- \int_{\mathcal{D}_1 \setminus \tilde{\mathcal{D}}_3} (z - y)^{-1} u(y) dy + \int_{\mathbb{R}^1 \setminus \mathcal{D}_1} [(x - y)^{-1} - (z - y)^{-1}] u(y) dy \equiv \sum_{i=1}^5 J_i(x, z). \end{aligned}$$

To estimate $J_1(x, z)$ note that

$$\begin{aligned} \tilde{\mathcal{D}}_2 &\subset B(x, 4|x - z|), \quad B(x, \frac{1}{2}(1 + |x|) - 2|x - z|) \subset \tilde{\mathcal{D}}_3, \\ \left(\frac{1}{2}(1 + |x|) \right)^\nu &- \left(\frac{1}{2}(1 + |x|) - 2|x - z| \right)^\nu \leq c|x - z|^\nu. \end{aligned}$$

Therefore

$$|J_1(x, z)| \leq c(1 + |x|)^{-\beta - \nu} |x - z|^\nu \|u\|.$$

$J_2(x, z)$ can be estimated quite similarly.

Further, $B(x, \frac{1}{2}(1 + |x|)) \subset B(z, \frac{1}{2}(1 + |x|) + |x - z|)$, so that

$$\begin{aligned} |J_4(x, z)| &\leq c(1 + |x|)^{-\beta} \ln \frac{1 + |x| + 2|x - z|}{1 + |x|} \|u\| \leq \\ &\leq c(1 + |x|)^{-\beta - \nu} |x - z|^\nu \|u\|. \end{aligned}$$

Note that if $y \in \tilde{\mathcal{D}}_1$, then $|x - y| \sim |z - y|$ and since

$$\left| \frac{1}{x - y} - \frac{1}{z - y} \right| = \frac{|x - z|}{|x - y| |z - y|}, \quad (1.2)$$

we shall have

$$\begin{aligned} |J_3(x, z)| &\leq c|x - z|(1 + |x|)^{-\beta - \nu} \int_{\tilde{\mathcal{D}}_3 \setminus \tilde{\mathcal{D}}_2} |y - z|^{\nu - 2} dy \|u\| \leq \\ &\leq c(1 + |x|)^{-\beta - \nu} |x - z|^\nu. \end{aligned}$$

Considering the sets \mathcal{D}_2 and \mathcal{D}_3 , we obtain by virtue of (1.2)

$$|J_5(x, z)| \leq c|x - z|^\nu (1 + |x|)^{-\beta - \nu}.$$

Thus we have proved (i).

(ii). From the proof of (i) it follows that B is a bounded operator from $H_\beta^{0,\nu}(R)$ into $H_{\beta+\gamma}^{0,\nu+\gamma}(R)$ where γ is an arbitrary positive number satisfying

$$\gamma < \min(\lambda, \nu_1 - \nu, 1 - \beta).$$

Indeed, it is obvious that

$$\begin{aligned} |v(x)| &\leq \int_{\mathcal{D}_1} |\mathring{b}(x) - \mathring{b}(y)| |x-y|^{-1} |u(y)| dy + \\ &+ \int_{\mathcal{D}_2 \cup \mathcal{D}_3} (|\mathring{b}(x)| + |\mathring{b}(y)|) |x-y|^{-1} |u(y)| dy. \end{aligned}$$

Hence in view of $\mathring{b} \in H_\gamma^{0,\gamma}(R)$ we obtain

$$|v(x)| \leq c(1 + |x|)^{-\beta-\gamma} \|u\|.$$

Now assume that $|x-z| \leq \frac{1}{8}(1+|x|)$ and estimate the difference $v(x) - v(z)$. We have

$$\begin{aligned} |v(x) - v(z)| &\leq \int_{\tilde{\mathcal{D}}_2} |\mathring{b}(x) - \mathring{b}(y)| |x-y|^{-1} |u(y)| dy + \\ &+ \int_{\mathcal{D}_2} |\mathring{b}(z) - \mathring{b}(y)| |z-y|^{-1} |u(y)| dy + \int_{R \setminus \tilde{\mathcal{D}}_2} (|\mathring{b}(x) - \mathring{b}(z)| |x-y|^{-1} |u(y)| dy + \\ &+ \int_{R \setminus \tilde{\mathcal{D}}_2} |\mathring{b}(z) - \mathring{b}(y)| |(x-y)^{-1} - (z-y)^{-1}| |u(y)| dy \equiv \sum_{i=1}^4 J_i(x, z) \end{aligned}$$

which yields

$$\begin{aligned} |J_i(x, z)| &\leq c(1 + |x|)^{-\beta-\nu_1-\lambda} |x-z|^{\nu_1} \|u\| \leq \\ &\leq c(1 + |x|)^{-\beta-\nu-2\gamma} |x-z|^{\nu+\gamma} \|u\| \quad (i = 1, 2). \end{aligned}$$

Represent term $J_3(x, z)$ as

$$\begin{aligned} J_3(x, z) &= |\mathring{b}(x) - \mathring{b}(z)| \left(\int_{\mathcal{D}_1 \setminus \mathcal{D}_2} |x-y|^{-1} |u(y)| dy + \int_{R \setminus \mathcal{D}_1} |x-y|^{-1} |u(y)| dy \right) \leq \\ &\leq c|x-z|^{\nu_1} (1 + |x|)^{-\nu_1-\lambda-\beta} \left(\ln \frac{1+|x|}{|x-z|} + c \right) \|u\| \leq \\ &\leq c(1 + |x|)^{-\beta-\nu-2\gamma} |x-z|^{\nu-\gamma} \|u\|. \end{aligned}$$

To estimate $J_4(x, z)$ we shall consider this integral on each of the sets $\mathcal{D}_1 \setminus \tilde{\mathcal{D}}_2$, \mathcal{D}_2 and \mathcal{D}_3 . By virtue of (1.2) we obtain

$$|J_4(x, z)| \leq c(1 + |x|)^{-\beta-\nu-2\gamma} |x-z| \|u\|.$$

Thus the operator B is bounded from $H_{\beta}^{0,\nu}(R)$ into $H_{\beta+\gamma}^{0,\nu+\gamma}(R)$. Now the validity of (ii) follows from the completely continuity of the embedding operator from $H_{\beta+\gamma}^{0,\nu+\gamma}(R)$ into $H_{\beta}^{0,\nu}(R)$. ■

The boundedness of singular operators in spaces $H_{\beta}^{0,\nu}$ is studied in [9]. Consider a one-dimensional matrix singular integral operator

$$\mathcal{K}(u)(x) = A(x)u(x) + \frac{1}{\pi i} B(x) \int_R \frac{u(y)}{y-x} dy,$$

$$A(x) = \|a_{ij}(x)\|_{n \times n}, \quad B(x) = \|b_{ij}(x)\|_{n \times n}, \quad u = (u, \dots, u_n).$$

Let $\Phi(\mathcal{K})$ denote the matrix symbol of the operator \mathcal{K} :

$$\Phi(\mathcal{K})(x, \xi) = A(x) + \frac{\xi}{|\xi|} B(x), \quad \xi \in R \setminus \{0\}.$$

Thus

$$A(x) = \frac{1}{2} [\Phi(\mathcal{K})(x, +1) + \Phi(\mathcal{K})(x, -1)],$$

$$B(x) = \frac{1}{2} [\Phi(\mathcal{K})(x, +1) - \Phi(\mathcal{K})(x, -1)].$$

Let $a_{ij} = a_{ij}(\infty) + \overset{\circ}{a}_{ij}$, $b_{ij} = b_{ij}(\infty) + \overset{\circ}{b}_{ij}$; $\overset{\circ}{a}_{ij}, \overset{\circ}{b}_{ij} \in H_{\lambda}^{k,\nu_1}(R)$ ($\lambda > 0$), $\det \Phi(\mathcal{K})(x, \xi) \neq 0$, $x \in \dot{R}$ ($\dot{R} = R \cup \{\infty\}$). Then

- (i) The operator \mathcal{K} is Noetherian both in the space $\prod_{i=1}^n H_{\beta_i}^{k,\nu}(R)$ ($0 < \beta_i < 1$, $\nu < \nu_1$) and in the space $\prod_{i=1}^n L_{p_i}(R)$ ($p_i > 1$) ($L_p(R)$ is the linear normed space of functions which are summable with degree p on R);
- (ii) any solution of the equation

$$\mathcal{K}(u)(x) = g(x), \quad g \in \prod_{i=1}^n H_{\beta_i}^{k,\nu}(R) \cap \prod_{i=1}^n L_{p_i}(R), \quad (1.3)$$

from the space $\prod_{i=1}^n L_{p_i}(R)$ belongs to the space $\prod_{i=1}^n H_{\beta_i}^{k,\nu}(R) \cap \prod_{i=1}^n L_{p_i}(R)$. In order that equation (1.3) to be solvable it is necessary and sufficient that

$$(g, v) \equiv \int_R \sum_{i=1}^n g_i \bar{v}_i dx = 0,$$

where v is an arbitrary solution of the equation

$$\mathcal{K}^*(v) = A^*(x)v(x) + \frac{1}{\pi i} \int_R \frac{B^*(y)v(y)}{y-x} dy$$

from the space $\prod_{i=1}^n H_{\beta_i}^{k,\nu}(R) \cap \prod_{i=1}^n L_{q_i}(R)$ ($\frac{1}{p_i} + \frac{1}{q_i} = 1$) and A^* (B^*) we denotes the conjugate matrix of A (B).

Proof. Let

$$v(x) = \int_R \frac{u(y) dy}{y-x}, \quad u \in H_{\beta}^{k,\nu}(R) \quad (\beta > 0).$$

Then

$$\partial^p v(x) = \int_R \frac{\partial^p u(y)}{y-x} dy, \quad p \leq k,$$

and since

$$\frac{1}{y-x} = \frac{1}{(x+i)\left(\frac{y+i}{x+i} - 1\right)} = - \sum_{r=0}^{p-1} \frac{(y+i)^r}{(x+i)^{r+1}} + \frac{(y+i)^p}{(x+i)^p(y-x)}, \quad (1.4)$$

we obtain

$$\begin{aligned} \partial^p v(x) &= - \sum_{r=0}^{p-1} \frac{1}{(x+i)^{r+1}} \int_R (y+i)^r \partial^p u(y) dy + \\ &+ \frac{1}{(x+i)^p} \int_R \frac{(y+i)^p \partial^p u(y)}{y-x} dy. \end{aligned}$$

Performing integration by parts we find that

$$\int_R (y+i)^r \partial^p u(y) dy = 0, \quad r \leq p-1.$$

Therefore

$$\partial^p v(x) = \frac{1}{(x+i)^p} \int_R \frac{(y+i)^p \partial^p u(y)}{y-x} dy. \quad (1.5)$$

Now Theorem 1.1 directly implies that in the conditions of the theorem the singular integral operator M defined by the equality $\Phi(M) = \Phi^{-1}(\mathcal{K})$ is a two-sided regularizer of the operator K in the space $\prod_{i=1}^n H_{\beta_i}^{k,\nu}(R)$. According to M. Riesz's and S. G. Mikhlin's theorems on boundedness of a singular integral and complete continuity of a commutator in spaces $L_p(R)$ (see [7], [12]), the operator M is a regularizer in the space $\prod_{i=1}^n L_{p_i}(R)$ as well.

The validity of (ii) follows from the fact that the operator \mathcal{K} has the same regularizer in a pair of densely embedded spaces (see [4]). ■

Remark 1.1. It is clear that the solvability conditions of equation (1.3) can be formulated using the adjoint operator

$$\mathcal{K}'(v) = A'(x)v(x) - \frac{1}{\pi i} \int_R \frac{B'(y)v(y)}{y-x} dy,$$

where A' (B') denotes the matrix obtained by transposing the matrix A (B).

If $AB = BA$, then the regularizer M of the operator \mathcal{K} can be defined by the equality

$$\Phi(M) = A(x) - \frac{\xi}{|\xi|} B(x).$$

Let σ, τ be natural numbers ($\sigma, \tau \geq 1$), $\delta = (\delta_1, \dots, \delta_n)$,

$$-\frac{1}{2} < \operatorname{Re} \delta_j \leq \frac{1}{2}, \quad j = 1, \dots, n, \quad \operatorname{Re} \delta_1 \geq \delta_2 \geq \dots \geq \operatorname{Re} \delta_n \quad (1.6)$$

and let ψ be an infinitely differentiable function on R satisfying $\psi(t) = 0$ for $|t| \leq \frac{1}{2}$, $\psi(t) = 1$ for $|t| \geq 1$.

Define the vector-function

$$\overset{s}{\psi}(\sigma, \tau, \delta; \cdot) = (\overset{s}{\psi}_1(\sigma, \tau, \delta; \cdot), \dots, \overset{s}{\psi}_n(\sigma, \tau, \delta; \cdot)) \quad (s = 1, \dots, n)$$

by the equality

$$\overset{s}{\psi}_i(\sigma, \tau, \delta; t) = \begin{cases} \psi(t) \sum_{p=1}^{\sigma} \sum_{q=0}^{p\tau} c_{+,i,s}^{p,q} (1+t)^{-p-\delta_i+\delta_s} \ln^q(1+t), & t \geq 0; \\ \psi(t) \sum_{p=1}^{\sigma} \sum_{q=0}^{p\tau} c_{-,i,s}^{p,q} (1+t)^{-p-\delta_i+\delta_s} \ln^q(1-t), & t < 0, \end{cases}$$

where $c_{\pm,i,s}^{p,q}$ are certain constants.

Assume further that the numbers β_i ($i = 1, \dots, n$) are chosen so that

$$\operatorname{Re} \delta_i - \operatorname{Re} \delta_n < \beta_i < 1 + \operatorname{Re} \delta_i - \operatorname{Re} \delta_1. \quad (1.7)$$

A function u defined on R belongs to the space $\overset{*}{H}_{\beta_i, \sigma, \tau, s}^{k, \nu}(R)$ iff $u \in H_{\beta_i}^{k, \nu}(R)$ and there exist constants $c_{i,s}^{p,q}$ ($p = 1, \dots, \sigma; q = 0, 1, \dots, p\tau$) depending on u such that

$$(u - \overset{s}{\psi}_i(\sigma, \tau, \delta; \cdot)) \in H_{\sigma+\beta_i}^{k, \nu}(R). \quad (1.8)$$

(1.6)–(1.8) imply that constants $c_{i,s}^{p,q}$ are uniquely defined by u .

In the linear space $\overset{*}{H}_{\beta_i, \sigma, \tau, s}^{k, \nu}(R)$ we shall define the norm by the equality

$$\|u\| = \|u\|_{H_{\beta_i}^{k, \nu}(R)} + \|u - \overset{s}{\psi}_i(\sigma, \tau, \delta; \cdot)\|_{H_{\sigma+\beta_i}^{k, \nu}(R)} + \sup_{p,q} |c_{i,s}^{p,q}|.$$

Now $\overset{*}{H}_{\beta_i, \sigma, \tau, s}^{k, \nu}(R)$ will be a Banach space.

By $(1+t)^\delta$ ($t \geq 0$, $\delta = (\delta_1, \dots, \delta_n)$) we denote a diagonal matrix of order n on whose diagonal there are functions $(1+t)^{\delta_i}$, i.e.

$$(1+t)^\delta = \operatorname{diag} [(1+t)^{\delta_1}, \dots, (1+t)^{\delta_n}].$$

Define the matrix-function $\theta(\sigma, \tau - 1, \delta; \cdot)$ by the equality

$$\begin{aligned} & \theta(\sigma, \tau - 1, \delta; t) = \\ & = \begin{cases} \psi(t) \sum_{p=1}^{\sigma} \sum_{q=0}^{\tau-1} (1+t)^{-\delta} T_{+}^{\nu,p} (1+t)^{-p} \ln^q(1+t)(1+t)^{\delta}, & t \geq 0, \\ \psi(t) \sum_{p=1}^{\sigma} \sum_{q=0}^{\tau-1} (1-t)^{-\delta} T_{-}^{\nu,p} (1-t)^{-p} \ln^q(1-t)(1-t)^{\delta}, & t < 0, \end{cases} \end{aligned}$$

where $T_{+}^{\nu,p}$ ($T_{-}^{\nu,p}$) denotes a constant matrix of order n .

Let the matrix $B \in H_{\lambda}^{k,\nu}(R)$ ($\lambda > 0$), $\det(I + B(x)) \neq 0$, $x \in \dot{R}$, and there exist matrices $\theta_1(\sigma, \tau - 1, \delta; \cdot)$, $\theta_2(\sigma, \tau - 1, \delta; \cdot)$ such that

$$\begin{aligned} & (B - \theta_1(\sigma, \tau - 1, \delta; \cdot)) \in H_{\sigma+\lambda}^{k,\nu}(R), \\ & ((I + B)^{-1} - I - \theta_2(\sigma, \tau - 1, \delta; \cdot)) \in H_{\sigma+\lambda}^{k,\nu}(R). \end{aligned}$$

Then the one-dimensional matrix singular integral operator

$$\mathcal{K}(u)(x) = (2I + B(x))u(x) + \frac{1}{\pi i} B(x) \int_R \frac{u(y)}{y-x} dy$$

is Noetherian in the space $\prod_{i=1}^n H_{\beta_i, \sigma, \tau, s}^{*k,\nu}(R)$ and any solution of the equation

$$\mathcal{K}(u) = g, \quad g \in \prod_{i=1}^n H_{\beta_i, \sigma, \tau, s}^{*k,\nu}(R) \cap \prod_{i=1}^n L_{p_i}(R) \quad (1.9)$$

from the space $\prod_{i=1}^n L_{p_i}(R)$ belongs to the space $\prod_{i=1}^n H_{\beta_i, \sigma, \tau, s}^{*k,\nu}(R) \cap \prod_{i=1}^n L_{p_i}(R)$.

In order that equation (1.9) to be solvable it is necessary and sufficient that $(g, v) = 0$ where v is an arbitrary solution of the equation $\mathcal{K}^*(v) = 0$ from the space $\prod_{i=1}^n H_{\beta_i, \sigma, \tau, s}^{*k,\nu}(R) \cap \prod_{i=1}^n L_{q_i}(R)$ ($\frac{1}{p_i} + \frac{1}{q_i} = 1$).

Proof. Applying equality (1.4), we represent the operator

$$B(x) \int_R (y-x)^{-1} u(y) dy$$

in the form

$$\begin{aligned} & B(x) \int_R (y-x)^{-1} u(y) dy = -[B(x) - \theta_1(\sigma, \tau - 1, \delta; x)] \times \\ & \quad \times \sum_{r=1}^{\sigma} (x+i)^{-r} \int_R (y+i)^{r-1} [u(y) - \overset{s}{\psi}(\sigma, \tau, \delta; y)] dy + \\ & \quad + B(x)(x+i)^{-\sigma} \int_R (y-x)^{-1} (y+i)^{\sigma} [u(y) - \overset{s}{\psi}(\sigma, \tau, \delta; y)] dy + \end{aligned}$$

$$\begin{aligned}
& + [B(x) - \theta_1(\sigma, \tau - 1, \delta; x)] \int_R (y - x)^{-1} \overset{s}{\psi}(\sigma, \tau, \delta; y) dy - \\
& - \theta_1(\sigma, \tau - 1, \delta; x) \sum_{r=1}^{\sigma} (x + i)^{-r} \int_R (y + i)^{r-1} [u(y) - \overset{s}{\psi}(\sigma, \tau, \delta; y)] dy + \\
& + \theta_1(\sigma, \tau - 1, \delta; x) \int_R (y - x)^{-1} \overset{s}{\psi}(\sigma, \tau, \delta; y) dy. \tag{1.10}
\end{aligned}$$

Similarly

$$\begin{aligned}
& \int_R (y - x)^{-1} B(y) u(y) dy = \\
& = - \sum_{r=1}^{\sigma} (x + i)^{-r} \int_R (y + i)^{r-1} B(y) [u(y) - \overset{s}{\psi}(\sigma, \tau, \delta; y)] dy + \\
& + (x + i)^{-\sigma} \int_R (y + i)^{\sigma} (y - x)^{-1} B(y) [u(y) - \overset{s}{\psi}(\sigma, \tau, \delta; y)] dy - \\
& - \sum_{r=1}^{\sigma} (x + i)^{-r} \int_R (y + i)^{r-1} [B(y) - \theta_1(\sigma, \tau - 1, \delta; y)] \overset{s}{\psi}(\sigma, \tau, \delta; y) dy + \\
& + (x + i)^{-\sigma} \int_R (y + i)^{\sigma} (y - x)^{-1} [B(y) - \theta_1(\sigma, \tau - 1, \delta; y)] \overset{s}{\psi}(\sigma, \tau, \delta; y) dy + \\
& + \int_R (y - x)^{-1} \theta_1(\sigma, \tau - 1, \delta; y) \overset{s}{\psi}(\sigma, \tau, \delta; y) dy. \tag{1.11}
\end{aligned}$$

It is not difficult to verify that the singular integral makes the asymptotics of the vectors $\overset{s}{\psi}(\sigma, \tau, \delta; \cdot)$, $\theta_1(\sigma, \tau, \delta; \cdot)$ worse only by a logarithm.

Representations (1.10), (1.11) immediately imply the boundedness of the operator \mathcal{K} and the complete continuity of the commutator in the space $\prod_{i=1}^n H_{\beta_i, \sigma, \tau, s}^{k, \nu}(R)$ for $k = 0$. When $k > 0$ these representations are based on equality (1.5). It is likewise clear that the operator M defined by

$$\Phi(M) = 2I + B - \frac{\xi}{|\xi|} B$$

regularizes the operator K . ■

Assume that

$$Z_+ = \{z = x_1 + ix_2, x_2 > 0\}, \quad Z_- = \{z = x_1 + ix_2, x_2 < 0\} \tag{1.12}$$

and consider the singular potential

$$v(z) = \frac{1}{2\pi i} \int_R \frac{u(y) dy}{y - z}.$$

We shall finish this section with

Let $u \in H_\beta^{k,\nu}(R)$ ($\beta > 0$). Then the function v is analytic both in the domain Z_+ and in the domain Z_- ,

$$\begin{aligned} \lim_{Z_+ \ni z \rightarrow x \in R} v(z) &= \frac{1}{2} u(x) + \frac{1}{2\pi i} \int_R \frac{u(y) dy}{y-x}, \\ \lim_{Z_- \ni z \rightarrow x \in R} v(z) &= -\frac{1}{2} u(x) + \frac{1}{2\pi i} \int_R \frac{u(y) dy}{y-x}. \end{aligned} \quad (1.13)$$

Moreover, the function v^+ (v^-) defined by

$$v^+(z) = \begin{cases} v(z), & z \in Z_+, \\ \lim_{Z_+ \ni \eta \rightarrow z} v(\eta), & z \in R \end{cases} \quad \left(v^-(z) = \begin{cases} v(z), & z \in Z_-, \\ \lim_{Z_- \ni \eta \rightarrow z} v(\eta), & z \in R \end{cases} \right)$$

belongs to the space $H_\beta^{k,\nu}(\overline{Z}_+)$ [$H_\beta^{k,\nu}(\overline{Z}_-)$], i. e.

$$\begin{aligned} |\partial_z^p v^\pm(z)| &\leq c(1+|z|)^{-p-\beta}, \quad p = 1, \dots, k, \\ |\partial_z^k v^\pm(z') - \partial_z^k v^\pm(z'')| &\leq c|z' - z''|^\nu \rho_{z',z''}^{-k-\beta-\nu}, \end{aligned}$$

where $\rho_{z',z''} = \min((1+|z'|), 1+|z''|)$.

The theorem can be proved using the equality

$$\frac{1}{2\pi i} \int_R \frac{dy}{y-z} = \begin{cases} \frac{1}{2}, & z \in Z_+, \\ -\frac{1}{2}, & z \in Z_- \end{cases}$$

and (1.5). The procedure is in the main similar to the one used in proving the corresponding properties of singular potentials in bounded domains (see [6]).

One can obtain quite complete information on one-dimensional singular integral operators from the momographs [7], [8].

§ 2. FACTORIZATION OF HOMOGENEOUS MATRIX-FUNCTIONS

Consider a matrix-function $A(\xi) = \|A_{ij}(\xi)\|_{n \times n}$. It will be assumed that

$$\begin{aligned} A(\lambda\xi) &= A(\xi) \quad (\lambda > 0), \quad A_{ij} \in C^\infty(R^m \setminus \{0\}) \quad (m \geq 2), \\ \det A(\xi) &\neq 0 \quad (\xi \neq 0). \end{aligned} \quad (2.1)$$

Put

$$A_0 = A^{-1}(0, \dots, 0, -1)A(0, \dots, 0, +1).$$

Let λ_j ($j = 1, \dots, s$) be the eigenvalues of the matrix A_0 , i.e. $\det(A_0 - \lambda_j I) = 0$, and r_j be their multiplicities ($\sum_{j=1}^s r_j = n$). Clearly $\lambda_j \neq 0$ ($j = 1, \dots, s$).

We introduce the matrices $B_r(\alpha) \equiv \|B_{\nu k}(\alpha)\|_{r \times r}$ where

$$B_{\nu k}(\alpha) = \begin{cases} 0, & \nu < k, \\ 1, & \nu = k, \\ \frac{\alpha^{\nu-k}}{(\nu-k)!}, & \nu > k, \end{cases}$$

$$B(r_i; \alpha) \equiv \text{diag} [B_{r_{i1}}(\alpha), \dots, B_{r_{ip_i}}(\alpha)]$$

$$(r_{i1} + \dots + r_{ip_i} = r_i).$$

It is easy to verify that

$$B_r(0) = I, \quad B_r(\alpha_1 + \alpha_2) = B_r(\alpha_1)B_r(\alpha_2). \quad (2.2)$$

This in particular implies that the matrices $B_r(\alpha_1)$, $B_r(\alpha_2)$ are commutative and $B_r(-\alpha) = B_r^{-1}(\alpha)$.

Following Jordan's theorem the matrix A_0 can be represented as

$$A_0 = gBg^{-1},$$

where $\det g \neq 0$, B is the modified Jordan form of A_0 and

$$B = \text{diag} [\lambda_1 B(r_1; 1), \dots, \lambda_s B(r_s; 1)].$$

Introduce the notation:

$$\delta'_k = \frac{1}{2\pi i} \ln \lambda_k, \quad k = 1, \dots, s;$$

$$\delta_j = \delta'_k \quad \text{for} \quad \sum_{\nu=1}^{k-1} r_\nu < j \leq \sum_{\nu=1}^k r_\nu, \quad j = 1, \dots, n.$$

It will be assumed that $\text{Re } \delta_1 \geq \text{Re } \delta_2 \geq \dots \geq \text{Re } \delta_n$,

$$\delta = (\delta_1, \dots, \delta_n), \quad \alpha_\pm(\xi) = \frac{1}{2\pi i} \ln \frac{\xi_m \pm i|\xi'|}{|\xi'|} \quad (\xi' = (\xi_1, \dots, \xi_{m-1})).$$

By $\ln z$ we denote the logarithm branch defined by $-\pi < \arg z \leq \pi$,

$$\left(\frac{\xi_m \pm i|\xi'|}{|\xi'|} \right)^{\delta_j} \equiv e^{\delta_j \ln \frac{\xi_m \pm i|\xi'|}{|\xi'|}},$$

$$\left(\frac{\xi_m \pm i|\xi'|}{|\xi'|} \right)^\delta \equiv \text{diag} \left[\left(\frac{\xi_m \pm i|\xi'|}{|\xi'|} \right)^{\delta_1}, \dots, \left(\frac{\xi_m \pm i|\xi'|}{|\xi'|} \right)^{\delta_n} \right],$$

$$B_\pm(\xi) \equiv \text{diag} [B(r_1; \alpha_\pm(\xi)), \dots, B(r_s; \alpha_\pm(\xi))].$$

Clearly

$$-\frac{1}{2} < \operatorname{Re} \delta_j \leq \frac{1}{2},$$

$$\lim_{\xi_m \rightarrow +\infty} [\alpha_+(\xi) - \alpha_-(\xi)] = 0, \quad \lim_{\xi_m \rightarrow -\infty} [\alpha_+(\xi) - \alpha_-(\xi)] = 1, \quad (2.3)$$

$$\lim_{\xi_m \rightarrow +\infty} (\xi_m + i|\xi'|)^{\delta_j} (\xi_m - i|\xi'|)^{-\delta_j} = 1,$$

$$\lim_{\xi_m \rightarrow -\infty} (\xi_m + i|\xi'|)^{\delta_j} (\xi_m - i|\xi'|)^{-\delta_j} = \lambda_k \quad \text{for} \quad \sum_{\nu=1}^{k-1} r_\nu < j \leq \sum_{\nu=1}^k r_\nu, \quad (2.4)$$

$$\lim_{\xi_m \rightarrow +\infty} B_+(\xi) \cdot B_-^{-1}(\xi) = I,$$

$$\lim_{\xi_m \rightarrow -\infty} B_+(\xi) \cdot B_-^{-1}(\xi) = \operatorname{diag} [B(r_1; 1), \dots, B(r_s; 1)]. \quad (2.5)$$

Put

$$A_*(\xi) \equiv \left(\frac{\xi_m - i|\xi'|}{|\xi'|} \right)^{-\delta} B_-^{-1}(\xi) g^{-1} A^{-1}(0, \dots, +1) \times$$

$$\times A(\xi) g B_+(\xi) \left(\frac{\xi_m + i|\xi'|}{|\xi'|} \right)^{\delta}, \quad \xi' \neq 0, \quad (2.6)$$

$$T(\xi) \equiv g^{-1} A^{-1}(0, \dots, +1) A(\xi) g -$$

$$- B_-(\xi) (\xi_m - i|\xi'|)^{\delta} (\xi_m + i|\xi'|)^{-\delta} B_+^{-1}(\xi),$$

$$T(0, \xi_m) \equiv \lim_{|\xi'| \rightarrow 0} T(\xi', \xi_m).$$

Then

$$A_*(\xi) - I = \left(\frac{\xi_m - i|\xi'|}{|\xi'|} \right)^{-\delta} B_-^{-1}(\xi) T(\xi) B_+(\xi) \left(\frac{\xi_m + i|\xi'|}{|\xi'|} \right)^{\delta}, \quad (2.7)$$

and from (2.3)–(2.5) we find that $\lim_{\xi_m \rightarrow \pm\infty} T(\xi) = 0$.

Introduce the set

$$G = \{(\eta', r, \xi_m) : \eta' \in R^{m-1}, \frac{1}{2} < |\eta'| < \frac{3}{2}, (r, \xi_m) \in R^2 \setminus \{0\}, r \geq 0\}$$

and define the function T_* on this set by the equality

$$T_*(\eta', r, \xi_m) = T(\eta' r, \xi_m).$$

It is easy to see that the matrix-function T_* is positively homogeneous of order zero with respect to variables (r, ξ_m) and is infinitely differentiable on G .

Let

$$\frac{\xi_m}{|\xi'|} = t, \quad B_{\pm} \left(\frac{\xi'}{|\xi'|}, t \right) \equiv B_{\pm}(t)$$

(this matrix-function does not depend on $\frac{\xi'}{|\xi'|}$). Then the matrix-function $A_*(\xi)$ can be rewritten as

$$A_*(\xi) = A_*\left(\frac{\xi'}{|\xi'|}, t\right) = I + (t-i)^{-\delta} B_-^{-1}(t) T_*\left(\frac{\xi'}{|\xi'|}, 1, t\right) B_+(t) (t+i)^\delta. \quad (2.8)$$

Define the new functions A_{**} and T_{**} on $(R^{m-1} \setminus \{0\}) \times R$ by the equality

$$A_{**}(\xi', t) = A_*\left(\frac{\xi'}{|\xi'|}, t\right), \quad T_{**}(\xi', t) = T_*\left(\frac{\xi'}{|\xi'|}, 1, t\right). \quad (2.9)$$

Then $A_{**}(\xi', \frac{\xi_m}{|\xi'|}) = A_*(\xi)$ and

$$A_{**}(\xi', t) = I + (t-i)^{-\delta} B_-^{-1}(t) T_{**}(\xi', t) B_+(t) (t+i)^\delta.$$

The following theorem is true.

*The matrix-function $A_{**}(\xi', t)$ is a positively homogeneous function of order zero with respect to the variable ξ' and is infinitely differentiable on $(R^{m-1} \setminus \{0\}) \times R$. When $t > 0$ ($t < 0$) it admits, for any natural number k , the expansion*

$$\begin{aligned} A_{**}(\xi', t) &= I + \sum_{\nu=1}^k \sum_{q=0}^{2n-2} (1+t)^{-\delta} T_{+}^{\nu,q}(\xi') (1+t)^{-\nu} \ln^q(1+t) (1+t)^\delta + \\ &\quad + \sum_{q=0}^{2n-2} (1+t)^{-\delta} T_{+}^{k+1,q}(\xi', t) (1+t)^{-k-1} \ln^q(1+t) (1+t)^\delta \\ (A_{**}(\xi', t) &= I + \sum_{\nu=1}^k \sum_{q=0}^{2n-2} (1-t)^{-\delta} T_{-}^{\nu,q}(\xi') (1-t)^{-\nu} \ln^q(1-t) (1-t)^\delta + \\ &\quad + \sum_{q=0}^{2n-2} (1-t)^{-\delta} T_{-}^{k+1,q}(\xi', t) (1-t)^{-k-1} \ln^q(1-t) (1-t)^\delta), \end{aligned} \quad (2.10)$$

where $T_{\pm}^{\nu,q}$ are positively homogeneous matrix-functions of order zero which are infinitely differentiable on $R^{m-1} \setminus \{0\}$, the matrix-functions $T_{+}^{k+1,q}$ ($T_{-}^{k+1,q}$) are positively homogeneous of order zero with respect to the variable ξ' and are infinitely differentiable on $(R^{m-1} \setminus \{0\}) \times R_+$ ($(R^{m-1} \setminus \{0\}) \times R_-$) and

$$|\partial_{\xi'}^p \partial_t^s T_{\pm}^{k+1,q}(\xi', t)| \leq c_{p,s} |\xi'|^{-|p|} (1+|t|)^{-s}. \quad (2.11)$$

Proof. The fact that the matrix-function $A_{**}(\xi', t)$ is a positively homogeneous function of order zero with respect to the variable ξ' and is infinitely differentiable on $(R^{m-1} \setminus \{0\}) \times R$ immediately follows from the definition of this function.

To obtain expansion (2.10) we first assume that $t > 0$, then apply the Taylor formula

$$u(y) = \sum_{|p|=0}^k \frac{1}{p!} \partial^p u(x) (y-x)^p + \sum_{|p|=k+1} \frac{k+1}{p!} (y-x)^p \int_0^1 \partial^p u(x + \tau(y-x)) (1-\tau)^k d\tau$$

and expand the matrix-function $T_*(\eta', r, t) = T_*(\eta', \frac{r}{r+t}, \frac{t}{r+t})$ in a series near the point $(\eta', 0, 1)$. Keeping in mind that $T_*(\eta', 0, 1) = 0$, we obtain

$$\begin{aligned} T_*(\eta', r, t) &= \sum_{p_1+p_2=1}^k \frac{(-1)^{p_2}}{p_1! p_2!} \frac{\partial^{p_1+p_2} T_*(\eta', 0, 1)}{\partial r^{p_1} \partial t^{p_2}} \frac{r^{p_1+p_2}}{(r+t)^{p_1+p_2}} + \\ &+ \sum_{p_1+p_2=k+1} \frac{(-1)^{p_2} (k+1)}{p_1! p_2!} \frac{r^{k+1}}{(r+t)^{k+1}} \times \\ &\times \int_0^1 \frac{\partial^{p_1+p_2} T_*(\eta', \tilde{r}, \tilde{t})}{\partial \tilde{r}^{p_1} \partial \tilde{t}^{p_2}} \bigg|_{\substack{\tilde{r}=\frac{r\tau}{r+t} \\ \tilde{t}=1-\frac{r\tau}{r+t}}} (1-\tau)^k d\tau. \end{aligned}$$

Hence

$$\begin{aligned} T_*\left(\frac{\xi'}{|\xi'|}, 1, t\right) &= \sum_{p_1+p_2=1}^k \frac{(-1)^{p_2}}{p_1! p_2!} \frac{\partial^{p_1+p_2} T_*\left(\frac{\xi'}{|\xi'|}, 0, 1\right)}{\partial r^{p_1} \partial t^{p_2}} (1+t)^{-p_1-p_2} + \\ &+ \sum_{p_1+p_2=k+1} \frac{(-1)^{p_2} (k+1)}{p_1! p_2!} (1+t)^{-(k+1)} \times \\ &\times \int_0^1 \frac{\partial^{p_1+p_2} T_*\left(\frac{\xi'}{|\xi'|}, \tilde{r}, \tilde{t}\right)}{\partial \tilde{r}^{p_1} \partial \tilde{t}^{p_2}} \bigg|_{\substack{\tilde{r}=\frac{\tau}{1+t} \\ \tilde{t}=1-\frac{\tau}{1+t}}} (1-\tau)^k d\tau. \end{aligned}$$

Now from (2.8) it follows that

$$\begin{aligned} A_{**}(\xi', t) &= I + \sum_{\nu=1}^k (t-i)^{-\delta} B_-^{-1}(t) T_1^\nu(\xi') (1+t)^{-\nu} B_+(t) (t+i)^\delta + \\ &+ (t-i)^{-\delta} B_-^{-1}(t) T_1^{k+1}(\xi', t) (1+t)^{-k-1} B_+(t) (t+i)^\delta, \end{aligned} \quad (2.12)$$

where T_1^ν is a positively homogeneous matrix-function of order zero which is infinitely differentiable on $R^{m-1} \setminus \{0\}$, T_1^{k+1} is a positively homogeneous matrix-function of order zero with respect to the variable ξ' and is infinitely differentiable on $(R^{m-1} \setminus \{0\}) \times R_+$ and

$$|\partial_{\xi'}^p \partial_t^s T_1^{k+1}(\xi', t)| \leq c_{p,s} |\xi'|^{-|p|} (1+t)^{-s}.$$

Keeping in mind the structure of matrices $B_{\pm}(t)$ and the fact that for large positive values of t we have

$$\begin{aligned}\ln(t \pm i) &= \ln(t + 1) + \sum_{k=1}^{\infty} c_k^{\pm} (1 + t)^{-k}, \\ (t \pm i)^{\delta_j} &= (t + 1)^{\delta_j} \sum_{k=0}^{\infty} \tilde{c}_k^{\pm} (1 + t)^{-k},\end{aligned}$$

we obtain the desired result from (2.12).

When $t < 0$ the matrix-function $T_+(\eta', r, t) = T_*(\eta', \frac{r}{r-t}, \frac{t}{r-t})$ should be expanded in a Taylor series near the point $(\eta', 0, -1)$. ■

*Theorem 2.1 is valid for the matrix-function $A_{**}^{-1}(\xi', t) = A_*^{-1}(\frac{\xi'}{|\xi'|}, t)$ as well.*

Let k, σ be natural numbers ($k \geq 0, \sigma \geq 1$), $0 < \nu < 1$ and $0 < \lambda < 1 - (\operatorname{Re} \delta_1 - \operatorname{Re} \delta_n)$. Then

$$\begin{aligned}A_{**}(\xi', \cdot) - I, \quad A_{**}^{-1}(\xi', \cdot) - I &\in H_{\lambda}^{k, \nu}(R); \\ A_{**}(\xi', \cdot) - I - \theta_1(\sigma, 2n - 2, \delta; \cdot) &\in H_{\sigma+\lambda}^{k, \nu}(R), \\ A_{**}^{-1}(\xi', \cdot) - I - \theta_2(\sigma, 2n - 2, \delta; \cdot) &\in H_{\sigma+\lambda}^{k, \nu}(R).\end{aligned}$$

*Here the j -th column of each of the matrices $A_{**}(\xi', \cdot) - I$ and $A_{**}^{-1}(\xi', \cdot) - I$ belongs to the space $\prod_{i=1}^n H_{\beta_i, \sigma, \tau, j}^{k, \nu}(R)$ ($\tau = 2n - 2$) where numbers β_i satisfy conditions (1.7).*

Now we are able to prove the main result of this section.

Let a positively homogeneous matrix-function $A(\xi) = \|A_{ij}(\xi)\|_{n \times n}$ of order zero be infinitely differentiable on $R^m \setminus \{0\}$ ($m \geq 2$) and be strongly elliptic (i.e.

$$\operatorname{Re} A(\xi) \eta \cdot \bar{\eta} \equiv \operatorname{Re} A_{ij}(\xi) \eta_j \bar{\eta}_i \neq 0 \quad (\xi \neq 0)$$

for any nonzero complex vector $\eta = (\eta_1, \dots, \eta_n)$). Then it admits the expansion

$$A(\xi) = c g A_-(\xi', \xi_m) \mathcal{D}(\xi) A_+(\xi', \xi_m) g^{-1},$$

where

$$\begin{aligned}c &= A(0, \dots, +1), \quad \mathcal{D}(\xi) = B_-(\xi) (\xi_m - i|\xi'|)^{\delta} (\xi_m + i|\xi'|)^{-\delta} B_+^{-1}(\xi), \\ A_{\pm}(\lambda \xi) &= A_{\pm}(\xi) \quad (\lambda > 0), \quad \det A_{\pm}(\xi) \neq 0 \quad (\xi \neq 0).\end{aligned}$$

The matrices A_+ , A_+^{-1} (A_- , A_-^{-1}) admit analytic continuations with respect to ξ_m to the lower (upper) complex half-plane, these continuations being bounded. Moreover, for any natural k the matrices A_{\pm} admit the expansions

$$A_{\pm}(\xi', \xi_m) = I + \sum_{p=1}^k \sum_{q=0}^{(p+1)(2n-1)} C_{\pm}^{p,q} \left(\frac{\xi'}{|\xi'|} \right) \left(\frac{\xi_m \pm i|\xi'|}{|\xi'|} \right)^{-p} \times \\ \times \ln^q \frac{\xi_m \pm i|\xi'|}{|\xi'|} + A^k(\xi), \quad (2.13)$$

where

$$C_{\pm}^{p,q} \in C^{\infty}(R^{m-1} \setminus \{0\}), \quad A^k(\lambda\xi) = A^k(\xi) \quad (\lambda > 0), \quad A^k \in C^k(R^m \setminus \{0\}).$$

The expansions of the same kind hold for the inverse matrices A_{\pm}^{-1} as well.

Proof. Let Z_+ , Z_- be the domains defined by (1.12) and A_{**} be the matrix-function defined by (2.9).

Consider the Hilbert homogeneous problem:

Find a matrix-function $\Phi(\xi, \cdot)$ which is analytic in $Z_+ \cup Z_-$, continuous on R from Z_+ and Z_- and satisfies the conditions

$$\Phi^-(\xi', t_0) = A_{**}(\xi', t_0)\Phi^+(\xi', t_0), \quad t_0 \in R, \\ \lim_{\bar{Z}_+ \ni z \rightarrow \infty} \Phi^+(\xi', z) = I, \quad \lim_{\bar{Z}_- \ni z \rightarrow \infty} \Phi^-(\xi', z) = I. \quad (2.14)$$

A solution will be sought for in the form

$$\Phi(\xi', z) = \frac{1}{2\pi i} \int_R \frac{\varphi(\xi', t)}{t - z} dt + I, \quad \varphi \in H_{\lambda}^{k,\nu}(R). \quad (2.15)$$

To define the matrix φ , by Theorem 1.4 we obtain a system of singular integral equations

$$(A_{**}(\xi', t_0) + I)\varphi(\xi', t_0) + \frac{1}{\pi i} (A_{**}(\xi', t_0) - I) \int_R \frac{\varphi(\xi', t)}{t - t_0} dt = \\ = 2(I - A_{**}(\xi', t_0)). \quad (2.16)$$

Denote by $\mathcal{K}_{\xi'}$ the singular integral operator corresponding to this system. It is clear that the determinant of the symbolic matrix of this operator is not zero and by Theorems 1.2, 1.3 and Corollary 2.2 $\mathcal{K}_{\xi'}$ is Noetherian both in the space $\prod_{i=1}^n H_{\beta_i}^{k,\nu}(R)$ and in the space $\prod_{i=1}^n H_{\beta_i, \sigma, \tau, s}^{k,\nu}(R)$ ($k \geq 0$, $\sigma \geq 1$, $0 < \nu < 1$, $\tau = 2n - 1$, $\text{Re } \delta_i - \text{Re } \delta_n < \beta_i < 1 + \text{Re } \delta_i - \text{Re } \delta_1$, $i = 1, \dots, n$). Note that since the matrix A is strongly elliptic, we have $|\text{Re } \delta_i| < \frac{1}{2}$, $i = 1, \dots, n$.

Show that the operator $\mathcal{K}_{\xi'}$ is invertible both in the space $\prod_{i=1}^n H_{\beta_i}^{k,\nu}(R)$ and in the space $\prod_{i=1}^n \overset{*}{H}_{\beta_i,\sigma,\tau,s}^{k,\nu}(R)$. To this effect, by Theorems 1.2, 1.3 it is enough to prove that the homogeneous equations $\mathcal{K}_{\xi'}(u) = 0$ and $\mathcal{K}'_{\xi'}(v) = 0$ ($\mathcal{K}'_{\xi'}$ denotes the adjoint operator of $\mathcal{K}_{\xi'}$) have only trivial solutions in the space $\prod_{i=1}^n H_{\beta_i}^{k,\nu}(R)$.

Let $\overset{\circ}{\varphi}$ be a solution of the equation $\mathcal{K}_{\xi'}(u) = 0$ from the space $\prod_{i=1}^n H_{\beta_i}^{k,\nu}(R)$. Then by Theorem 1.2 $\overset{\circ}{\varphi}$ will belong to the space $[H_{\beta}^{k,\nu}(R)]^n$ where

$$\frac{1}{2} + \max |\operatorname{Re} \delta_j| < \beta < 1.$$

Put

$$\overset{\circ}{\Phi}(\xi', z) = \frac{1}{2\pi i} \int_R \frac{\overset{\circ}{\varphi}(\xi', t)}{t - z} dt.$$

$\overset{\circ}{\Phi}(\xi', \cdot)$ is analytic in $Z_+ \cup Z_-$, satisfies the boundary condition

$$\overset{\circ}{\Phi}^-(\xi', t) = A_{**}(\xi', t) \overset{\circ}{\Phi}^+(\xi', t) \quad (2.17)$$

and by virtue of Theorem 1.4

$$\begin{aligned} \int_R |\overset{\circ}{\Phi}^+(\xi', t + i\tau)|^2 (1 + |t| + \tau)^{2s} dt &\leq c, \\ \int_R |\overset{\circ}{\Phi}^-(\xi', t - i\tau)|^2 (1 + |t| + \tau)^{2s} dt &\leq c, \\ \tau \geq 0, \quad \frac{1}{2} + \max |\operatorname{Re} \delta_j| &< \frac{1}{2} + s < \beta. \end{aligned} \quad (2.18)$$

Recalling now that

$$\begin{aligned} A_{**}(\xi', t) &= A_*\left(\frac{\xi'}{|\xi'|}, t\right) = \\ &= (t - i)^{-\delta} B_-^{-1}(t) g^{-1} A^{-1}(0, \dots, +1) A\left(\frac{\xi'}{|\xi'|}, t\right) g B_+(t) (t + i)^{\delta}, \end{aligned}$$

from (2.17) we obtain

$$A(0, \dots, +1) g B_-(t) (t - i)^{\delta} \overset{\circ}{\Phi}^-(\xi', t) = A\left(\frac{\xi'}{|\xi'|}, t\right) g B_+(t) (t + i)^{\delta} \overset{\circ}{\Phi}^+(\xi', t).$$

Due to inequalities (2.18) the conditions of the Paley–Wiener theorem (see [3]) is fulfilled for the vector-functions

$$\begin{aligned} A(0, \dots, +1) g B_-(t) (t - i)^{\delta} \overset{\circ}{\Phi}^-(\xi', t), \\ g B_+(t) (t + i)^{\delta} \overset{\circ}{\Phi}^+(\xi', t), \end{aligned}$$

and if we denote by $F(u)$ the Fourier transform of the function u ,

$$F(u)(x) = \int_R e^{ix \cdot y} u(y) dy,$$

then we shall get

$$\begin{aligned} A(0, \dots, +1)gB_-(t)(t-i)^\delta \overset{\circ}{\Phi}^-(\xi', t) &= F(f^-(\xi', \cdot))(t), \\ gB_+(t)(t+i)^\delta \overset{\circ}{\Phi}^+(\xi', t) &= F(f^+(\xi', \cdot))(t), \\ f^\pm(\xi', \cdot) &\in L_2^\pm(R) \end{aligned}$$

($L_2^\pm(R)$ denotes a subspace of the space $L_2(R)$ consisting of functions with support on the closed semi-axis \overline{R}_\pm).

We have

$$\begin{aligned} &\int_R A\left(\frac{\xi'}{|\xi'|}, t\right) F(f^+(\xi', \cdot))(t) \cdot \overline{F(f^+(\xi', \cdot))(t)} dt = \\ &= \int_R F(f^-(\xi', \cdot))(t) \cdot \overline{F(f^+(\xi', \cdot))(t)} dt = 2\pi \int_R f^-(\xi', t) \cdot \overline{f^+(\xi', t)} dt = 0 \end{aligned}$$

Hence since the matrix A is strongly elliptic, $F(f^+(\xi', \cdot))(t) \equiv 0$. Therefore $\overset{\circ}{\Phi}^\pm(\xi', t) \equiv 0$ and from Theorem 1.4 it follows that $\overset{\circ}{\varphi}(\xi', \cdot) = 0$.

Now assume that $\overset{\circ}{\psi}(\xi', \cdot)$ is a solution of the equation $\mathcal{K}_{\xi'}(v) = 0$ from the space $\prod_{i=1}^n H_{\beta_i}^{k, \nu}(R)$, i.e.

$$(A'_{**}(\xi', t_0) + I) \overset{\circ}{\psi}(\xi', t_0) - \frac{1}{\pi i} \int_R \frac{A'_{**}(\xi', t) - I}{t - t_0} \overset{\circ}{\psi}(\xi', t) dt = 0.$$

Put

$$\overset{\circ}{\Psi}(\xi', z) = \frac{1}{2\pi i} \int_R \frac{A'_{**}(\xi', t) - I}{t - t_0} \overset{\circ}{\psi}(\xi', t) dt.$$

$\overset{\circ}{\Psi}(\xi', z)$ is analytic in $Z_+ \cup Z_-$ and satisfies by virtue of (1.13) the boundary condition

$$\overset{\circ}{\Psi}^-(\xi', t_0) = [A'_{**}(\xi', t_0)]^{-1} \overset{\circ}{\Psi}^+(\xi', t_0).$$

Hence, in view of the fact that the strong ellipticity of the matrix A implies that the matrix $[A']^{-1}$ is strongly elliptic, we find, as above, that $\overset{\circ}{\psi}(\xi', \cdot) = 0$.

Thus the operator $\mathcal{K}_{\xi'}$ is invertible both in the space $\prod_{i=1}^n H_{\beta_i}^{k, \nu}(R)$ and in the space $\prod_{i=1}^n H_{\beta_i, \sigma, \tau, s}^{k, \nu}(R)$.

Therefore, by virtue of Corollary 2.2, for any $\xi' \in (R^{m-1} \setminus \{0\})$ there exists a matrix $\varphi(\xi', \cdot) = \|\varphi_{ij}(\xi', \cdot)\|_{n \times n}$ which satisfies system (2.16) and

whose j -th column belong both to the space $\prod_{i=1}^n H_{\beta_i}^{k,\nu}(R)$ and to the space

$\prod_{i=1}^n H_{\beta_i, \sigma, \tau, s}^{*k,\nu}(R)$ ($\tau = 2n - 1$); this matrix is unique.

The uniqueness of a solution of system (2.16) implies that φ is a positively homogeneous matrix-function of order zero with respect to a variable ξ' .

Taking into account the differential properties of coefficients of the operator $\mathcal{K}_{\xi'}$, we find that this operator is infinitely differentiable with respect to the parameter ξ' in the norm of operators in the spaces $\prod_{i=1}^n H_{\beta_i}^{k,\nu}(R)$,

$\prod_{i=1}^n H_{\beta_i, \sigma, \tau, s}^{*k,\nu}(R)$.

The invertibility of the operator $\mathcal{K}_{\xi'}$ implies that the inverse operator $\mathcal{K}_{\xi'}^{-1}$ also possesses this property, which in view of Theorem 2.1 implies in turn that the matrix $\varphi(\xi', t)$ is infinitely differentiable with respect to the variable ξ' ($\xi' \neq 0$) and the j -th column of the matrix $\partial_{\xi'}^p \varphi(\xi', \cdot)$ belongs both to the space $\prod_{i=1}^n H_{\beta_i}^{k,\nu}(R)$ and to the space $\prod_{i=1}^n H_{\beta_i, \sigma, \tau, s}^{*k,\nu}(R)$ uniformly with respect to ξ' near $|\xi'| = 1$.

In particular we have

$$|\partial_{\xi'}^p \partial_t^s \varphi_{ij}(\xi', t)| \leq \frac{c}{|\xi'|^{|p|} (1 + |t|)^{s + \beta_i}}, \quad |p|, s = 0, 1, \dots, \quad (2.19)$$

and there exist positively homogeneous and infinitely differentiable on $R^{m-1} \setminus \{0\}$ functions $c_{\pm i, j}^{p, q}(\xi')$ such that if we define the function $\psi_{ij}(\sigma, \tau, \delta; \cdot)$ by

$$\begin{aligned} & \psi_{ij}(\sigma, \tau, \delta; \xi', t) = \\ & = \begin{cases} \psi(t) \sum_{p=1}^{\sigma} \sum_{q=0}^{p\tau} c_{+i, j}^{p, q} (1+t)^{-p-\delta_i+\delta_j} \ln^q(1+t), & t \geq 0, \\ \psi(t) \sum_{p=1}^{\sigma} \sum_{q=0}^{p\tau} c_{-i, j}^{p, q} (1+t)^{-p-\delta_i+\delta_j} \ln^q(1-t), & t < 0, \end{cases} \end{aligned}$$

then

$$|\partial_{\xi'}^p \partial_t^r (\varphi_{ij}(\xi', t) - \psi_{ij}(\sigma, \tau, \delta; \xi', t))| \leq \frac{c}{|\xi'|^{|\alpha|} (1 + |t|)^{\sigma + r + \beta_i}}, \quad (2.20)$$

$$|\alpha|, r = 0, 1, \dots$$

Now consider the matrix $\Phi(\xi', z)$ defined by equality (2.15)

$$\Phi(\xi', z) = \frac{1}{2\pi i} \int_R \frac{\varphi(\xi', t)}{t - z} dt + I.$$

By Theorem 1.4 it follows from (2.19) that $\Phi(\xi', \cdot)$ is analytic in $Z_+ \cup Z_-$ and

$$|\partial_{\xi'}^p \partial_z^s (\Phi_{ij}^{\pm}(\xi', z) - \delta_{ij})| \leq \frac{c}{|\xi'|^{|p|} (1 + |z|)^{s+\beta_i}}. \quad (2.21)$$

Show that $\det \Phi^{\pm}(\xi', z) \neq 0$, $z \in \overline{Z}_{\pm}$. Assume the contrary. For the sake of definiteness let $\det \Phi^+(\xi', z_0) = 0$ for some $z_0 \in \overline{Z}_+$. Then there exist numbers $\gamma_1, \dots, \gamma_n$ not all equal to zero and such that the linear combination $\sum_{i=1}^n \gamma_i \Phi^+(\xi', z)$ of the columns of the matrix $\Phi^+(\xi', z)$ vanishes at $z = z_0$. We put

$$\psi(\xi', z) = \frac{\sum_{i=1}^n \gamma_i \Phi^+(\xi', z)}{z - z_0}.$$

$\psi(\xi', \cdot)$ is analytic in $Z_+ \cup Z_-$ and satisfies the boundary condition

$$A_{**}(\xi', t)\psi^+(\xi', t) = \psi^-(\xi', t)$$

(except at the point $t = z_0$ when $z_0 \in R$), and by virtue of (2.21) conditions (2.18) are fulfilled for ψ^{\pm} . Therefore, as shown above, $\psi^+(\xi', z) \equiv 0$, $z \in \overline{Z}_+$, and thus $\sum_{i=1}^n \gamma_i \Phi^+(\xi', z) \equiv 0$, $z \in \overline{Z}_+$. Passing to the limit when $\overline{Z}_+ \ni z \rightarrow \infty$, we obtain $\gamma_i = 0$, $i = 1, \dots, n$, which is impossible. Therefore the matrix $\Phi^{-1}(\xi', z)$ is also analytic in $Z_+ \cup Z_-$ and (2.21) is valid for its elements. Further, by (2.20) it is not difficult to show that the elements of the matrices $\Phi^{\pm}(\xi', z)$, $z \in Z_{\pm}$, admit the expansion

$$\begin{aligned} \Phi_{ij}^{\pm}(\xi', z) &= \delta_{ij} + \sum_{p=1}^{\sigma} \sum_{q=0}^{p\tau+1} C_{\pm ij}^{p,q}(\xi') (z \pm i)^{-p-\delta_i+\delta_j} \ln^q(z \pm i) + \\ &\quad + \Phi_{ij,\sigma}^{\pm}(\xi', z), \quad z \in Z_{\pm}, \end{aligned} \quad (2.22)$$

where

$$C_{\pm ij}^{p,q}(\lambda \xi') = C_{\pm ij}^{p,q}(\xi'), \quad \Phi_{ij,\sigma}^{\pm}(\lambda \xi', z) = \Phi_{ij,\sigma}^{\pm}(\xi', z) \quad (\lambda > 0),$$

$C_{\pm ij}^{p,q} \in C^{\infty}(R^{m-1} \setminus \{0\})$, $\Phi_{ij,\sigma}^{\pm}(\xi', \cdot)$ is analytic in Z_{\pm} and

$$|\partial_{\xi'}^p \partial_z^s \Phi_{ij,\sigma}^{\pm}(\xi', z)| \leq \frac{c}{|\xi'|^{|p|} (1 + |z|)^{\sigma+s+\beta_i}}, \quad z \in \overline{Z}_{\pm}. \quad (2.23)$$

In order to obtain a similar expansion for the elements of the matrix $[\Phi^{\pm}(\xi', z)]^{-1}$ we should proceed as follows.

Clearly the matrix $[\Phi'(\xi', z)]^{-1}$ is analytic in $Z_+ \cup Z_-$ and satisfies the boundary condition

$$\begin{aligned} [(\Phi^-(\xi', t))']^{-1} &= [A'_{**}(\xi', t)]^{-1} [(\Phi^+(\xi', t))']^{-1}, \\ \lim_{\overline{Z}_+ \ni z \rightarrow \infty} [(\Phi^+(\xi', z))']^{-1} &= I, \quad \lim_{\overline{Z}_- \ni z \rightarrow \infty} [(\Phi^-(\xi', z))']^{-1} = I. \end{aligned} \quad (2.24)$$

On other hand, consider the problem: by the boundary condition (2.24) find an analytic in $Z_+ \cup Z_-$ matrix-function $\Psi'(\xi', \cdot)$ which is completely continuous on R from Z_+ and Z_- . Repeat our reasoning above, we can find the expansion of $\Psi^\pm(\xi', \cdot)$. But since the solution of the boundary value problem (2.24) is unique in the space of nonsingular matrices, we have $\Psi = [\Phi']^{-1}$.

Thus the elements $\psi_{ij}^\pm(\xi', z)$ of matrices $[\Phi^\pm(\xi', z)]^{-1}$ admit the expansion

$$\begin{aligned} \psi_{ij}^\pm(\xi', z) = & \delta_{ij} + \sum_{p=1}^{\sigma} \sum_{q=0}^{p\tau+1} C_{\pm ij}^{p,q}(\xi')(z \pm i)^{-p-\delta_i+\delta_j} \ln^q(z \pm i) + \\ & + \psi_{ij,\sigma}^\pm(\xi', z), \quad z \in Z_\pm, \end{aligned} \quad (2.25)$$

where

$$C_{\pm ij}^{p,q}(\lambda\xi') = C_{\pm ij}^{p,q}(\xi'), \quad \psi_{ij,\sigma}^\pm(\lambda\xi', z) = \psi_{ij,\sigma}^\pm(\xi', z) \quad (\lambda > 0),$$

$C_{\pm ij}^{p,q} \in C^\infty(R^{m-1} \setminus \{0\})$, $\psi_{ij,\sigma}^\pm(\xi', \cdot)$ is analytic in Z_\pm and

$$\begin{aligned} |\partial_{\xi'}^p \partial_z^s \psi_{ij,\sigma}^\pm(\xi', z)| & \leq \frac{c}{|\xi'|^{|p|} (1+|z|)^{\sigma+s+\gamma_j}}, \\ \operatorname{Re} \delta_1 - \operatorname{Re} \delta_j & < \gamma_j < 1 - (\operatorname{Re} \delta_j - \operatorname{Re} \delta_n). \end{aligned} \quad (2.26)$$

From now on expansion (2.13) is obtained in an elementary way. ■

Remark 2.1. The fact that the partial indices of the strongly elliptic matrix A are zero is proved in [3]. In the same paper an expansion of form (2.13) is obtained when A is a scalar function.

§ 3. REGULARIZATION OF SINGULAR INTEGRAL OPERATORS ON MANIFOLDS WITH BOUNDARY

Let $\mathcal{G}(R^m)$ be the space of infinitely differentiable functions u on R^m quickly decreasing at infinity, i.e. possessing the property that

$$\sup_{x \in R^m} (1+|x|)^N \sum_{|\alpha| \leq n} |\partial^\alpha u(x)| < \infty$$

for any nonnegative integers n, N .

Denote by $F(\varphi)$ the Fourier transform of the function φ , i.e.

$$F(\varphi)(x) = \int_{R^m} e^{ix \cdot y} \varphi(y) dy \equiv F_{y \rightarrow x} \varphi(y).$$

The inverse Fourier transform F^{-1} is written as

$$F^{-1}(\psi)(x) = \frac{1}{(2\pi)^m} \int_{R^m} e^{-ix \cdot y} \psi(y) dy.$$

Define the functions ℓ_{\pm} by the equalities

$$\ell_+(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0, \end{cases} \quad \ell_-(t) = \begin{cases} 0, & t > 0, \\ 1, & t < 0. \end{cases}$$

Let $m \geq 2$ and consider the pseudodifferential operator

$$A(u)(x) = \frac{1}{(2\pi)^m} \int_{R^m} e^{-ix \cdot \xi} A(\xi) F(u)(\xi) d\xi, \quad u \in \mathcal{G}(R^m), \quad (3.1)$$

where

$$A(\xi) = a(\xi') \left(\frac{\xi_m}{|\xi'|} \pm i \right)^{-n} \ln^p \left(\frac{\xi_m}{|\xi'|} \pm i \right), \quad \xi' = (\xi_1, \dots, \xi_{m-1}),$$

n, p ($n \geq 1, p \geq 0$) are natural numbers and a is a positively homogeneous function of order zero which is infinitely differentiable on $R^{m-1} \setminus \{0\}$.

The pseudodifferential operator A defined by (3.1) can be represented as a singular integral operator

$$A(u)(x) = a_0 u(x) + \int_{R^m} K(x-y) u(y) dy, \quad (3.2)$$

where

$$K(x) = \sum_{j=1}^{m-1} \sum_{q=0}^p \ell_{\pm} \left(\frac{x_m}{|x|} \right) \partial_{x_j} \left[K_{j,q}(x) \left(\frac{|x_m|}{|x|} \right)^{n-1} \ln^q \frac{|x_m|}{|x|} \right],$$

$K_{j,q}$ is a positively homogeneous function of order $1-m$ which is infinitely differentiable on $R^m \setminus \{0\}$ and

$$a_0 = [\text{mes } S(0,1)]^{-1} \int_{S(0,1)} A(\xi) ds.$$

Proof. We shall prove the theorem only for the case “+”, since the case “-” is proved similarly. After differentiating the equality

$$(\sigma + i\tau)^\mu = c(\mu) \int_0^\infty t^{-\mu-1} e^{-\tau t} e^{it\sigma} dt, \quad \text{Re } \mu < 0, \quad \tau > 0, \quad (3.3)$$

p -times with respect to the parameter μ , where $c(\mu) = i^\mu / \Gamma(-\mu)$ (see, for example, [3]), we obtain

$$(\sigma + i\tau)^\mu \ln^p(\sigma + i\tau) = \sum_{q=0}^p c_q F_{t \rightarrow \sigma}(\ell_+(t) t^{-\mu-1} e^{-\tau t} \ln^q t).$$

Therefore

$$\begin{aligned} \left(\frac{\xi_m}{|\xi'}| + i\right)^{-n} \ln^p \left(\frac{\xi_m}{|\xi'}| + i\right) &= \sum_{q=0}^p c_q \int_0^\infty t^{n-1} e^{-t} e^{i\frac{\xi_m}{|\xi'} t} \ln^q t dt = \\ (t = |\xi'|x_m) & \\ &= |\xi'|^n \sum_{q=0}^p c_q \int_0^\infty x_m^{n-1} e^{-|\xi'|t} e^{i\xi_m \cdot x_m} \ln^q (|\xi'|x_m) dx_m. \end{aligned}$$

Apply this equality to get

$$\begin{aligned} A(u)(x) &= F_{\xi' \rightarrow x'}^{-1} a(\xi') F_{\xi_m \rightarrow x_m}^{-1} \left[\left(\frac{\xi_m}{|\xi'}| + i\right)^{-n} \ln^p \left(\frac{\xi_m}{|\xi'}| + i\right) F(u)(\xi) \right] = \\ &= F_{\xi' \rightarrow x'}^{-1} a(\xi') |\xi'|^n \sum_{q=0}^p c_q \int_{-\infty}^\infty \ell_+(x_m - y_m) (x_m - y_m)^{n-1} \times \\ &\quad \times e^{-|\xi'|(x_m - y_m)} \ln^q |\xi'| (x_m - y_m) \cdot \overset{\square}{u}(\xi', y_m) dy_m, \end{aligned}$$

where

$$\overset{\square}{u}(\xi', y_m) = F_{y' \rightarrow \xi'}(u(y', y_m)).$$

Noting that $|\xi'| = \sum_{j=1}^{m-1} \frac{\xi_j}{|\xi'}| \xi_j$ and assuming that $a_{j,q}(\xi') = a(\xi') \frac{\xi_j}{|\xi'}| c_q$, we find

$$\begin{aligned} \ell_+(x_m - y_m) (x_m - y_m)^{n-1} F_{\xi' \rightarrow x'}^{-1} a_{j,q}(\xi') |\xi'|^{n-1} e^{-|\xi'|(x_m - y_m)} \ln^q |\xi'| (x_m - y_m) &= \\ &= \frac{\ell_+(x_m - y_m) (x_m - y_m)^{n-1}}{(2\pi)^{m-1}} \int_0^\infty r^{n+m-3} \ln^q r (x_m - y_m) dr \times \\ &\quad \times \int_{|\xi'|=1} a_{j,q}(\xi') e^{-r((x_m - y_m) + ix' \cdot \xi')} d_{\xi'} S = \\ &= \frac{\ell_+(x_m - y_m) (x_m - y_m)^{1-m}}{(2\pi)^{m-1}} \int_{|\xi'|=1} a_{j,q}(\xi') d_{\xi'} S \int_0^\infty e^{-r \frac{x_m - y_m + ix' \cdot \xi'}{x_m - y_m}} r^{n+m-3} \ln^q r dr. \end{aligned}$$

But

$$\begin{aligned} &\int_0^\infty e^{-r \frac{x_m - y_m + ix' \cdot \xi'}{x_m - y_m}} r^{n+m-3} \ln^q r dr = \\ &= \frac{(x_m - y_m)^{n+m-2}}{((x_m - y_m) + ix' \cdot \xi')^{n+m-2}} \sum_{s=0}^q c_s \ln^s \frac{x_m - y_m}{x_m - y_m + ix' \cdot \xi'}. \end{aligned}$$

Therefore

$$A(u)(x) = \sum_{j=1}^{m-1} \sum_{q=0}^p \int_{R^m} Q_{j,q}(x - y) \partial y_j u(y) dy, \quad (3.4)$$

where

$$Q_{j,q}(x) = \ell_+(x_m) x_m^{n-1} \int_{|\xi'|=1} \frac{b_{j,q}(\xi') \ln^q \frac{x_m}{x_m + i x' \cdot \xi'}}{(x_m + i x' \cdot \xi')^{n+m-2}} d_{\xi'} S$$

and $b_{j,q}$ is a positively homogeneous function of order zero which is infinitely differentiable on $R^{m-1} \setminus \{0\}$.

Rewrite $Q_{j,q}(x)$ as

$$Q_{j,q}(x) = \ell_+(x_m) \left(\frac{x_m}{|x|} \right)^{n-1} \frac{1}{|x|^{m-1}} \sum_{s=0}^q Q_{j,q,s}(x) \ln^s \frac{x_m}{|x|},$$

where

$$Q_{j,q,s}(x) = \int_{|\xi'|=1} \frac{b_{j,q,s}(\xi') \ln^{q-s} \left(\frac{x_m}{|x|} + i \frac{x'}{|x|} \cdot \xi' \right)}{\left(\frac{x_m}{|x|} + i \frac{x'}{|x|} \cdot \xi' \right)^{n+m-2}} d_{\xi'} S, \quad x_m \neq 0.$$

Show that $Q_{j,q,s}$ is infinitely differentiable on $R^m \setminus \{0\}$ ($Q_{j,q,s}(x', 0) = \lim_{x_m \rightarrow 0} Q_{j,q,s}(x)$).

Let $\psi \in C^\infty(R)$, $\psi(t) = 1$ for $|t| < \frac{1}{4}$ and $\psi(t) = 0$ for $|t| > \frac{1}{2}$. It will be assumed that $x = (x', x_m)$, $|x| \geq c > 0$. The fact that the function $Q_{j,q,s}$ is infinitely differentiable in the domain $|x_m| > |x'|$ is beyond any doubt. Assume now that $|x'| > \frac{1}{2}|x_m|$ and represent $Q_{j,q,s}(x)$ as

$$\begin{aligned} Q_{j,q,s}(x) &= \int_{|\xi'|=1} \frac{b_{j,q,s}(\xi') \ln^{q-s} \left(\frac{x_m}{|x|} + i \frac{x'}{|x|} \cdot \xi' \right) \psi \left(\frac{x'}{|x'|} \cdot \xi' \right)}{\left(\frac{x_m}{|x|} + i \frac{x'}{|x|} \cdot \xi' \right)^{n+m-2}} d_{\xi'} S + \\ &+ \int_{|\xi'|=1} \frac{b_{j,q,s}(\xi') \ln^{q-s} \left(\frac{x_m}{|x|} + i \frac{x'}{|x|} \cdot \xi' \right) (1 - \psi \left(\frac{x'}{|x'|} \cdot \xi' \right))}{\left(\frac{x_m}{|x|} + i \frac{x'}{|x|} \cdot \xi' \right)^{n+m-2}} d_{\xi'} S = \\ &= Q'_{j,q,s}(x) + Q''_{j,q,s}(x). \end{aligned}$$

It is clear that $Q''_{j,q,s}$ is infinitely differentiable. Introducing the new spherical coordinates $(t, \eta'') : t = \frac{x'}{|x'|} \cdot \xi'$, $\eta'' = (\eta_1, \dots, \eta_{m-2})$ and applying integration by parts we ascertain that $Q'_{j,q,s}$ is also infinitely differentiable in the domain $|x'| > \frac{1}{2}|x_m|$.

From now on the theorem is proved by means of equality (3.4). ■

Let $m \geq 2$, $-1 < \operatorname{Re} \delta < 1$. Then the pseudodifferential operator (3.1), where

$$A(\xi) = (\xi_m + i|\xi'|)^\delta (\xi_m - i|\xi'|)^{-\delta}$$

can be represented as the singular integral operator (3.2), where

$$K(x) = c(m, \delta) \frac{\text{sign} \frac{x_m}{|x|}}{|x|^m} + \frac{\partial}{\partial x_m} \left[\ell_+ \left(\frac{x_m}{|x|} \right) K_+(x) + \ell_- \left(\frac{x_m}{|x|} \right) K_-(x) \right] + \\ + \sum_{j=1}^{m-1} \sum_{q=0}^1 \frac{\partial}{\partial x_j} \left[\left(\ell_+ \left(\frac{x_m}{|x|} \right) Q_{j,q}^+(x) + \ell_- \left(\frac{x_m}{|x|} \right) Q_{j,q}^-(x) \right) \ln^q \frac{|x_m|}{|x|} \right],$$

K_{\pm} , $Q_{j,q}^{\pm}$ are positively homogeneous functions of order $1 - m$ which are infinitely differentiable on $R^m \setminus \{0\}$, and $K_{\pm}(0, x') = 0$.

Proof. Assume that $0 < \text{Re } \delta < 1$. Then

$$(\xi_m + i|\xi'|)^{\delta} = (\xi_m + i|\xi'|)(\xi_m + i|\xi'|)^{\delta-1}$$

and by virtue of equality (3.3)

$$A(u)(x) = c(m, \delta) F_{\xi' \rightarrow x'}^{-1} \int_{-\infty}^{\infty} \ell_+(x_m - t)(x_m - t)^{-\delta} e^{-|\xi'|(x_m - t)} \times \\ \times \left(\frac{\partial}{\partial t} + |\xi'| \right) \int_{-\infty}^{\infty} \ell_-(t - y_m)(y_m - t)^{\delta-1} e^{|\xi'|(t - y_m)} \bar{u}(\xi', y_m) dy_m dt = \\ = c(m, \delta) F_{\xi' \rightarrow x'}^{-1} \int_{-\infty}^{x_m} (x_m - t)^{-\delta} e^{-|\xi'|(x_m - t)} \times \\ \times \int_t^{\infty} (y_m - t)^{\delta-1} e^{|\xi'|(t - y_m)} \left(\frac{\partial}{\partial y_m} + |\xi'| \right) \bar{u}(\xi', y_m) dy_m dt.$$

Changing the integration order, we get

$$A(u)(x) = c(m, \delta) F_{\xi' \rightarrow x'}^{-1} \left[\int_{-\infty}^{x_m} \left(\frac{\partial}{\partial y_m} + |\xi'| \right) \bar{u}(\xi', y_m) dy_m \times \right. \\ \times \int_{-\infty}^{y_m} (x_m - t)^{-\delta} (y_m - t)^{\delta-1} e^{-|\xi'|(x_m + y_m - 2t)} dt + \\ \left. + \int_{x_m}^{\infty} \left(\frac{\partial}{\partial y_m} + |\xi'| \right) \bar{u}(\xi', y_m) dy_m \times \right. \\ \left. \times \int_{-\infty}^{x_m} (x_m - t)^{-\delta} (y_m - t)^{\delta-1} e^{-|\xi'|(x_m + y_m - 2t)} dt \right] = \\ = c(m, \delta) F_{\xi' \rightarrow x'}^{-1} \left[\int_{-\infty}^{x_m} \left(\frac{\partial}{\partial y_m} + |\xi'| \right) \bar{u}(\xi', y_m) dy_m \times \right. \\ \left. \times \int_0^{\infty} t^{\delta-1} (x_m - y_m + t)^{-\delta} e^{-|\xi'|(x_m - y_m + 2t)} dt + \right.$$

$$\begin{aligned}
& + \int_{x_m}^{\infty} \left(\frac{\partial}{\partial y_m} + |\xi'| \right) \overset{\square}{u}(\xi', y_m) dy_m \times \\
& \times \int_0^{\infty} t^{-\delta} (y_m - x_m + t)^{\delta-1} e^{-|\xi'|(y_m - x_m + 2t)} dt \Big].
\end{aligned}$$

Note that

$$\begin{aligned}
F_{\xi' \rightarrow x'}^{-1} e^{-x_m |\xi'|} &= c(m) \frac{x_m}{|x|^m}, \quad x_m > 0, \\
F_{\xi' \rightarrow x'}^{-1} \frac{\xi_i}{|\xi'|} e^{-x_m |\xi'|} &= c_1(m) \frac{x_i}{|x|^m}, \quad i = 1, \dots, m-1, \quad x_m > 0,
\end{aligned} \tag{3.5}$$

Therefore

$$A(u)(x) = \sum_{j=1}^m \int_{R^m} K_j(x-y) \partial y_j u(y) dy,$$

where

$$\begin{aligned}
K_j(x) &= c_j(m, \delta) [\ell_+(x_m) \overset{+}{K}_j(x) + \ell_-(x_m) \overset{-}{K}_j(x)], \\
\overset{+}{K}_j(x) &= \int_0^{\infty} t^{\delta-1} (x_m + t)^{-\delta} \frac{x_j dt}{((x_m + 2t)^2 + |x'|^2)^{\frac{m}{2}}}, \quad x_m > 0, \\
&\quad j = 1, \dots, m-1; \\
\overset{+}{K}_m(x) &= \int_0^{\infty} t^{\delta-1} (x_m + t)^{-\delta} \frac{x_m + 2t}{((x_m + 2t)^2 + |x'|^2)^{\frac{m}{2}}} dt, \quad x_m > 0, \\
\overset{-}{K}_j(x) &= \int_0^{\infty} t^{-\delta} (-x_m + t)^{\delta-1} \frac{x_j dt}{((-x_m + 2t)^2 + |x'|^2)^{\frac{m}{2}}}, \quad x_m < 0, \\
&\quad j = 1, \dots, m-1; \\
\overset{-}{K}_m(x) &= \int_0^{\infty} t^{-\delta} (-x_m + t)^{\delta-1} \frac{-x_m + 2t}{((-x_m + 2t)^2 + |x'|^2)^{\frac{m}{2}}} dt, \quad x_m < 0.
\end{aligned}$$

Using the transform $t = x_m t_1$, we rewrite the functions $\overset{+}{K}_j(x)$ as follows:

$$\begin{aligned}
\overset{+}{K}_j(x) &= \int_0^{\infty} t^{\delta-1} (1+t)^{-\delta} \frac{x_j dt}{(x_m^2 (1+2t)^2 + |x'|^2)^{\frac{m}{2}}}, \\
\overset{+}{K}_m(x) &= \int_0^{\infty} t^{\delta-1} (1+t)^{-\delta} \frac{x_m (1+2t)}{(x_m^2 (1+2t)^2 + |x'|^2)^{\frac{m}{2}}} dt.
\end{aligned}$$

Similarly

$$\begin{aligned}
\overset{-}{K}_j(x) &= \int_0^{\infty} t^{-\delta} (1+t)^{\delta-1} \frac{x_j dt}{(x_m^2 (1+2t)^2 + |x'|^2)^{\frac{m}{2}}}, \\
\overset{-}{K}_m(x) &= \int_0^{\infty} t^{-\delta} (1+t)^{\delta-1} \frac{-x_m (1+2t)}{(x_m^2 (1+2t)^2 + |x'|^2)^{\frac{m}{2}}} dt.
\end{aligned}$$

Simple transformations lead us to

$$\begin{aligned} K_{\pm j}(x) &= K_{\pm j,1} + K_{\pm j,2} \ln \frac{|x_m|}{|x|}, \quad j = 1, \dots, m-1, \\ K_{\pm m}(x) &= \int_{|x_m|}^{\infty} \frac{dt}{(t^2 + |x'|^2)^{\frac{m}{2}}} + K_{\pm m}(x), \end{aligned}$$

where $K_{\pm j,1}$, $K_{\pm j,2}$, $K_{\pm m}$ are positively homogeneous functions of order $1 - m$ which are infinitely differentiable on $R^m \setminus \{0\}$, and $K_{\pm m}(0, x') = 0$.

The theorem is proved for the case $0 < \operatorname{Re} \delta < 1$. The case $-1 < \operatorname{Re} \delta < 0$ is treated similarly.

If $\operatorname{Re} \delta = 0$, then we have to consider a pseudodifferential operator with the symbol

$$A(\xi) = (\xi_m + i|\xi'|)^{\varepsilon + \delta} (\xi_m - i|\xi'|)^{-\varepsilon - \delta}$$

and the theorem is proved by passing to the limit. ■

In a similar manner we prove our next theorem.

Let $m \geq 2$. Then the pseudodifferential operator (3.1) with the symbol

$$A(\xi) = \ln(\xi_m + i|\xi'|) - \ln(\xi_m - i|\xi'|)$$

can be represented as the singular integral operator (3.2), where

$$\begin{aligned} K(x) &= \frac{\partial}{\partial x_m} \left[c_1(m) \frac{|x_m|}{|x|^m} + c_2(m) \frac{|x_m|}{|x|^m} \ln \frac{|x_m|}{|x|} \right] + \\ &+ \sum_{j=1}^{m-1} \sum_{q=0}^1 \frac{\partial}{\partial x_j} \left[\left(\ell_+ \left(\frac{|x_m|}{|x|} \right) Q_{j,q}^+(x) + \ell_- \left(\frac{|x_m|}{|x|} \right) Q_{j,q}^-(x) \right) \ln^q \frac{|x_m|}{|x|} \right], \end{aligned}$$

and $Q_{\pm j,q}$ are positively homogeneous functions of order $1 - m$ which are infinitely differentiable on $R^m \setminus \{0\}$.

Let

$$\begin{aligned} H(u)(\xi) \frac{1}{\pi i} &\equiv \int_R \frac{u(\xi', \eta_m)}{\eta_m - \xi_m} d\eta_m, \\ P_+ &= \frac{1}{2}(I + H), \quad P_- = \frac{1}{2}(I - H), \\ \omega_+(\xi) &= (\xi_m + i|\xi'|)^{\delta}, \quad \omega_-(\xi) = (\xi_m - i|\xi'|)^{\delta}. \end{aligned}$$

We know (see, for example, [3]) that

$$F(\ell_+ u) = P_+(F(u)), \quad F(\ell_- u) = P_-(F(u)). \quad (3.6)$$

There is

Let $m \geq 2$, n ($n \geq 0$) be a natural number and $-1 < \operatorname{Re} \delta < 1$. Then the operator

$$M(u)(x) = F_{\xi \rightarrow x}^{-1} (\xi_m + i|\xi'|)^{-\delta} \times \\ \times \frac{1}{\pi i} \int_R \frac{(\eta_m + i|\xi'|)^\delta (\ln(\eta_m + i|\xi'|) - \ln(\xi_m + i|\xi'|))^n}{\eta_m - \xi_m} F(u)(\xi', \eta_m) d\eta_m, \quad (3.7) \\ u \in \mathcal{G}(R^m),$$

can be represented as the integral operator

$$M(u)(x) = \begin{cases} \sum_{p=0}^n c_p(m, \delta) \int_{R^m} |x-y|^{-m} \left(\frac{|y_m|}{x_m}\right)^{-\delta} \times \\ \quad \times \ln^p \frac{|y_m|}{x_m} u(y) dy + \delta_{n0} u(x), & x_m > 0, \\ -\delta_{n0} u(x), & x_m < 0, \end{cases} \quad (3.8)$$

where δ_{nk} is the Kronecker symbol.

Proof. Let $0 < \operatorname{Re} \delta < 1$. Assume first that $n = 0$ and consider the operator

$$B(u)(x) = F_{\xi \rightarrow x}^{-1} \omega_+^{-1}(\xi) H(\omega_+ F(u))(\xi). \quad (3.9)$$

If $x_m < 0$, then by (3.3) and (3.6)

$$B(u)(x) = 2F_{\xi \rightarrow x}^{-1} \omega_+^{-1} P_+(\omega_+ F(u))(\xi) - u(x) = \\ = 2c(m, \delta) F_{\xi' \rightarrow x'}^{-1} \int_{-\infty}^{\infty} \ell_+(x_m - t)(x_m - t)^{\delta-1} e^{-|\xi'|(x_m - t)} dt \times \\ \times \ell_+(t) \int_{-\infty}^{\infty} \ell_+(t - y_m)(t - y_m)^{-\delta} e^{-|\xi'|(t - y_m)} \times \\ \times \left(\frac{\partial}{\partial y_m} + |\xi'| \right) \bar{u}(\xi', y_m) dy_m - u(x) = -u(x).$$

If however $x_m > 0$, then by (3.3) and (3.6) we get

$$B(u)(x) = -2F_{\xi \rightarrow x}^{-1} \omega_+^{-1} P_-(\omega_+ F(u))(\xi) + u(x) = \\ = c_1(m, \delta) F_{\xi' \rightarrow x'}^{-1} \int_{-\infty}^{\infty} \ell_+(x_m - t)(x_m - t)^{\delta-1} e^{-|\xi'|(x_m - t)} dt \times \\ \times \ell_-(t) \int_{-\infty}^{\infty} \ell_+(t - y_m)(t - y_m)^{-\delta} e^{-|\xi'|(t - y_m)} \times \\ \times \left(\frac{\partial}{\partial y_m} + |\xi'| \right) \bar{u}(\xi', y_m) dy_m + u(x) = \\ = c_1(m, \delta) F_{\xi' \rightarrow x'}^{-1} \int_{-\infty}^0 (x_m - t)^{\delta-1} e^{-|\xi'|(x_m - t)} dt \times \\ \times \int_{-\infty}^t (t - y_m)^{-\delta} e^{-|\xi'|(t - y_m)} \left(\frac{\partial}{\partial y_m} + |\xi'| \right) \bar{u}(\xi', y_m) dy_m + u(x).$$

Changing of the integration order leads us to

$$\begin{aligned}
B(u)(x) &= c_1(m, \delta) F_{\xi' \rightarrow x'}^{-1} \int_{-\infty}^0 \left(\frac{\partial}{\partial y_m} + |\xi'| \right) \bar{u}(\xi', y_m) dy_m \times \\
&\quad \times \int_{y_m}^0 (x_m - t)^{\delta-1} (t - y_m)^{-\delta} e^{-|\xi'| |x_m - y_m|} dt + u(x) = \\
&= c_1(m, \delta) F_{\xi' \rightarrow x'}^{-1} \int_{-\infty}^0 \left(\frac{\partial}{\partial y_m} + |\xi'| \right) \bar{u}(\xi', y_m) dy_m \times \\
&\quad \times \int_0^1 \left(\frac{x_m - y_m}{-y_m} - t \right)^{\delta-1} t^{-\delta} e^{-|\xi'| |x_m - y_m|} dt + u(x).
\end{aligned}$$

Integration by parts gives us

$$\begin{aligned}
B(u)(x) &= c_2(m, \delta) F_{\xi' \rightarrow x'}^{-1} \int_{-\infty}^0 \bar{u}(\xi', y_m) dy_m \times \\
&\quad \times \frac{x_m}{y_m^2} \int_0^1 \left(\frac{x_m - y_m}{-y_m} - t \right)^{\delta-2} t^{-\delta} e^{-|\xi'| |x_m - y_m|} dt + u(x).
\end{aligned}$$

Hence, taking into account that

$$\frac{x_m}{y_m^2} \int_0^1 \left(\frac{x_m - y_m}{-y_m} - t \right)^{\delta-2} t^{-\delta} dt = c \left(\frac{x_m}{|y_m|} \right)^\delta (x_m - y_m)^{-1},$$

we find by equality (3.5) that

$$B(u)(x) = c(m, \delta) \int_{R^m} |x - y|^{-m} \left(\frac{|y_m|}{x_m} \right)^{-\delta} u(y) dy + u(x), \quad x_m > 0.$$

For $n > 0$ equality (3.8) is obtained by differentiating equality (3.9) n -times with respect to the parameter δ .

The proof of the theorem for the case $-1 < \operatorname{Re} \delta < 0$ is similar.

The case $\operatorname{Re} \delta = 0$ is proved by passing to the limit. ■

By the same technique we prove our next

Let $m \geq 2$, n ($n \geq 0$) be a natural number and $-1 < \operatorname{Re} \delta < 1$. Then the operator

$$\begin{aligned}
M(u)(x) &= F_{\xi \rightarrow x}^{-1} (\xi_m - i|\xi'|)^{-\delta} \times \\
&\quad \times \frac{1}{\pi i} \int_R \frac{(\eta_m - i|\xi'|)^\delta (\ln(\eta_m - i|\xi'|) - \ln(\xi_m - i|\xi'|))^n}{\eta_m - \xi_m} F(u)(\xi', \eta_m) d\eta_m, \\
&\quad u \in \mathcal{G}(R^m),
\end{aligned}$$

can be represented as the integral operator

$$M(u)(x) = \begin{cases} \delta_{n0}u(x), & x_m > 0, \\ \sum_{p=0}^n c_p(m, \delta) \int_{R_+^m} |x-y|^{-m} \left(\frac{y_m}{|x_m|}\right)^{-\delta} \times \\ \quad \times \ln^p \frac{y_m}{|x_m|} u(y) dy - \delta_{n0}u(x), & x_m < 0. \end{cases}$$

Put

$$\Gamma = \{x : x \in R^m, x_m = 0\}.$$

A function u defined on $R^m \setminus \Gamma$ belongs to the space $H_{\alpha, \beta}^\nu(R^m \setminus \Gamma)$ ($0 < \nu, \alpha < 1, \beta > 0, \alpha + \beta < m$) iff

- (i) $\forall x \in R^m \setminus \Gamma \quad |u(x)| \leq c|x_m|^{-\alpha}(1+|x|)^{-\beta};$
- (ii) $\forall x \in R^m \setminus \Gamma, \quad \forall y \in B(x, \frac{1}{2}|x_m|)$

$$|u(x) - u(y)| \leq c|x_m|^{-(\alpha+\nu)}(1+|x|)^{-\beta}|x-y|^\nu.$$

The norm in $H_{\alpha, \beta}^\nu(R^m \setminus \Gamma)$ is defined by

$$\begin{aligned} \|u\| &= \sup_{x \in R^m \setminus \Gamma} |x_m|^\alpha (1+|x|)^\beta |u(x)| + \\ &+ \sup_{\substack{x \in R^m \setminus \Gamma \\ y \in B(x, \frac{1}{2}|x_m|)}} |x_m|^{\alpha+\nu} (1+|x|)^\beta \frac{|u(x) - u(y)|}{|x-y|^\nu}. \end{aligned}$$

$H_{\alpha, \beta}^\nu(R^m \setminus \Gamma)$ is a Banach space.

The following theorem on the boundedness of a singular integral operator

$$\begin{aligned} A(u)(x) &= \int_{R^m} K(x, x-y)u(y) dy, \\ K(x, z) &= f\left(x, \frac{z}{|z|}\right)|z|^{-m} \end{aligned} \tag{3.10}$$

in a Hölder space with weight is proved in [5].

Let the characteristic f of the singular integral operator (3.10) defined on $(R^m \setminus \Gamma) \times (S(0, 1) \setminus \Gamma)$ satisfy the conditions:

- (a) $\forall x \in R^m \setminus \Gamma \quad \int_{S(0,1)} f(x, z) d_z S = 0;$
- (b) $\forall x \in R^m \setminus \Gamma, \quad \forall z \in S(0, 1) \setminus \Gamma \quad |f(x, z)| \leq c|z_m|^{-\sigma} \quad (0 \leq \sigma \leq \alpha);$
- (c) $\forall x, y \in R^m \setminus \Gamma, \quad \forall z, \theta, \omega \in S(0, 1) \setminus \Gamma$

$$\begin{aligned} |f(x, z) - f(y, z)| &\leq c|x-y|^\nu (\min(|x_m|, |y_m|))^{-\nu} |z_m|^{-\sigma}, \\ |f(x, \theta) - f(x, \omega)| &\leq c|\theta - \omega|^{\nu_1} (\min(|\theta_m|, |\omega_m|))^{-\nu_1 - \sigma}, \\ \nu_1 &> \nu, \quad \nu_1 + \sigma < 1. \end{aligned}$$

Then operator (3.10) is bounded in the space $H_{\alpha, \beta}^\nu(R^m \setminus \Gamma)$.

By virtue of this theorem and the Calderon–Zygmund theorem [1] we can readily formulate the conditions whose fulfillment will make the singular integral operators from Theorems 3.1–3.3 bounded in the spaces $H_{\alpha,\beta}^{\nu}(R^m \setminus \Gamma)$ and $L_p(R^m)$.

Introduce the notation:

$$\begin{aligned} \bar{H}_{\alpha,\beta}^{\nu}(R^m) &= \{u : u \in H_{\alpha,\beta}^{\nu}(R^m \setminus \Gamma), \text{ supp } u \in \bar{R}_{\pm}\}, \\ L_p^{\pm}(R^m) &= \{u : u \in L_p(R^m), \text{ supp } u \in \bar{R}_{\pm}\}. \end{aligned}$$

Let the integral operator M be defined by

$$M(u)(x) = \begin{cases} \int_{R_-^m} |x-y|^{-m} \left(\frac{|y_m|}{x_m}\right)^{-\delta} \ln^s \frac{|y_m|}{x_m} u(y) dy, & x_m > 0, \\ 0, & x_m < 0, \end{cases}$$

where s is a non-negative integer, $-1 < \text{Re } \delta < 1$. Then

(i) If $\text{Re } \delta < \alpha < 1 + \text{Re } \delta$, $\alpha + \beta < m + \text{Re } \delta$, then the integral operator M is bounded from the space $\bar{H}_{\alpha,\beta}^{\nu}(R^m)$ into the space $\bar{H}_{\alpha,\beta}^{\nu}(R^m)$;

(ii) If $\text{Re } \delta < \frac{1}{p} < 1 + \text{Re } \delta$, then M is bounded from the space $L_p^-(R^m)$ into the space $L_p^+(R^m)$ ($p > 1$).

Proof. (i) is proved in the same manner as Theorem 3.6. (ii) is proved in a rather simple way by using the Hölder integral inequality. Indeed, let $\delta_1 = \text{Re } \delta \leq 0$ and $\frac{1}{p} < 1 + \text{Re } \delta$. Choose a positive number γ from the conditions

$$\begin{cases} \gamma q < 1 + \text{Re } \delta, \\ \gamma p > -\text{Re } \delta, \\ \gamma p < 1 - \text{Re } \delta \end{cases} \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

By the Hölder inequality we have

$$\begin{aligned} |A(u)(x)| &\leq \left(\int_{R_-^m} |x-y|^{-m} \left(\frac{|y_m|}{x_m}\right)^{\delta_1} \left| \ln^s \frac{|y_m|}{x_m} \right| |y_m|^{\gamma p} |u(y)|^p dy \right)^{\frac{1}{p}} \times \\ &\times \left(\int_{R_-^m} |x-y|^{-m} \left(\frac{|y_m|}{x_m}\right)^{\delta_1} \left| \ln^s \frac{|y_m|}{x_m} \right| |y_m|^{-\gamma q} dy \right)^{\frac{1}{q}} \leq \\ &\leq c x_m^{-\gamma} \left(\int_{R_-^m} |x-y|^{-m} \left(\frac{|y_m|}{x_m}\right)^{\delta_1} \left| \ln^s \frac{|y_m|}{x_m} \right| |y_m|^{\gamma p} |u(y)|^p dy \right)^{\frac{1}{p}}. \end{aligned}$$

Hence

$$\begin{aligned} \int_{R_+^m} |A(u)(x)|^p dx &\leq c \int_{R_-^m} |u(y)|^p dy \times \\ &\times |y_m|^{\gamma p} \int_{R_+^m} |x-y|^m x_m^{-\gamma p} \left(\frac{|y_m|}{x_m}\right)^{\delta_1} \left| \ln^s \frac{|y_m|}{x_m} \right| dx \leq c \int_{R_-^m} |u(y)|^p dy. \end{aligned}$$

The case $0 < \text{Re } \delta < 1$ can be treated similarly. ■

Consider the matrix singular integral operator

$$\begin{aligned} A(u)(x) &= au(x) + \int_{R^m} K(x-y)u(y) dy, & (3.11) \\ a &= \|a_{ij}\|_{n \times n}, \quad K(z) = \|K_{ij}(z)\|_{n \times n}, \\ u &= (u_1, \dots, u_n), \quad K(\lambda z) = \lambda^{-m} K(z) \quad (\lambda > 0). \end{aligned}$$

It will be assumed that $K \in C^\infty(R^m \setminus \{0\})$ and $\int_{|z|=1} K(z) dS = 0$.

Denote by $\Phi(A)(\xi)$ the symbolic matrix of the operator A and let δ_i ($i = 1, \dots, n$) be the expansion indices of the matrix $\Phi(A)$ (see Theorem 2.2).

Now we are ready to prove

(i) *If the symbolic matrix $\Phi(A)$ of the operator A is strongly elliptic and*

$$\begin{aligned} \max_i \operatorname{Re} \delta_i &< \frac{1}{p} < 1 + \min_i \operatorname{Re} \delta_i \quad (p > 1), \\ \max_i \operatorname{Re} \delta_i &< \alpha < 1 + \min_i \operatorname{Re} \delta_i, \quad \alpha + \beta < m + \min_i \operatorname{Re} \delta_i, \end{aligned}$$

then the operator $\ell_+ A$ ($(\ell_+ A)(u)(x) = \ell_+(x)A(u)(x)$) is invertible both in the space $[L_p^+(R^m)]^n$ and in the space $[\overset{+}{H}_{\alpha, \beta}^\nu(R^m)]^n$;

(ii) *In particular if the strongly elliptic symbolic matrix $\Phi(A)$ of the operator A is additionally assumed to be Hermitian or even, then the integral operator $\ell_+ A$ is invertible in spaces $[L_p^+(R^m)]^n$ ($p > 1$), $[\overset{+}{H}_{\alpha, \beta}^\nu(R^m)]^n$ ($0 < \nu, \alpha < 1, \beta > 0, \alpha + \beta < m$).*

Proof. By Theorem 2.2 the matrix $\Phi(A)$ admits the expansion

$$\Phi(A)(\xi) = cgA_-(\xi)\mathcal{D}_-(\xi)\mathcal{D}_+(\xi)A_+(\xi)g^{-1}, \quad (3.12)$$

where

$$\mathcal{D}_-(\xi) = B_-(\xi) \left(\frac{\xi_m - i|\xi'|}{|\xi'|} \right)^\delta, \quad \mathcal{D}_+(\xi) = \left(\frac{\xi_m + i|\xi'|}{|\xi'|} \right)^{-\delta} B_+^{-1}(\xi).$$

Denote by A_- the singular integral operator with the symbol $cgA_-(\xi)$, and by A_+ the singular integral operator with the symbol $A_+(\xi)g^{-1}$. Assuming further that M denotes the operator

$$M(u)(x) = F_{\xi' \rightarrow x}^{-1} \mathcal{D}_+^{-1} P_+ (\mathcal{D}_-^{-1} F(u))(\xi),$$

we make the composition

$$B = A_+^{-1} \circ M \circ A_-^{-1}. \quad (3.13)$$

From Theorems 2.2, 3.1–3.7 it follows that B is a bounded operator in the spaces $[L_p^+(R^m)]^n$ and $[\overset{+}{H}_{\alpha, \beta}^\nu(R^m)]^n$ (note that in the case (ii) we have $\operatorname{Re} \delta_i = 0$, $i = 1, \dots, n$).

By (3.12) and (3.13) it is easy to prove that B inverse to $\ell_+ A$. ■

Let M be an $(m - 1)$ -dimensional compact manifold without boundary of the class $C^{1,\delta}$ ($0 \leq \delta \leq 1$) in R^m . Put

$$d(x) \equiv d(x, M) = \inf_{y \in M} |x - y|, \quad x \in R^m.$$

A function u defined on $R^m \setminus M$ belongs to the space $H_{\alpha,\beta}^\nu(R^m \setminus M)$ ($0 \leq \nu, \alpha < 1, \beta \geq 0, \alpha + \beta < m$) iff

- (i) $\forall x \in R^m \setminus M \quad |u(x)| \leq cd^{-\alpha}(x)(1 + |x|)^{-\beta}$;
- (ii) $\forall x \in R^m \setminus M, \forall y \in B(x, \frac{1}{2}d(x))$

$$|u(x) - u(y)| \leq cd^{-(\nu+\alpha)}(x)(1 + |x|)^{-\beta}|x - y|^\nu.$$

The norm in the space $H_{\alpha,\beta}^\nu(R^m \setminus M)$ is defined by

$$\begin{aligned} \|u\| = & \sup_{x \in R^m \setminus M} d^\alpha(x)(1 + |x|)^\beta |u(x)| + \\ & + \sup_{\substack{x \in R^m \setminus M \\ y \in B(x, \frac{1}{2}d(x))}} d^{\nu+\alpha}(x)(1 + |x|)^\beta \frac{|u(x) - u(y)|}{|x - y|^\nu}. \end{aligned}$$

$H_{\alpha,\beta}^\nu(R^m \setminus M)$ is a Banach space.

Now present some theorems proved in [5].

Let $M \in C^{1,\delta}$ ($0 \leq \delta \leq 1$) and the characteristic f of the singular integral operator (3.10) be defined on $(R^m \setminus M) \times S(0, 1)$ and satisfy the conditions:

- (a) $\forall x \in R^m \setminus M, \forall z \in S(0, 1)$

$$|f(x, z)| \leq c, \quad \int_{S(0,1)} f(x, z) d_z S = 0;$$

- (b) $\forall x, y \in R^m \setminus M, \forall \theta, \omega \in S(0, 1)$

$$\begin{aligned} |f(x, \theta) - f(y, \theta)| & \leq c|x - y|^\nu (\min(d(x), d(y)))^{-\nu}, \\ |f(x, \theta) - f(x, \omega)| & \leq c|\theta - \omega|^{\nu_1}, \quad \nu_1 > \nu. \end{aligned}$$

Then operator (3.10) is bounded in the space $H_{\alpha,\beta}^\nu(R^m \setminus M)$.

Let the characteristic f of the singular integral operator (3.10) satisfy the conditions of Theorem 3.6 with $\sigma < \alpha$ and the first inequality of the condition (c) fulfilled in a stronger form

$$|f(x, z) - f(y, z)| \leq c|x - y|^{\nu_1} (\min(|x_m|, |y_m|))^{-\nu_1} |z_m|^{-\sigma}.$$

Assume, moreover, that the function a is represented as $a(x) = a(\infty) + \overset{\circ}{a}(x)$ where $\overset{\circ}{a} \in H_\lambda^{0,\nu_1}(R^m)$ ($\lambda > 0$) (see Definition 1.1). Then the integral operator

$$C(u)(x) = \int_{R^m} [a(x) - a(y)]K(x, x - y)u(y) dy$$

is completely continuous in the space $H_{\alpha,\beta}^\nu(R^m \setminus \Gamma)$.

Let $M \in C^{1,\delta}$ ($0 \leq \delta \leq 1$) and the characteristic f of the singular integral operator (3.10) satisfy the conditions of Theorem 3.9 with the first inequality of the condition (b) replaced by a stronger one

$$|f(x, \theta) - f(y, \theta)| \leq c|x - y|^{\nu_1} (\min(d(x), d(y)))^{-\nu_1}.$$

Let the function a satisfy the conditions of Theorem 3.10. Then the integral operator C is completely continuous in the space $H_{\alpha,\beta}^\nu(R^m \setminus M)$.

Let \mathcal{D} be a finite or infinite domain in R^m bounded by a compact manifold M without boundary of the class C^{1,ν_1} .

We shall consider the matrix singular integral operator

$$A(u)(x) = a(x)u(x) + \int_{\mathcal{D}} f\left(x, \frac{x-y}{|x-y|}\right) |x-y|^{-m} u(y) dy, \quad (3.14)$$

where

$$a(x) = \|a_{ij}(x)\|_{n \times n}, \quad f(x, z) = \|f_{ij}(x, z)\|_{n \times n}, \quad u = (u_1, \dots, u_n),$$

in the spaces $[H_{\alpha,\beta}^\nu(\mathcal{D})]^n$ ($0 < \alpha, \nu < 1$, $\nu < \nu_1$, $\beta > 0$, $\alpha + \beta < m$) and $[L_p(\mathcal{D}, (1 + |x|)^\gamma)]^n$ ($p > 1$, $-\frac{m}{p} < \gamma < \frac{m}{q}$, $q = \frac{p}{p-1}$),

$$u \in L_p(\mathcal{D}, (1 + |x|)^\nu) \equiv \int_{\mathcal{D}} |u(x)|^p (1 + |x|)^{\nu p} dx < \infty.$$

Taking into account the character of a bounded operator acting in a space with two norms, the theorems we have proved above enable us to prove

Let $a \in H_\lambda^{\nu_1}(\overline{\mathcal{D}})$, $f(x, \cdot) \in C^\infty(R^m \setminus \{0\})$, $\int_{S(0,1)} f(x, z) d_z S = 0$, $\partial_z^p f(\cdot, z) \in H_\lambda^{\nu_1}(\overline{\mathcal{D}})$, $|p| = 0, 1, \dots$ if \mathcal{D} is bounded. Let there exists $\lim_{|x| \rightarrow \infty} a(x) \equiv a(\infty)$, $\lim_{|x| \rightarrow \infty} f(x, z) = f(\infty, z)$ and $(a - a(\infty)) \in H_\lambda^{\nu_1}(\overline{\mathcal{D}})$, $\partial_z^p (f(\cdot, z) - f(\infty, z)) \in H_\lambda^{\nu_1}(\overline{\mathcal{D}})$, $|p| = 1, 2, \dots$ if \mathcal{D} is unbounded. Let, moreover, the determinant of the symbolic matrix $\Phi(A)(x, \xi)$ of the singular integral operator (3.14) be not zero, $\forall x \in M$ the matrix $\Phi(A)(x, \xi)$ be strongly elliptic and

$$\begin{aligned} \max_{i,x} \operatorname{Re} \delta_i(x) &< \frac{1}{p} < 1 + \min_{i,x} \operatorname{Re} \delta_i(x) \quad (p > 1), \\ \max_{i,x} \operatorname{Re} \delta_i(x) &< \alpha < 1 + \min_{i,x} \operatorname{Re} \delta_i(x), \\ \alpha + \beta &< m + \min_{i,x} \operatorname{Re} \delta_i(x). \end{aligned}$$

Then operator (3.14) is Noetherian both in the space $[L_p(\mathcal{D}, (1 + |x|)^\gamma)]^n$ and in the space $[H_{\alpha,\beta}^\nu(\mathcal{D})]^n$ and any solution of the equation

$$A(u)(x) = g(x), \quad g \in [L_p(\mathcal{D}, (1 + |x|)^\gamma)]^n \cap [H_{\alpha,\beta}^\nu(\mathcal{D})]^n \quad (3.15)$$

from the space $[L_p(\mathcal{D}, (1 + |x|)^\gamma)]^n$ belongs to the space $[L_p(\mathcal{D}, (1 + |x|)^\gamma)]^n \cap [H_{\alpha, \beta}^\nu(\mathcal{D})]^n$.

In order that equation (1.15) to be solvable it is necessary and sufficient that $(g, v) = 0$, where v is an arbitrary solution of the formally conjugate equation $A^*(v) = 0$ from the space $[L_p(\mathcal{D}, (1 + |x|)^{-\gamma})]^n \cap [H_{\alpha, \beta}^\nu(\mathcal{D})]^n$.

Proof. The fact that A is Noetherian is proved by constructing a regularizer. After investigating the invertibility of local representations of the operator A , the regularization problem is solved in the standard manner (see [3]). We take the finite covering of $\overline{\mathcal{D}}$ by open neighbourhoods Q_j ($j = 1, \dots, k$). In such neighbourhoods Q_j , where $Q_j \cap M \neq \emptyset$, the regularizer is constructed by means of an operator of form (3.13), whereas in inner neighbourhoods the regularizer is constructed by means of the singular integral operator B_1 with the symbol $[\Phi(A)(x, \xi)]^{-1}$. ■

For the investigation of multidimensional integral operators on manifolds with boundary in spaces L_p see [2], [3], [10], [11], [13].

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