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ON PERIODIC PROBLEMS WITH NONLINEAR CAPUTO TEMPERED FRACTIONAL DIFFERENTIAL EQUATIONS

Abstract. The main goal of this paper is to study the existence and uniqueness of periodic solutions for a problem with fractional differential equation involving the Caputo tempered fractional derivative. The proofs are based upon the coincidence degree theory of Mawhin. To show the efficiency of the stated result, two illustrative examples will be demonstrated.

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1 Introduction

Fractional calculus extends differentiation and integration to non-integer orders, gaining attention in theoretical studies and practical applications across research domains. Its versatility has made it a crucial tool in the field. In the previous decades, more and more researchers have paid their attention to fractional calculus, since they found that the fractional order integrals and derivatives were more suitable for the description of the phenomena in the real world, such as viscoelastic systems, dielectric polarization, electromagnetic waves, heat conduction, robotics, biological systems, finance and so on (see, e.g., [4, 5, 7, 17, 19, 20, 36, 37]). Owing to great efforts of researchers, there have been rapid developments on the theory of fractional calculus and its applications, including well-posedness, stability, bifurcation and chaos in fractional differential equations and their control. Recently, there has been a significant increase in research on fractional calculus, exploring various outcomes under different conditions and forms of fractional differential equations and inclusions [1–3]. For more details on the applications of fractional calculus, the reader is directed to the books of Herrmann [15], Hilfer [16], Kilbas *et al.* [18] and Samko *et al.* [34]. In [8, 9], Benchohra *et al.* demonstrated the existence, uniqueness and stability results for various classes of problems with different conditions with some form of extension of the well-known Hilfer fractional derivative which unifies the Riemann–Liouville and Caputo fractional derivatives.

Tempered fractional calculus has emerged as an important class of fractional calculus operators in recent years. This class can generalize various forms of fractional calculus and possesses analytic kernels, making it an extension of fractional calculus that can describe the transition between normal and anomalous diffusion. The definitions of fractional integration with weak singular and exponential kernels were initially established by Buschman in [12], and further elaboration on this topic can be found in [6, 21–23, 26, 27, 29, 33, 35]. Although the Caputo tempered fractional derivative has not been extensively explored in the literature, it holds the potential to significantly contribute to this field. By studying this derivative, we aim to better understand its properties and potential applications in this unique mathematical notion, thus advancing fractional calculus.

In [32], the authors investigated the following class of Caputo tempered fractional differential equation:

$$\begin{cases} \left({}^{C}_{\kappa_1}\mathfrak{D}^{\zeta,\varepsilon}_{t}y\right)(t) = f\Big(t, y(t), \left({}^{C}_{\kappa_1}\mathfrak{D}^{\zeta,\varepsilon}_{t}y\right)(t)\Big), \quad t \in \Xi := [\kappa_1, \kappa_2], \\ \imath y(\kappa_1) + \jmath y(\kappa_2) = \varsigma y(\eta) + \varrho, \end{cases}$$

where $0 < \zeta < 1$, $\varepsilon \geq 0$, ${}^{C}_{\kappa_1}\mathfrak{D}^{\zeta,\varepsilon}_{t}$ is the Caputo tempered fractional derivative, $f : \Xi \times \mathbb{R} \times \mathbb{R}$ is a continuous function, $\kappa_1 < \eta < \kappa_2 < +\infty$, $\imath, \jmath, \varrho, \varsigma$ are real constants.

In [22], the authors investigated the following class of Caputo tempered fractional differential equation with finite delay:

$$\begin{cases} \left({}^{C}_{0}\mathfrak{D}^{\zeta,\varepsilon}_{t}y\right)(t) = f\big(t, y_t, \mathfrak{D}^{\zeta}_{0}y(t)\big), \quad t \in \Theta := [0, \varpi], \\ y(t) = \wp(t), \quad t \in [-\kappa, 0], \\ \alpha_1 y(0) + \alpha_2 y(\varpi) = \alpha_3, \end{cases}$$

where $0 < \zeta < 1$, $\varepsilon \geq 0$, ${}^{C}_{0}\mathfrak{D}^{\zeta,\varepsilon}_{t}$ is the Caputo tempered fractional derivative, $f : \Theta \times C([-\kappa, 0], \mathbb{R}) \times \mathbb{R}$ is a continuous function, $\wp \in C([-\kappa, \varpi], \mathbb{R})$, $0 < \varpi < +\infty$, $\alpha_1, \alpha_2, \alpha_3$ are real constants, and $\kappa > 0$ is the time delay. The results are based on the fixed point theorems of Banach, Schauder and Schaefer. Notice that this problem include initial, terminal and anti-periodic problems, however the used method doesn't provide the results for the periodic problem.

The concept of the coincidence degree theory introduced by Mawhin [14, 25] has found extensive application in analyzing various categories of nonlinear differential equations. This approach proves to be particularly valuable when conventional methods like the fixed point principle are inapplicable. In [10, 11, 13, 30, 31], the utilization of coincidence degree theory yielded results for nonlinear differential equations of fractional order that would have been unattainable through other means such as the fixed point principle.

In [10], by using the coincidence degree theory of Mawhin, the authors studied the nonlinear pantograph fractional equations with Ψ-Hilfer fractional derivative:

$$\begin{cases} {}^{H}\mathfrak{D}_{\mathfrak{a}^+}^{\varrho,\beta;\Psi}y(t) = f\big(t, y(t), y(\varepsilon t)\big), & t \in (0, \varkappa], \\ \mathfrak{I}_{0^+}^{1-\nu,\Psi}y(0) = \mathfrak{I}_{0^+}^{1-\nu,\Psi}y(\varkappa), \end{cases}$$

where ${}^{H}\mathfrak{D}_{0^+}^{\varrho,\beta;\Psi}$ denote the Ψ-Hilfer fractional derivative of order $0 < \varrho \leq 1$, $0 < \varepsilon < 1$ and type $\beta \in [0,1]$. $\mathfrak{I}_{0^+}^{1-\nu,\Psi}$ is the Ψ-Riemann–Liouville fractional integral of order $1-\nu$, $(\nu = \varrho + \beta - \varrho\beta)$. Moreover, $f : (0, \varkappa] \times \mathfrak{R}^2 \to \mathfrak{R}$ is a given continuous function.

In this paper, we study the existence and uniqueness of periodic solutions for the problem with fractional differential equation involving the Caputo tempered fractional derivative:

$$\big({}_0^C\mathfrak{D}_t^{\varrho,\wp}y\big)(t) = f(t, y(t)), \quad t \in \nabla := [0, \varkappa], \tag{1.1}$$

$$y(0) = y(\varkappa) = 0, \tag{1.2}$$

where $0 < \varrho < 1$, $\wp \geq 0$, ${}_0^C\mathfrak{D}_t^{\varrho,\wp}$ is the Caputo tempered fractional derivative, $f : \nabla \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The organization of this paper is outlined as follows. In Section 2, specific notations and preliminary explanations regarding the tempered fractional derivatives utilized in this manuscript are introduced. Section 3 is dedicated to presenting the existence and uniqueness outcomes for the problem stated in equations (1.1), (1.2); these results are derived by using Mawhin's theory of coincidence degree. The final section offers illustrative examples that serve to reinforce the findings obtained in this study.

2 Preliminaries

We denote by $C(\nabla, \mathbb{R})$ the Banach space of all continuous functions from ∇ into $\mathbb{R}$ with the following norm:

$$\|f\|_\infty = \sup_{t\in\nabla} \big\{|f(t)|\big\}.$$

As usual, $AC(\nabla)$ denotes the space of absolutely continuous functionsfrom ∇ into $\mathbb{R}$. For any $\alpha \in \mathbb{N}^*$, we denote by $AC^\alpha(\nabla)$ the space defined by

$$AC^\alpha(\nabla) := \Big\{y : \nabla \to \mathbb{R} : \ \frac{d^\alpha}{dt^\alpha}\, y(t) \in AC(\nabla)\Big\}.$$

Consider the space $X_b^p(0, \varkappa)$ $(b \in \mathbb{R},\ 1 \leq p \leq \infty)$ of those real-valued Lebesgue measurable functions y on $[0, \varkappa]$ for which $\|y\|_{X_b^p} < \infty$, where the norm is defined by

$$\|y\|_{X_b^p} = \left(\int_0^\varkappa |t^b y(t)|^p \,\frac{dt}{t}\right)^{\frac{1}{p}} \quad (1 \leq p < \infty, \ \ b \in \mathbb{R}).$$

Definition 2.1 ([23, 29, 35])**.** Suppose that the real function y is piecewise continuous on $[0, \varkappa]$ and $y \in X_b^p(0, \varkappa)$, $\wp > 0$. Then the Riemann–Liouville tempered fractional integral of order ϱ is defined by

$${}_0\mathcal{I}_t^{\varrho,\wp}y(t) = e^{-\wp t}\ {}_0\mathcal{I}_t^{\varrho}(e^{\wp t}y(t)) = \frac{1}{\Gamma(\varrho)}\int_0^t \frac{e^{-\wp(t-s)}y(s)}{(t-s)^{1-\varrho}}\, ds, \tag{2.1}$$

where ${}_0\mathcal{I}_t^{\varrho}$ denotes the Riemann–Liouville fractional integral [18] defined by

$${}_0\mathcal{I}_t^{\varrho}y(t) = \frac{1}{\Gamma(\varrho)}\int_0^t \frac{y(s)}{(t-s)^{1-\varrho}}\, ds. \tag{2.2}$$

Obviously, the tempered fractional integral (2.1) reduces to the Riemann–Liouville fractional integral (2.2) if $\wp = 0$.

Definition 2.2 ([23, 29])**.** For $\alpha - 1 < \varrho < \alpha$; $\alpha \in \mathbb{N}^+$, $\wp \geq 0$, the Riemann–Liouville tempered fractional derivative is defined by

$$ {}_0\mathfrak{D}_t^{\varrho,\wp} y(t) = e^{-\wp t}\, {}_0\mathfrak{D}_t^{\varrho}(e^{\wp t} y(t)) = \frac{e^{-\wp t}}{\Gamma(\alpha-\varrho)} \frac{d^\alpha}{dt^\alpha} \int_0^t \frac{e^{\wp s} y(s)}{(t-s)^{\varrho-\alpha+1}}\, ds, $$

where ${}_0\mathfrak{D}_t^{\varrho}(e^{\wp t} y(t))$ denotes the Riemann–Liouville fractional derivative [18] given by

$$ {}_0\mathfrak{D}_t^{\varrho}(e^{\wp t} y(t)) = \frac{d^\alpha}{dt^\alpha}\left({}_0\mathcal{I}_t^{\alpha-\varrho}(e^{\wp t} y(t))\right) = \frac{1}{\Gamma(\alpha-\varrho)} \frac{d^\alpha}{dt^\alpha} \int_0^t \frac{(e^{\wp s} y(s))}{(t-s)^{\varrho-\alpha+1}}\, ds. $$

Definition 2.3 ([23, 35])**.** For $\alpha - 1 < \varrho < \alpha$; $\alpha \in \mathbb{N}^+, \wp \geq 0$, the Caputo tempered fractional derivative is defined as

$$ {}_0^C\mathfrak{D}^{\varrho,\wp} y(t) = e^{-\wp t}\, {}_0^C\mathfrak{D}^{\varrho}\big(e^{\wp t} y(t)\big) = \frac{e^{-\wp t}}{\Gamma(\alpha-\varrho)} \int_0^t \frac{1}{(t-s)^{\varrho-\alpha+1}} \frac{d^\alpha\, (e^{\wp s} y(s))}{ds^\alpha}\, ds, $$

where ${}_0^C\mathfrak{D}_t^{\varrho,\wp}\,(e^{\wp t} y(t))$ denotes the Caputo fractional derivative [18] given by

$$ {}_0^C\mathfrak{D}_t^{\varrho}(e^{\wp t} y(t)) = \frac{1}{\Gamma(\alpha-\varrho)} \int_0^t \frac{1}{(t-s)^{\varrho-\alpha+1}} \frac{d^\alpha (e^{\wp s} y(s))}{ds^\alpha}\, ds. $$

Lemma 2.1 ([23])**.** *For a constant* C*,*

$$ {}_0\mathfrak{D}_t^{\varrho,\wp} C = C e^{-\wp t}\; {}_0\mathfrak{D}_t^{\varrho} e^{\wp t}, \quad {}_0^C\mathfrak{D}_t^{\varrho,\wp} C = C e^{-\wp t}\; {}_0^C\mathfrak{D}_t^{\varrho} e^{\wp t}. $$

Obviously, ${}_0\mathfrak{D}_t^{\varrho,\wp}(C) \neq {}_0^C\mathfrak{D}_t^{\varrho,\wp}(C)$*, and* ${}_0^C\mathfrak{D}_t^{\varrho,\wp}(C)$ *is no longer equal to zero, being different from* ${}_0^C\mathfrak{D}_t^{\varrho}(C) \equiv 0$.

Lemma 2.2 ([23,35])**.** *Let* $y(t) \in AC^\alpha[0, \varkappa]$ *and* $\alpha - 1 < \varrho < \alpha$*. Then the Caputo tempered fractional derivative and the Riemann–Liouville tempered fractional integral have the composite properties*

$$ {}_0\mathcal{I}_t^{\varrho,\wp}\big[{}_0^C\mathfrak{D}_t^{\varrho,\wp} y(t)\big] = y(t) - \sum_{\imath=0}^{\alpha-1} e^{-\wp t} \frac{t^\imath}{\imath!} \left[\frac{d^\imath (e^{\wp t} y(t))}{dt^\imath}\Big|_{t=0}\right] $$

and

$$ {}_0^C\mathfrak{D}_t^{\varrho,\wp}\big[{}_0\mathcal{I}_t^{\varrho,\wp} y(t)\big] = y(t) \;\; \text{for} \;\; \varrho \in (0,1). $$

Theorem 2.1 ([24])**.** *Let* $y, \mathfrak{x} \in AC^\alpha(\nabla, \mathfrak{R})$*,* $\alpha - 1 < \varrho \leq \alpha$ $(\alpha \in \mathbb{N})$*,* $\wp \in [0, +\infty)$ *and* $\Psi \in C^\alpha(\nabla, \mathfrak{R})$ *be a non-decreasing function such that* $\Psi' \neq 0$ *on* ∇*. Then we have*

$$ {}_0^C\mathfrak{D}_{\Psi(t)}^{\varrho,\wp} y(t) = {}_0^C\mathfrak{D}_{\Psi(t)}^{\varrho,\wp} \mathfrak{x}(t) \Longleftrightarrow y(t) = \mathfrak{x}(t) + e^{-\wp \Psi(t)} \sum_{\imath=0}^{\alpha-1} c_\imath (\psi(t) - \psi(0))^\imath, \;\; t \in \nabla, $$

where

$$ c_\imath = \frac{1}{\imath!}\left[\Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^\imath \big(e^{\wp \Psi(t)}[y(t) - \mathfrak{x}(t)]\big)\right]_{t=0}. $$

Remark 2.1. If we put $\omega = y - \mathfrak{x} \in C^1(\nabla, \mathfrak{R})$, $\Psi(t) = t$ and $0 < \varrho \leq 1$, then we have

$$ {}_0^C\mathfrak{D}_t^{\varrho,\wp} \omega(t) = 0 \Longleftrightarrow \omega(t) = e^{-\wp t} \omega(0), \;\; t \in \nabla. $$

Definition 2.4 ([14, 25])**.** We consider the normed spaces $\Im$ and $\widehat{\Im}$. A Fredholm operator of index zero is a linear operator $\mho : \mathrm{Dom}(\mho) \subset \Im \to \widehat{\Im}$ such that

(a) $\mathfrak{dim}\ker\mho = \mathfrak{codim}\,\mathfrak{Img}\,\mho < +\infty$;

(b) $\mathfrak{Img}\,\mho$ is a closed subset of $\widehat{\Im}$.

By Definition 2.4, there exist continuous projectors $\widehat{\mho} : \widehat{\Im} \to \widehat{\Im}$ and $\overline{\mho} : \Im \to \Im$ satisfying

$$\mathfrak{Img}\,\mho = \ker\widehat{\mho}, \quad \ker\mho = \mathfrak{Img}\,\overline{\mho}, \quad \widehat{\Im} = \mathfrak{Img}\,\widehat{\mho} \oplus \mathfrak{Img}\,\mho, \quad \Im = \ker\overline{\mho} \oplus \ker\mho.$$

Thus the restriction of $\mho$ to $\operatorname{Dom}\mho \cap \ker\overline{\mho}$, denoted by $\mho_{\overline{\mho}}$, is an isomorphism onto its image.

Definition 2.5 ([14, 25])**.** Let $\mathfrak{Z} \subseteq \Im$ be a bounded subset and $\mho$ be a Fredholm operator of index zero with $\operatorname{Dom}\mho \cap \mathfrak{Z} \neq \varnothing$. Then the operator $\mathbb{k} : \overline{\mathfrak{Z}} \to \widehat{\Im}$ is said to be $\mho$-compact in $\overline{\mathfrak{Z}}$ if

(a) the mapping $\widehat{\mho}\mathbb{k} : \overline{\mathfrak{Z}} \to \widehat{\Im}$ is continuous and $\widehat{\mho}\mathbb{k}(\overline{\mathfrak{Z}}) \subseteq \widehat{\Im}$ is bounded;

(b) the mapping $(\mho_{\overline{\mho}})^{-1}(id - \widehat{\mho})\mathbb{k} : \overline{\mathfrak{Z}} \to \Im$ is completely continuous.

Lemma 2.3 ([28])**.** *Let $\Im$, $\widehat{\Im}$ be the Banach spaces, $\mathfrak{Z} \subset \Im$ be a bounded open set symmetric with respect to $0 \in \mathfrak{Z}$. Suppose that $\mho : \operatorname{Dom}\mho \subset \Im \to \widehat{\Im}$ is a Fredholm operator of index zero with $\operatorname{Dom}\mho \cap \overline{\mathfrak{Z}} \neq \varnothing$ and $\mathbb{k} : \Im \to \widehat{\Im}$ is a $\mho$-compact operator on $\overline{\mathfrak{Z}}$. Assume, moreover, that*

$$\mho y - \mathbb{k}y \neq -\kappa(\mho y + \mathbb{k}(-y))$$

for any $y \in \operatorname{Dom}\mho \cap \partial\mathfrak{Z}$ and any $\kappa \in (0, 1]$, where $\partial\mathfrak{Z}$ is the boundary of $\mathfrak{Z}$ with respect to $\Im$. If these conditions are fulfilled, then there exists at least one solution of the equation $\mho y = \mathbb{k}y$ on $\operatorname{Dom}\mho \cap \overline{\mathfrak{Z}}$.

3 Main results

Let the spaces

$$\Im = \Big\{ y \in C(\nabla, \mathfrak{R}) : \; y(t) = {}_0\mathcal{I}_t^{\varrho,\wp}\mathfrak{x}(t) : \; \mathfrak{x} \in C(\nabla, \mathfrak{R}) \Big\}$$

and

$$\widehat{\Im} = C(\nabla, \mathfrak{R}),$$

be endowed with the norms

$$\|y\|_{\Im} = \|y\|_{\widehat{\Im}} = \|y\|_\infty = \sup_{t\in\nabla} |y(t)|.$$

Let $\mho : \operatorname{Dom}\mho \subseteq \Im \to \widehat{\Im}$ be given by

$$\mho y := {}_0^C\mathfrak{D}_t^{\varrho,\wp} y, \tag{3.1}$$

where

$$\operatorname{Dom}\mho = \Big\{ y \in \Im : \; {}_0^C\mathfrak{D}_t^{\varrho,\wp} y \in \widehat{\Im} : y(0) = y(\varkappa) = 0 \Big\}.$$

Lemma 3.1. *Using the definition of $\mho$ given in* (3.1)*, we have*

$$\ker\mho = \big\{ y \in \Im : \; y(t) = 0, \; t \in \nabla \big\}$$

and

$$\mathfrak{Img}\,\mho = \Big\{ \mathfrak{x} \in \widehat{\Im} : \; \int_0^{\varkappa} (\varkappa - s)^{\varrho - 1} e^{-\wp(\varkappa - s)} \mathfrak{x}(s)\, ds = 0 \Big\}.$$

Proof. By Remark 2.1, we have that for all $y \in \operatorname{Dom}\mho \subset \Im$ the equation $\mho y = {}_0^C\mathfrak{D}_t^{\varrho,\wp} y = 0$ in ∇ has a solution of the form

$$y(t) = e^{-\wp t} y(0) = 0, \;\; t \in \nabla,$$

then

$$\ker\mho = \big\{ y \in \Im : \; y(t) = 0, \; t \in \nabla \big\}.$$

For $\mathfrak{x} \in \mathfrak{Img}\,\mho$, there exists $y \in \mathrm{Dom}\,\mho$ such that $\mathfrak{x} = \mho y \in \widehat{\Im}$. Using Lemma 2.2, we obtain

$$y(t) = e^{-\wp t} y(0) + {}_0\mathcal{I}_t^{\varrho,\wp}\mathfrak{x}(t) = e^{-\wp t} y(0) + \frac{1}{\Gamma(\varrho)} \int_0^t (t-s)^{\varrho-1} e^{-\wp(t-s)} \mathfrak{x}(s)\, ds$$

for every $t \in \nabla$. Since $y \in \mathrm{Dom}\,\mho$, we have $y(0) = y(\varkappa) = 0$. Thus

$$\int_0^{\varkappa} (\varkappa - s)^{\varrho-1} e^{-\wp(\varkappa-s)} \mathfrak{x}(s)\, ds = 0.$$

Furthermore, if $\mathfrak{x} \in \widehat{\Im}$ and satisfies

$$\int_0^{\varkappa} (\varkappa - s)^{\varrho-1} e^{-\wp(\varkappa-s)} \mathfrak{x}(s)\, ds = 0,$$

then for any $y(t) = {}_0\mathcal{I}_t^{\varrho,\wp}\mathfrak{x}(t)$, using Lemma 2.2, we get $\mathfrak{x}(t) = {}_0^C\mathfrak{D}_t^{\varrho,\wp} y(t)$. Therefore,

$$y(\varkappa) = y(0) = 0,$$

which implies that $y \in \mathrm{Dom}\,\mho$. So, $\mathfrak{x} \in \mathfrak{Img}\,\mho$. Hence

$$\mathfrak{Img}\,\mho = \left\{ \mathfrak{x} \in \widehat{\Im} : \int_0^{\varkappa} (\varkappa - s)^{\varrho-1} e^{-\wp(\varkappa-s)} \mathfrak{x}(s)\, ds = 0 \right\}.$$

This completes the proof. □

Lemma 3.2. *Let $\mho$ be defined by (3.1). Then $\mho$ is a Fredholm operator of index zero, and the linear continuous projector operators $\widehat{\mho} : \widehat{\Im} \to \widehat{\Im}$ and $\overline{\mho} : \Im \to \Im$ can be written as*

$$\widehat{\mho}(\mathfrak{x}) = \frac{1}{\varpi(\varkappa)} \int_0^{\varkappa} (\varkappa - s)^{\varrho-1} e^{-\wp(\varkappa-s)} \mathfrak{x}(s)\, ds$$

and

$$\overline{\mho}(y) = 0,$$

where

$$\varpi(\varkappa) = \int_0^{\varkappa} (\varkappa - s)^{\varrho-1} e^{-\wp(\varkappa-s)}\, ds.$$

Furthermore, the operator $\mho_{\overline{\mho}}^{-1} : \mathfrak{Img}\,\mho \to \Im \cap \ker \overline{\mho}$ can be written as

$$\mho_{\overline{\mho}}^{-1}(\mathfrak{x})(t) = {}_0\mathcal{I}_t^{\varrho,\wp}\mathfrak{x}(t), \quad t \in \nabla.$$

Proof. Obviously, for each $\mathfrak{x} \in \widehat{\Im}$, $\widehat{\mho}^2\mathfrak{x} = \widehat{\mho}\mathfrak{x}$ and $\mathfrak{x} = \widehat{\mho}(\mathfrak{x}) + (\mathfrak{x} - \widehat{\mho}(\mathfrak{x}))$, where $(\mathfrak{x} - \widehat{\mho}(\mathfrak{x})) \in \ker \widehat{\mho} = \mathfrak{Img}\,\mho$. Using the fact that $\mathfrak{Img}\,\mho = \ker \widehat{\mho}$ and $\widehat{\mho}^2 = \widehat{\mho}$, we have $\mathfrak{Img}\,\mho \cap \mathfrak{Img}\,\widehat{\mho} = 0$. So,

$$\widehat{\Im} = \mathfrak{Img}\,\mho \oplus \mathfrak{Img}\,\widehat{\mho}.$$

By the same way we get that $\mathfrak{Img}\,\overline{\mho} = \ker \mho$ and $\overline{\mho}^2 = \overline{\mho}$. It follows for each $y \in \Im$ that $y = (y - \overline{\mho}(y)) + \overline{\mho}(y)$, then $\Im = \ker \overline{\mho} + \ker \mho$. Clearly, we have $\ker \overline{\mho} \cap \ker \mho = 0$. So,

$$\Im = \ker \overline{\mho} \oplus \ker \mho.$$

Using Rank-nullity theorem, we get

$$\mathfrak{codim}\,\mathfrak{Img}\,\mho = \dim\widehat{\Im} - \dim\mathfrak{Img}\,\mho = \left[\dim\ker\widehat{\mho} + \dim\mathfrak{Img}\,\widehat{\mho}\right] - \dim\mathfrak{Img}\,\mho,$$

and since $\mathfrak{Img}\,\mho = \ker\widehat{\mho}$, we have

$$\mathfrak{codim}\,\mathfrak{Img}\,\mho = \dim\mathfrak{Img}\,\widehat{\mho}. \tag{3.2}$$

Using also Rank-nullity theorem, we obtain

$$\dim\ker\mho = \dim\Im - \dim\mathfrak{Img}\,\mho = \mathfrak{codim}\,\mathfrak{Img}\mho,$$

which implies that

$$\dim\ker\mho = \mathfrak{codim}\,\mathfrak{Img}\,\mho. \tag{3.3}$$

By (3.2) and (3.3),

$$\dim\ker\mho = \mathfrak{codim}\,\mathfrak{Img}\,\mho = \dim\mathfrak{Img}\,\widehat{\mho},$$

and since $\dim\mathfrak{Img}\,\widehat{\mho} < \infty$, we have

$$\dim\ker\mho = \mathfrak{codim}\,\mathfrak{Img}\,\mho < \infty.$$

And since $\mathfrak{Img}\,\mho$ is a closed subset of $\widehat{\Im}$, $\mho$ is a Fredholm operator of index zero.

Now, we will show that the inverse of $\mho|_{\mathrm{Dom}\,\mho\cap\ker\overline{\mho}}$ is $\mho_{\overline{\mho}}^{-1}$. Effectively, for $\mathfrak{x}\in\mathfrak{Img}\,\mho$, by Lemma 2.2, we have

$$\mho\mho_{\overline{\mho}}^{-1}(\mathfrak{x}) = {}_0^C\mathfrak{D}_t^{\varrho,\wp}({}_0\mathcal{I}_t^{\varrho,\wp}\mathfrak{x}) = \mathfrak{x}. \tag{3.4}$$

Furthermore, for $y\in\mathrm{Dom}\,\mho\cap\ker\overline{\mho}$ we get

$$\mho_{\overline{\mho}}^{-1}(\mho(y(t))) = {}_0\mathcal{I}_t^{\varrho,\wp}({}_0^C\mathfrak{D}_t^{\varrho,\wp}y(t)) = y(t) - e^{-\wp t}y(0), \quad t\in\nabla.$$

Using the fact that $y\in\mathrm{Dom}\,\mho\cap\ker\overline{\mho}$, we have

$$y(0) = 0.$$

Thus

$$\mho_{\overline{\mho}}^{-1}\mho(y) = y. \tag{3.5}$$

Using (3.4) and (3.5) together, we get $\mho_{\overline{\mho}}^{-1} = (\mho|_{\mathrm{Dom}\,\mho\cap\ker\overline{\mho}})^{-1}$, which completes the demonstration. □

Let us assume the following hypothesis:

(H1) Assume $f(t,0)\neq 0$ for $t\in\nabla$, and there exists a positive constant γ with

$$|f(t,y) - f(t,\overline{y})| \leqslant \gamma|y-\overline{y}|$$

for every $t\in\nabla$ and $y,\overline{y}\in\mathfrak{R}$.

Define $\Bbbk:\Im\to\widehat{\Im}$ by

$$\Bbbk y(t) := f(t,y(t)), \quad t\in\nabla.$$

Then problem (1.1), (1.2) is equivalent to the problem $\mho y = \Bbbk y$.

Lemma 3.3. *Suppose that* (H1) *is satisfied, then for any bounded open set* $\mathfrak{Z}\subset\Im$*, the operator* $\Bbbk$ *is* $\mho$*-compact.*

Proof. We consider for $\mathfrak{K} > 0$ the bounded open set $\mathfrak{Z} = \{y \in \Im : \|y\|_{\Im} < \mathfrak{K}\}$. We split the proof into three steps:

Step 1: *$\mathcal{QN}$ is continuous.*

Let $(\mathfrak{x}_\alpha)_{\alpha\in\mathbb{N}}$ be a sequence such that $\mathfrak{x}_\alpha \to \mathfrak{x}$ in $\widehat{\Im}$, then for each $t \in \nabla$, we have

$$\big|\mathcal{QN}(\mathfrak{x}_\alpha)(t) - \mathcal{QN}(\mathfrak{x})(t)\big| \leqslant \frac{1}{\varpi(\varkappa)} \int_0^{\varkappa} (\varkappa - s)^{\varrho-1} e^{-\wp(\varkappa-s)} \big|\mathbb{k}(\mathfrak{x}_\alpha)(s) - \mathbb{k}(\mathfrak{x})(s)\big| \, ds.$$

By (H1), we have

$$\big|\mathcal{QN}(\mathfrak{x}_\alpha)(t) - \mathcal{QN}(\mathfrak{x})(t)\big| \leqslant \frac{\gamma}{\varpi(\varkappa)} \int_0^{\varkappa} (\varkappa - s)^{\varrho-1} e^{-\wp(\varkappa-s)} |\mathfrak{x}_\alpha(s) - \mathfrak{x}(s)| \, ds \leqslant \gamma \|\mathfrak{x}_\alpha - \mathfrak{x}\|_{\widehat{\Im}}.$$

Thus for each $t \in \nabla$, we get

$$\big|\mathcal{QN}(\mathfrak{x}_\alpha)(t) - \mathcal{QN}(\mathfrak{x})(t)\big| \to 0 \;\text{ as }\; \alpha \to +\infty,$$

and hence

$$\big\|\mathcal{QN}(\mathfrak{x}_\alpha) - \mathcal{QN}(\mathfrak{x})\big\|_{\widehat{\Im}} \to 0 \;\text{ as }\; \alpha \to +\infty.$$

We deduce that $\mathcal{QN}$ is continuous.

Step 2: *$\mathcal{QN}(\overline{\mathfrak{Z}})$ is bounded.*

For $t \in \nabla$ and $\mathfrak{x} \in \overline{\mathfrak{Z}}$, we have

$$\begin{aligned}
\big|\mathcal{QN}(\mathfrak{x})(t)\big| &\leqslant \frac{1}{\varpi(\varkappa)} \int_0^{\varkappa} (\varkappa - s)^{\varrho-1} e^{-\wp(\varkappa-s)} |\mathbb{k}(\mathfrak{x})(s)| \, ds \\
&\leqslant \frac{1}{\varpi(\varkappa)} \int_0^{\varkappa} (\varkappa - s)^{\varrho-1} e^{-\wp(\varkappa-s)} \big|f(s,\mathfrak{x}(s)) - f(s,0)\big| \, ds \\
&\qquad + \frac{1}{\varpi(\varkappa)} \int_0^{\varkappa} (\varkappa - s)^{\varrho-1} e^{-\wp(\varkappa-s)} |f(s,0)| \, ds \\
&\leqslant f^* + \frac{\gamma}{\varpi(\varkappa)} \int_0^{\varkappa} (\varkappa - s)^{\varrho-1} e^{-\wp(\varkappa-s)} |\mathfrak{x}(s)| \, ds \leqslant f^* + \gamma\mathfrak{K},
\end{aligned}$$

where $f^* = \|f(\,\cdot\,,0)\|_\infty$.

Thus

$$\|\mathcal{QN}(\mathfrak{x})\|_{\widehat{\Im}} \leqslant f^* + \gamma\mathfrak{K}.$$

So, $\mathcal{QN}(\overline{\mathfrak{Z}})$ is a bounded set in $\widehat{\Im}$.

Step 3: *$\mho_{\overline{\mho}}^{-1}(id - \widehat{\mho})\mathbb{k} : \overline{\mathfrak{Z}} \to \Im$ is completely continuous.*

We will use the Arzelà–Ascoli theorem, so we have to show that $\mho_{\overline{\mho}}^{-1}(id - \widehat{\mho})\mathbb{k}(\overline{\mathfrak{Z}}) \subset \Im$ is equicontinuous and bounded. First, for any $y \in \overline{\mathfrak{Z}}$ and $t \in \nabla$, we get

$$\begin{aligned}
\mho_{\overline{\mho}}^{-1}\big(\mathbb{k}y(t) - \mathcal{QN}y(t)\big) &= {}_0\mathcal{I}_t^{\varrho,\wp}\left[f(t,y(t)) - \frac{1}{\varpi(\varkappa)} \int_0^{\varkappa} (\varkappa - s)^{\varrho-1} e^{-\wp(\varkappa-s)} f(s,y(s)) \, ds\right] \\
&= \frac{1}{\Gamma(\varrho)} \int_0^{t} (t - s)^{\varrho-1} e^{-\wp(t-s)} f(s,y(s)) \, ds - \frac{\varpi(t)}{\Gamma(\varrho)\varpi(\varkappa)} \int_0^{\varkappa} (\varkappa - s)^{\varrho-1} e^{-\wp(\varkappa-s)} f(s,y(s)) \, ds.
\end{aligned}$$

For all $y \in \overline{\mathfrak{Z}}$ and $t \in \nabla$, we get

$$\begin{aligned}
\left|\mho_{\varpi}^{-1}(id-\widehat{\mho})\Bbbk y(t)\right| &\leqslant \frac{1}{\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}e^{-\wp(t-s)}\left|f(s,y(s))-f(s,0)\right| ds \\
&\quad + \frac{1}{\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}e^{-\wp(t-s)}|f(s,0)|\, ds \\
&\quad + \frac{\varpi(t)}{\Gamma(\varrho)\varpi(\varkappa)}\int_0^\varkappa (\varkappa-s)^{\varrho-1}e^{-\wp(\varkappa-s)}\left|f(s,y(s))-f(s,0)\right| ds \\
&\quad + \frac{\varpi(t)}{\Gamma(\varrho)\varpi(\varkappa)}\int_0^\varkappa (\varkappa-s)^{\varrho-1}e^{-\wp(\varkappa-s)}|f(s,0)|\, ds, \\
&\leqslant \frac{2f^*\varpi(t)}{\Gamma(\varrho)} + \frac{\gamma}{\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}e^{-\wp(t-s)}|y(s)|\, ds \\
&\quad + \frac{\gamma\varpi(t)}{\Gamma(\varrho)\varpi(\varkappa)}\int_0^\varkappa (\varkappa-s)^{\varrho-1}e^{-\wp(\varkappa-s)}|y(s)|\, ds \\
&\leqslant \frac{2\varkappa^\varrho}{\Gamma(\varrho+1)}\left[f^*+\gamma\mathfrak{K}\right].
\end{aligned}$$

Therefore,

$$\left\|\mho_{\varpi}^{-1}(id-\widehat{\mho})\Bbbk y\right\|_{\mathfrak{I}} \leqslant \frac{2\varkappa^\varrho}{\Gamma(\varrho+1)}\left[f^*+\gamma\mathfrak{K}\right].$$

This means that $\mho_{\varpi}^{-1}(id-\widehat{\mho})\Bbbk(\overline{\mathfrak{Z}})$ is uniformly bounded in $\mathfrak{I}$. It remains to show that $\mho_{\varpi}^{-1}(id-\widehat{\mho})\Bbbk(\overline{\mathfrak{Z}})$ is equicontinuous.

For $0 < t_1 < t_2 \leqslant \varkappa$, $y \in \overline{\mathfrak{Z}}$, we have

$$\begin{aligned}
&\left|\mho_{\varpi}^{-1}(id-\widehat{\mho})\Bbbk y(t_2) - \mho_{\varpi}^{-1}(id-\widehat{\mho})\Bbbk y(t_1)\right| \\
&\leqslant \frac{1}{\Gamma(\varrho)}\int_0^{t_1} \left|(t_2-s)^{\varrho-1}e^{-\wp(t_2-s)} - (t_1-s)^{\varrho-1}e^{-\wp(t_1-s)}\right| |f(s,y(s))|\, ds \\
&\quad + \frac{1}{\Gamma(\varrho)}\int_{t_1}^{t_2} (t_2-s)^{\varrho-1}e^{-\wp(t_2-s)}|f(s,y(s))|\, ds \\
&\quad + \frac{|\varpi(t_2)-\varpi(t_1)|}{\Gamma(\varrho)\varpi(\varkappa)}\int_0^\varkappa (\varkappa-s)^{\varrho-1}e^{-\wp(\varkappa-s)}|f(s,y(s))|\, ds \\
&\leqslant \frac{1}{\Gamma(\varrho)}\int_0^{t_1} \left|(t_2-s)^{\varrho-1}e^{-\wp(t_2-s)} - (t_1-s)^{\varrho-1}e^{-\wp(t_1-s)}\right|\left|f(s,y(s))-f(s,0)\right| ds \\
&\quad + \frac{1}{\Gamma(\varrho)}\int_0^{t_1} \left|(t_2-s)^{\varrho-1}e^{-\wp(t_2-s)} - (t_1-s)^{\varrho-1}e^{-\wp(t_1-s)}\right| |f(s,0)|\, ds \\
&\quad + \frac{1}{\Gamma(\varrho)}\int_{t_1}^{t_2} (t_2-s)^{\varrho-1}e^{-\wp(t_2-s)}\left|f(s,y(s))-f(s,0)\right| ds
\end{aligned}$$

$$+\frac{1}{\Gamma(\varrho)}\int_{t_1}^{t_2}(t_2-s)^{\varrho-1}e^{-\wp(t_2-s)}|f(s,0)|\,ds$$
$$+\frac{|\varpi(t_2)-\varpi(t_1)|}{\Gamma(\varrho)\varpi(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho-1}e^{-\wp(\varkappa-s)}\big|f(s,y(s))-f(s,0)\big|\,ds$$
$$+\frac{|\varpi(t_2)-\varpi(t_1)|}{\Gamma(\varrho)\varpi(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho-1}e^{-\wp(\varkappa-s)}|f(s,0)|\,ds$$
$$\leqslant\frac{\gamma\mathfrak{K}+f^*}{\Gamma(\varrho)}\int_0^{t_1}\Big|(t_2-s)^{\varrho-1}e^{-\wp(t_2-s)}-(t_1-s)^{\varrho-1}e^{-\wp(t_1-s)}\Big|\,ds$$
$$+\frac{2\gamma\mathfrak{K}+f^*}{\Gamma(\varrho)}\int_{t_1}^{t_2}(t_2-s)^{\varrho-1}e^{-\wp(t_2-s)}\,ds+\frac{\gamma\mathfrak{K}+f^*}{\Gamma(\varrho)}\big|\varpi(t_2)-\varpi(t_1)\big|.$$

The operator $\mho_{\overline{\mho}}^{-1}(id-\widehat{\mho})\Bbbk(\overline{\mathfrak{Z}})$ is equicontinuous in $\Im$ because the right-hand side of the above inequality tends to zero as $t_1\to t_2$ and the limit is independent of y. The Arzelà–Ascoli theorem implies that $\mho_{\overline{\mho}}^{-1}(id-\widehat{\mho})\Bbbk(\overline{\mathfrak{Z}})$ is relatively compact in $\Im$. As a consequence of steps 1 to 3, we get that $\Bbbk$ is $\mho$-compact in $\overline{\mathfrak{Z}}$. □

Lemma 3.4. *Assume* (H1). *If the condition*

$$\frac{\gamma\varkappa^{\varrho}}{\Gamma(\varrho+1)}<\frac{1}{2}\tag{3.6}$$

is satisfied, then there exists $\mathfrak{W}>0$ *independent of* κ *such that*

$$\mho(y)-\Bbbk(y)=-\kappa\big[\mho(y)+\Bbbk(-y)\big]\Longrightarrow\|y\|_{\Im}\leqslant\mathfrak{W},\quad\kappa\in(0,1].$$

Proof. Let $y\in\Im$ satisfy

$$\mho(y)-\Bbbk(y)=-\kappa\mho(y)-\kappa\Bbbk(-y),$$

then

$$\mho(y)=\frac{1}{1+\kappa}\Bbbk(y)-\frac{\kappa}{1+\kappa}\Bbbk(-y).$$

So, from the expression of $\mho$ and $\Bbbk$, for any $t\in\nabla$ we get

$$\mho y(t)={}_0^C\mathfrak{D}_t^{\varrho,\wp}y(t)=\frac{1}{1+\kappa}f(t,y(t))-\frac{\kappa}{1+\kappa}f(t,-y(t)).$$

By Lemma 2.2, we have

$$y(t)=e^{-\wp t}y(0)+\frac{1}{\kappa+1}\Big[{}_0\mathcal{I}_t^{\varrho,\wp}(f(s,y(s)))(t)-\kappa\,{}_0\mathcal{I}_t^{\varrho,\wp}(f(s,-y(s)))(t)\Big].$$

Thus for every $t\in\nabla$ we obtain

$$|y(t)|\leqslant|y(0)|+\frac{1}{(\kappa+1)\Gamma(\varrho)}\int_0^t(t-s)^{\varrho-1}e^{-\wp(t-s)}|f(s,y(s))|\,ds$$
$$+\frac{\kappa}{(\kappa+1)\Gamma(\varrho)}\int_0^t(t-s)^{\varrho-1}e^{-\wp(t-s)}\big|f(s,-y(s))\big|\,ds$$
$$\leqslant|y(0)|+\frac{1}{(\kappa+1)\Gamma(\varrho)}\int_0^t(t-s)^{\varrho-1}e^{-\wp(t-s)}\big|f(s,y(s))-f(s,0)\big|\,ds$$

$$
\begin{aligned}
&+\frac{1}{(\kappa+1)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}e^{-\wp(t-s)}|f(s,0)|\,ds \\
&+\frac{\kappa}{(\kappa+1)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}e^{-\wp(t-s)}\big|f(s,-y(s))-f(s,0)\big|\,ds \\
&+\frac{\kappa}{(\kappa+1)\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}e^{-\wp(t-s)}|f(s,0)|\,ds \\
\leqslant\ & |y(0)|+\frac{2f^*\varkappa^\varrho}{\Gamma(\varrho+1)}+\frac{2\gamma\varkappa^\varrho}{\Gamma(\varrho+1)}\,\|y\|_{\Im},
\end{aligned}
$$

hence

$$\|y\|_{\Im}\leqslant |y(0)|+\frac{2f^*\varkappa^\varrho}{\Gamma(\varrho+1)}+\frac{2\gamma\varkappa^\varrho}{\Gamma(\varrho+1)}\,\|y\|_{\Im}.$$

We deduce that

$$\|y\|_{\Im}\leqslant\frac{|y(0)|+\frac{2f^*\varkappa^\varrho}{\Gamma(\varrho+1)}}{[1-\frac{2\gamma\varkappa^\varrho}{\Gamma(\varrho+1)}]}:=\mathfrak{W}.$$

□

Lemma 3.5. *If conditions* (H1) *and* (3.6) *are satisfied, then there exists a bounded open set* $\mathfrak{Z}\subset\Im$ *with*

$$\mho(y)-\Bbbk(y)\neq-\kappa\big[\mho(y)+\Bbbk(-y)\big],$$

for any $y\in\partial\mathfrak{Z}$ *and any* $\kappa\in(0,1]$.

Proof. Using Lemma 3.4, there exists a positive constant $\mathfrak{W}$ independent of κ such that if y satisfies

$$\mho(y)-\Bbbk(y)=-\kappa[\mho(y)+\Bbbk(-y)],\quad \kappa\in(0,1],$$

then $\|y\|_{\Im}\leqslant\mathfrak{W}$. So, if

$$\mathfrak{Z}=\big\{y\in\Im;\ \|y\|_{\Im}<\vartheta\big\} \tag{3.7}$$

such that $\vartheta>\mathfrak{W}$, we deduce that

$$\mho(y)-\Bbbk(y)\neq-\kappa\big[\mho(y)-\Bbbk(-y)\big]$$

for all $y\in\partial\mathfrak{Z}=\{y\in\Im;\ \|y\|_{\Im}=\vartheta\}$ and $\kappa\in(0,1]$. □

Theorem 3.1. *Assume* (H1) *and* (3.6) *hold, then problem* (1.1), (1.2) *has a unique solution in* $\operatorname{Dom}\mho\cap\overline{\mathfrak{Z}}$.

Proof. It is clear that the set $\mathfrak{Z}$ defined in (3.7) is symmetric, $0\in\mathfrak{Z}$ and $\Im\cap\overline{\mathfrak{Z}}=\overline{\mathfrak{Z}}\neq\varnothing$. In addition, by Lemma 3.5, we have

$$\mho(y)-\Bbbk(y)\neq-\kappa[\mho(y)-\Bbbk(-y)]$$

for each $y\in\Im\cap\partial\mathfrak{Z}=\partial\mathfrak{Z}$ and each $\kappa\in(0,1]$. By Lemma 2.3, problem (1.1), (1.2) has at least one solution in $\operatorname{Dom}\mho\cap\overline{\mathfrak{Z}}$.

Now, we prove the uniqueness result. Suppose that problem (1.1), (1.2) has two different solutions $y_1,y_2\in\operatorname{Dom}\mho\cap\overline{\mathfrak{Z}}$. Then for each $t\in\nabla$ we have

$${}_0^C\mathfrak{D}_t^{\varrho,\wp}y_1(t)=f(t,y_1(t)),\quad {}_0^C\mathfrak{D}_t^{\varrho,\wp}y_2(t)=f(t,y_2(t))$$

and

$$y_1(0)=y_1(\varkappa)=0,\quad y_2(0)=y_2(\varkappa)=0.$$

Let

$$\mathfrak{U}(t)=y_1(t)-y_2(t)\ \text{ for all }\ t\in\nabla.$$

Then
$$\mho\mathfrak{U}(t) = {}_0^C\mathfrak{D}_t^{\varrho,\wp}\mathfrak{U}(t) = {}_0^C\mathfrak{D}_t^{\varrho,\wp}y_1(t) - {}_0^C\mathfrak{D}_t^{\varrho,\wp}y_2(t) = f(t, y_1(t)) - f(t, y_2(t)). \tag{3.8}$$
On the other hand, by Lemma 2.2, we have
$${}_0\mathcal{I}_t^{\varrho,\wp}{}_0^C\mathfrak{D}_t^{\varrho,\wp}\mathfrak{U}(t) = \mathfrak{U}(t) - e^{-\wp t}\mathfrak{U}(0) = \mathfrak{U}(t),$$
By (3.8) and (H1), for all $t \in \nabla$ we have
$$\begin{aligned}|\mathfrak{U}(t)| = \left|{}_0\mathcal{I}_t^{\varrho,\wp}{}_0^C\mathfrak{D}_t^{\varrho,\wp}\mathfrak{U}(t)\right| &\leq {}_0\mathcal{I}_t^{\varrho,\wp}\Big[\left|f(s, y_1(s)) - f(s, y_2(s))\right|\Big](t)\\ &\leq \frac{1}{\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}e^{-\wp(t-s)}\left|f(s, y_1(s)) - f(s, y_2(s))\right| ds\\ &\leq \frac{\gamma}{\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}e^{-\wp(t-s)}|\mathfrak{U}(s)|\, ds \leq \frac{\gamma\varkappa^{\varrho}}{\Gamma(\varrho+1)}\|\mathfrak{U}\|_{\mathfrak{I}}\end{aligned}$$

Therefore,
$$\|\mathfrak{U}\|_{\mathfrak{I}} \leqslant \frac{\gamma\varkappa^{\varrho}}{\Gamma(\varrho+1)}\|\mathfrak{U}\|_{\mathfrak{I}}.$$
Hence, by (3.6), we conclude that
$$\|\mathfrak{U}\|_{\mathfrak{I}} = 0.$$
As a result, for any $t \in \nabla$, we get
$$\mathfrak{U}(t) = 0 \Longrightarrow y_1(t) = y_2(t).$$
□

4 Examples

Example 4.1. Consider for NFDE the following problem:
$$\begin{gathered}{}_0^C\mathfrak{D}_t^{\frac{1}{3};2}y(t) = f(t, y(t)), \quad t \in \nabla := [0, 2],\\ y(0) = y(2) = 0,\end{gathered}$$
where
$$f(t, y(t)) = \frac{\sqrt{t^2+1}}{\ln(1+t)}\frac{1}{45} + \sin(y(t)).$$
Here $\varrho = \frac{1}{3}$, $\wp = 2$ and $\varkappa = 2$. It is clear that the function $f \in C([0, 2], \mathfrak{R})$. Let $y, \overline{y} \in \mathfrak{R}$ and $t \in \nabla$, then
$$|f(t, y) - f(t, \overline{y})| \leqslant \frac{1}{45}|y - \overline{y}|,$$
which implies that (H1) is satisfied with $\gamma = \frac{1}{45}$. Furthermore, by some simple calculations, we see that
$$\frac{\gamma\varkappa^{\varrho}}{\Gamma(\varrho+1)} \approx 0.0314 < \frac{1}{2}.$$
Using Theorem 3.1, our problem has a unique solution.

Example 4.2. Consider for NFDE the following problem:
$$\begin{gathered}{}_0^C\mathfrak{D}_t^{\frac{1}{2};4}y(t) = \widehat{f}(t, y(t)), \quad t \in \nabla := [0, 1],\\ y(0) = y(1) = 0,\end{gathered}$$
where
$$\widehat{f}(t, y(t)) = \ln(\sqrt{t} + e) + \frac{1}{37e^{t+2}(1 + y(t))}.$$

Here $\varrho = \frac{1}{2}$, $\wp = 4$ and $\varkappa = 1$. It is easy to see that $\widehat{f} \in C([0,1],\mathfrak{R})$. Let $y, \overline{y} \in \mathfrak{R}$ and $t \in \nabla$, then

$$|\widehat{f}(t,y) - \widehat{f}(t,\overline{y})| \leqslant \frac{1}{37e^2}\,|y-\overline{y}|.$$

Hence the assumption (H1) is satisfied with $\gamma = \frac{1}{37e^2}$. By simple calculations, we see that

$$\frac{\gamma\varkappa^{\varrho}}{\Gamma(\varrho+1)} \approx 0.00413 < \frac{1}{2}\,.$$

So, by Theorem 3.1, our problem has a unique solution.

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