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# A STUDY ON SOME NEW CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS WITH RETARDATION AND ANTICIPATION

**Abstract.** In this paper, we present some results on the existence and uniqueness for the class of problems for nonlinear fractional differential equations with some new generalized conformable derivatives with retardation and anticipation. For our proofs, we employ some suitable fixed point theorems. Finally, we provide two illustrative examples.

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**Key words and phrases.** Implicit differential equations, Riemann–Liouville fractional derivative, Caputo fractional derivative, fractional differential equations, improved conformable fractional derivative, existence, retardation, anticipation, fixed point.

# 1 Introduction

Fractional calculus has long been an appealing research topic in functional space theory due to its applicability in the modeling and practical understanding of natural phenomena. Indeed, various applications in viscoelasticity and electrochemistry have been explored. Noninteger derivatives of fractional order have been effectively applied to generalize the fundamental laws of nature. For more details, we recommend [1–3, 10, 12, 15, 21–23, 25–30] and the references therein. Many papers and monographs have lately been published in which the authors studied a wide range of results for various forms of fractional differential equations, inclusions with different types of conditions. One may see the papers [5–9, 17, 18] and the references therein.

Recently in [20], Khalil *et al.* gave a novel definition of fractional derivative which is a natural extension to the standard first derivative. The conformable fractional derivative is natural and satisfies most of the properties that the classical integral derivative has, such as linearity, product rule, quotient rule, power rule, chain rule, and it bring us great convenience when it is applied for modeling many physical problems [4, 11, 24].

Very recently, in [13], F. Gao and C. Chi claimed that there are still shortcomings or disadvantages for the conformable derivative and in order to overcome this difficulty, they proposed an improved conformable fractional derivative. The benefit of the improved conformable derivative is that its physical behavior is closer than the conformable fractional derivative of Riemann–Liouville and Caputo. This improved conformable fractional derivative has a great potential in simulating various physical problems that typically employ the fractional derivative of Riemann–Liouville and Caputo.

In [16], the authors introduced a new conformable fractional derivative which obeys all the abovementioned classical properties. It can be considered as a generalization of the conformable derivatives introduced by Khalil *et al.* [20] and Katugampola in [19]. Furthermore, because there are currently few studies in the literature focusing on the generalized conformable fractional derivative, we have an opportunity to make a substantial contribution to the field. We think that by researching the conformable fractional derivative, we may obtain a better grasp of its traits and capabilities, as well as contribute to the continued advancement of fractional calculus.

In this paper, we study the existence and uniqueness of solutions for the impulsive initial value problem with nonlinear fractional differential equation involving the improved Caputo-type conformable fractional derivative with retardation and anticipation:

$$ {}^{C}_{t_k}\widetilde{\mathcal{T}}_{\alpha} y(t) = f(t, y^t(\,\cdot\,)), \ \ t \in \overline{\Theta} := \Theta \setminus \{t_1, \dots, t_m\}, \ \ \Theta := [\kappa_1, \kappa_2], \tag{1.1} $$

$$ \Delta y\big|_{t=t_k} = \Phi_k(y(t_k^-)); \ \ k = 1, \dots, m, \tag{1.2} $$

$$ y(t) = \chi(t), \ \ t \in [\kappa_1 - r, \kappa_1], \ \ r > 0, \tag{1.3} $$

$$ y(t) = \widetilde{\chi}(t), \ \ t \in [\kappa_2, \kappa_2 + \delta], \ \ \delta > 0, \tag{1.4} $$

where $0 < \alpha < 1$, and ${}^{C}_{t_k}\widetilde{\mathcal{T}}_{\alpha}$ is the improved Caputo-type conformable fractional derivative defined in [13], $f : \Theta \times \mathcal{PC}([-r, \delta]) \to \mathbb{R}$ and $\Phi_k : \mathbb{R} \to \mathbb{R}$ are the given functions to be specified later, $\widetilde{\chi} \in C([\kappa_2, \kappa_2 + \delta], \mathbb{R})$ and $\chi \in C([\kappa_1 - r, \kappa_1], \mathbb{R})$ with

$$ \chi(\kappa_1) = 0, \ \ \kappa_1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = \kappa_2 < \infty, $$
$$ y(t_k^+) = \lim_{\epsilon \to 0^+} y(t_k + \epsilon) \ \text{ and } \ y(t_k^-) = \lim_{\epsilon \to 0^-} y(t_k + \epsilon) $$

represent the right- and left-hand limits of $y(t)$ at $t = t_k$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$. By $y^t$ we denote the element of $\mathcal{PC}([-r, \delta])$ defined by

$$ y^t(s) = y(t + s), \ \ s \in [-r, \delta]. $$

Next, we consider the impulsive boundary value problem with implicit nonlinear fractional differential equation involving a generalization of the conformable fractional derivative with retardation and anticipation:

$$ {}^{e}_{t_k}\mathcal{D}^{\alpha} y(t) = g(t, y^t(\,\cdot\,), {}^{e}_{t_k}\mathcal{D}^{\alpha} y(t)), \ \ t \in \overline{\Theta} := \Theta \setminus \{t_1, \dots, t_m\}, \ \ \Theta := [\kappa_1, \kappa_2], \tag{1.5} $$

$$\Delta y\big|_{t=t_k} = \Psi_k(y(t_k^-)); \quad k=1,\dots,m, \tag{1.6}$$

$$\vartheta_1 y(\kappa_1) + \vartheta_2 y(\kappa_2) = \vartheta_3, \tag{1.7}$$

$$y(t) = \chi(t), \quad t\in[\kappa_1 - r, \kappa_1], \quad r>0, \tag{1.8}$$

$$y(t) = \widetilde{\chi}(t), \quad t\in[\kappa_2, \kappa_2+\delta], \quad \delta>0, \tag{1.9}$$

where $0<\alpha<1$, and ${}^{e}_{t_k}\mathcal{D}^{\alpha}$ is a new generalized conformable fractional derivative defined in [16], $g:\Theta\times\mathcal{PC}([-r,\delta])\times\mathbb{R}\to\mathbb{R}$ and $\Psi_k:\mathbb{R}\to\mathbb{R}$ are the given functions to be specified later, $\widetilde{\chi}\in C([\kappa_2,\kappa_2+\delta],\mathbb{R})$ and $\chi\in C([\kappa_1-r,\kappa_1],\mathbb{R})$, $\vartheta_1,\vartheta_2,\vartheta_3\in\mathbb{R}$ such that $\vartheta_1+\vartheta_2\neq 0$.

The structure of this paper is as follows. Section 2 presents certain notations and preliminaries about the improved conformable fractional derivatives used throughout this paper. In Section 3, we present the existence and uniqueness result for problem (1.1)–(1.4) that is based on the Banach contraction principle. Section 4 deals with the existence result of problem (1.5)–(1.9) using Schauder's fixed point theorem. In the last section, illustrative examples are provided in support of the obtained results.

## 2 Preliminaries

First, we give the definitions and the notations that we will use throughout this paper. We denote by $C(\Theta,\mathbb{R})$ the Banach space of all continuous functions from $\Theta$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup_{t\in\Theta}\{|y(t)|\}.$$

$AC(\Theta,\mathbb{R})$ is the space of absolutely continuous functions on $\Theta$. Consider the Banach space

$$\mathcal{PC}(\Theta,\mathbb{R}) = \Big\{y:\Theta\to\mathbb{R}: \ y\in C((t_k,t_{k+1}],\mathbb{R}); \ k=0,\dots,m, \ \text{and there exist} \ y(t_k^-) \ \text{and} \ y(t_k^+); \ k=1,\dots,m, \ \text{with} \ y(t_k^-)=y(t_k)\Big\},$$

with the norm

$$\|y\|_{\mathcal{PC}} = \sup_{t\in\Theta}|y(t)|.$$

Consider the Banach space

$$\mathcal{PC}([-r,\delta]) = \Big\{y:[-r,\delta]\to\mathbb{R}: \ \tau\to y(\tau)\in C((\tau_k,\tau_{k+1}],\mathbb{R}); \ k=0,\dots,m, \ \text{and there exist} \ y(\tau_k^-) \ \text{and} \ y(\tau_k^+); \ k=1,\dots,m, \ \text{with} \ y(\tau_k^-)=y(\tau_k) \ \text{and} \ \tau_k=t_k-t \ \text{for each} \ t\in(t_k,t_{k+1}]\Big\}$$

with the norm

$$\|y\|_{[-r,\delta]} = \sup_{t\in[-r,\delta]}|y(t)|.$$

Also, define the following space:

$$\mathcal{C} = \Big\{y:[\kappa_1-r,\kappa_2+\delta]\mapsto\mathbb{R}: \ y\,|_{[\kappa_1-r,\kappa_1]}\in C([\kappa_1-r,\kappa_1]), \ y\big|_{[\kappa_1,\kappa_2]}\in\mathcal{PC}(\Theta,\mathbb{R}) \ \text{and} \ y\big|_{[\kappa_2,\kappa_2+\delta]}\in C([\kappa_2,\kappa_2+\delta])\Big\}$$

with the norm

$$\|y\|_{\mathcal{C}} = \sup\big\{|y(t)|: \ \kappa_1-r\le t\le\kappa_2+\delta\big\}.$$

Consider the space $X_b^p(\kappa_1,\kappa_2)$ ($b\in\mathbb{R}$, $1\le p\le\infty$) of those real-valued Lebesgue measurable functions $f$ on $[\kappa_1,\kappa_2]$ for which $\|f\|_{X_b^p}<\infty$, where the norm is defined by

$$\|f\|_{X_b^p} = \left(\int_{\kappa_1}^{\kappa_2}|t^b f(t)|^p\,\frac{dt}{t}\right)^{\frac{1}{p}} \quad (1\le p<\infty, \ b\in\mathbb{R}).$$

**Definition 2.1** (The conformable fractional derivative [20]). Let $f:[0,+\infty)\to\mathbb{R}$ be a given function, then the conformable fractional derivative of $f$ of order $\alpha$ is defined by

$$\mathcal{T}_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$$

for $t>0$ and $\alpha\in(0,1]$. If $f$ is $\alpha$-differentiable in some $(0,a)$, $a>0$, and $\lim\limits_{t\to 0+}\mathcal{T}_a(f)(t)$ exists, then define

$$\mathcal{T}_\alpha(f)(0)=\lim_{t\to 0+}\mathcal{T}_\alpha(f)(t).$$

If the conformable fractional derivative of $f$ of order $\alpha$ exists, then we simply say that $f$ is $\alpha$-differentiable. It is easy to see that if $f$ is differentiable, then

$$\mathcal{T}_\alpha(f)(t)=t^{1-\alpha}f'(t).$$

**Definition 2.2** (Generalized conformable fractional derivative [20]). Let $f:[0,+\infty)\to\mathbb{R}$ be a given function, then the new generalized conformable fractional derivative of $f$ of order $\alpha$ is defined by

$${}^{e}\mathcal{D}^\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon e^{(\alpha-1)t})-f(t)}{\varepsilon}$$

for $t>0$ and $\alpha\in(0,1]$.

**Definition 2.3** (The improved Caputo-type conformable fractional derivative [13]). Let $f:\mathbb{R}\to\mathbb{R}$ be a given function. The improved Caputo-type conformable fractional derivative of $f$ of order $\alpha$ is defined by

$${}^{C}_{a}\widetilde{\mathcal{T}}_\alpha(f)(t)=\lim_{\varepsilon\to 0}\left[(1-\alpha)(f(t)-f(a))+\alpha\,\frac{f(t+\varepsilon(t-a)^{1-\alpha})-f(t)}{\varepsilon}\right],$$

where $-\infty<a<t<+\infty, a$ is a given number and $\alpha\in[0,1]$.

**Definition 2.4** (The improved Riemann–Liouville-type conformable fractional derivative [13]). Let $f:\mathbb{R}\to\mathbb{R}$ be a given function. The improved Riemann–Liouville-type conformable fractional derivative of $f$ of order $\alpha$ is defined by

$${}^{RL}_{a}\widetilde{\mathcal{T}}_\alpha(f)(t)=\lim_{\varepsilon\to 0}\left[(1-\alpha)f(t)+\alpha\frac{f(t+\varepsilon(t-a)^{1-\alpha})-f(t)}{\varepsilon}\right],$$

where $-\infty<a<t<+\infty$, $a$ is a given number and $\alpha\in[0,1]$.

**Lemma 2.1** ([13]). *If $\alpha\in[0,1]$, $f$ and $g$ are two $\alpha$-differentiable functions at a point $t$ and $m$, $n$ are two given numbers, then the improved conformable fractional derivative satisfies the following properties:*

- ${}^{C}_{a}\widetilde{\mathcal{T}}_\alpha(\lambda)=0$ *for any constant* $\lambda$*;*
- ${}^{C}_{a}\widetilde{\mathcal{T}}_\alpha(mf+ng)=m{}^{C}_{a}\widetilde{\mathcal{T}}_\alpha(f)+n{}^{C}_{a}\widetilde{\mathcal{T}}_\alpha(g)$*;*
- ${}^{RL}_{a}\widetilde{\mathcal{T}}_\alpha(mf+ng)=m{}^{RL}_{a}\widetilde{\mathcal{T}}_\alpha(f)+n{}^{RL}_{a}\widetilde{\mathcal{T}}_\alpha(g)$*;*
- ${}^{RL}_{a}\widetilde{\mathcal{T}}_\alpha(fg)=(1-\alpha){}^{RL}_{a}\widetilde{\mathcal{T}}_\alpha(f)g+f{}^{RL}_{a}\widetilde{\mathcal{T}}_\alpha(g)-(1-\alpha)fg$*;*
- ${}^{RL}_{a}\widetilde{\mathcal{T}}_\alpha(f(g(t)))=(1-\alpha)f(g(t))+\alpha f'(g(t))\mathcal{T}_\alpha(g(t))$.

In order to proceed with our proofs, we have first to define the following fractional integral.

**Definition 2.5** (The $\alpha$-fractional integral). For $\alpha\in(0,1]$ and a continuous function $f$, let

$$(\mathcal{I}^\alpha_{a^+}f)(t)=\frac{1}{\alpha}\int_a^t\frac{f(s)}{(s-a)^{1-\alpha}}\,e^{\frac{1-\alpha}{\alpha^2}[(s-a)^\alpha-(t-a)^\alpha]}\,ds.$$

It is worth noting that when $\alpha=1$, $\mathcal{I}^1_{a^+}(f)=\int\limits_a^t f(s)\,ds$ coincides with the usual Riemann integral.

**Lemma 2.2.** *If $\alpha \in [0,1]$, $f$ is an $\alpha$-differentiable function at a point $t$, then we have:*

- $(\mathcal{I}^{\alpha}_{a^+}\, {}^{C}_{a}\widetilde{\mathcal{T}}_{\alpha}(f))(t) = f(t) - f(a)$;
- ${}^{C}_{a}\widetilde{\mathcal{T}}_{\alpha}(\mathcal{I}^{\alpha}_{a^+} f)(t) = f(t)$.

*Proof.* Let $\alpha \in [0,1]$. Then for $t > a$, we have

$$
\begin{aligned}
(\mathcal{I}_{\alpha}\, {}^{C}_{a}\widetilde{\mathcal{T}}_{\alpha} f)(t) &= \frac{1}{\alpha}\int_a^t \frac{({}^{C}_{a}\widetilde{\mathcal{T}}_{\alpha} f)(s)}{(s-a)^{1-\alpha}}\, e^{\frac{1-\alpha}{\alpha^2}[(s-a)^{\alpha}-(t-a)^{\alpha}]}\, ds \\
&= \frac{1}{\alpha}\int_a^t \frac{(1-\alpha)(f(s)-f(a)) + \alpha(s-a)^{1-\alpha} f'(s)}{(s-a)^{1-\alpha}}\, e^{\frac{1-\alpha}{\alpha^2}[(s-a)^{\alpha}-(t-a)^{\alpha}]}\, ds \\
&= \frac{1-\alpha}{\alpha}\int_a^t \frac{f(s)-f(a)}{(s-a)^{1-\alpha}}\, e^{\frac{1-\alpha}{\alpha^2}[(s-a)^{\alpha}-(t-a)^{\alpha}]}\, ds + \int_a^t f'(s) e^{\frac{1-\alpha}{\alpha^2}[(s-a)^{\alpha}-(t-a)^{\alpha}]}\, ds \\
&= \frac{1-\alpha}{\alpha}\int_a^t \frac{f(s)-f(a)}{(s-a)^{1-\alpha}}\, e^{\frac{1-\alpha}{\alpha^2}[(s-a)^{\alpha}-(t-a)^{\alpha}]}\, ds + \left[f(s) e^{\frac{1-\alpha}{\alpha^2}[(s-a)^{\alpha}-(t-a)^{\alpha}]}\right]_a^t \\
&\quad - \frac{1-\alpha}{\alpha}\int_a^t \frac{f(s)}{(s-a)^{1-\alpha}}\, e^{\frac{1-\alpha}{\alpha^2}[(s-a)^{\alpha}-(t-a)^{\alpha}]}\, ds \\
&= f(t) - f(a) e^{\frac{\alpha-1}{\alpha^2}(t-a)^{\alpha}} - f(a)\,\frac{1-\alpha}{\alpha}\int_a^t \frac{1}{(s-a)^{1-\alpha}}\, e^{\frac{1-\alpha}{\alpha^2}[(s-a)^{\alpha}-(t-a)^{\alpha}]}\, ds \\
&= f(t) - f(a).
\end{aligned}
$$

Now, let us consider the following equation:

$$
(1-\alpha)y(t) + \alpha(t-a)^{1-\alpha} y'(t) = f(t). \tag{2.1}
$$

By a variation of the constant method, we can, on the one hand, obtain

$$
y(t) = \int_a^t \frac{f(s)}{\alpha(s-a)^{1-\alpha}}\, e^{\frac{1-\alpha}{\alpha^2}[(s-a)^{\alpha}-(t-a)^{\alpha}]}\, ds. \tag{2.2}
$$

On the other hand, we have

$$
(\mathcal{I}^{\alpha}_{a^+} f)(t) = \frac{1}{\alpha}\int_0^t \frac{f(s)}{(s-a)^{1-\alpha}}\, e^{\frac{1-\alpha}{\alpha^2}[(s-a)^{\alpha}-(t-a)^{\alpha}]}\, ds. \tag{2.3}
$$

From equations (2.1), (2.2) and (2.3), we can deduce that

$$
{}^{C}_{a}\widetilde{\mathcal{T}}_{\alpha}(\mathcal{I}^{\alpha}_{a^+} f)(t) = {}^{RL}_{a}\widetilde{\mathcal{T}}_{\alpha}(\mathcal{I}^{\alpha}_{a^+} f)(t) = f(t).
$$

□

**Definition 2.6** ([16]). For $\alpha \in (0,1]$ and a continuous function $f$, let

$$
(\mathcal{J}^{\alpha}_{a^+} f)(t) = \int_a^t f(s) e^{(1-\alpha)s}\, ds.
$$

**Lemma 2.3** ([16]). *If $\alpha \in [0,1]$ and $f$ is a continuous function, then for $t > a$, we have:*

- $(\mathcal{J}^{\alpha}_{a+}\ {}^{e}_{a}\mathcal{D}^{\alpha}(f))(t) = f(t) - f(a)$;
- ${}^{e}_{a}\mathcal{D}^{\alpha}(\mathcal{J}^{\alpha}_{a+} f)(t) = f(t)$.

**Lemma 2.4.** *Let $f : [\kappa_1, \kappa_2] \times \mathcal{PC}([-r, \delta]) \to \mathbb{R}$ be a continuous function. Then $y \in AC([\kappa_1, \kappa_2])$ is a solution of the differential equation*

$$
{}^{C}_{\kappa_1}\widetilde{\mathcal{T}}_{\alpha} y(t) = f(t, y^t(\cdot)), \quad t \in [\kappa_1, \kappa_2], \quad 0 < \alpha < 1, \tag{2.4}
$$

*if and only if $y$ satisfies the following equation:*

$$
y(t) = y(\kappa_1) + \frac{1}{\alpha} \int_{\kappa_1}^{t} \frac{f(s, y^s(\cdot))}{(s-\kappa_1)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-\kappa_1)^{\alpha} - (t-\kappa_1)^{\alpha}]}\, ds. \tag{2.5}
$$

*Proof.* To obtain the integral equation (2.5), we apply the $\alpha$-fractional integral to both sides of (2.4), and by Lemma 2.2, we get

$$
y(t) = y(\kappa_1) + \frac{1}{\alpha} \int_{\kappa_1}^{t} \frac{f(s, y^s(\cdot))}{(s-\kappa_1)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-\kappa_1)^{\alpha} - (t-\kappa_1)^{\alpha}]}\, ds.
$$

Now, we apply the improved Caputo-type conformable fractional derivative of order $\alpha$ to both sides of (2.5), for $t \in \Theta$ and by Lemma 2.1 and Lemma 2.2, we obtain

$$
{}^{C}_{\kappa_1}\widetilde{\mathcal{T}}_{\alpha} y(t) = f(t, y^t(\cdot)).
$$

□

Following the same steps as in the preceding lemma (i.e., by using Lemma 2.3), we can obtain the following necessary result.

**Lemma 2.5.** *Let $g : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a continuous function. Then $y \in AC([\kappa_1, \kappa_2])$ is a solution of the differential equation*

$$
{}^{e}_{\kappa_1}\mathcal{D}^{\alpha} y(t) = g(t), \quad t \in [\kappa_1, \kappa_2], \quad 0 < \alpha < 1,
$$

*if and only if $y$ satisfies the following equation:*

$$
y(t) = y(\kappa_1) + \int_{\kappa_1}^{t} g(s) e^{(1-\alpha)s}\, ds.
$$

## 3 Existence and uniqueness results for the first problem

**Lemma 3.1.** *Let $0 < \alpha < 1$, $\widetilde{\chi} \in C([\kappa_2, \kappa_2 + \delta], \mathbb{R})$ and $\chi \in C([\kappa_1 - r, \kappa_1], \mathbb{R})$ with $\chi(\kappa_1) = 0$, and $f : \Theta \times \mathcal{PC}([-r, \delta]) \to \mathbb{R}$ and $\Phi_k : \mathbb{R} \to \mathbb{R}$ be the given continuous functions. Then problem (1.1)–(1.4) has the following solution:*

$$
y(t) = \begin{cases}
\chi(t) \quad \text{if } t \in [\kappa_1 - r, \kappa_1], \\[4pt]
\dfrac{1}{\alpha} \displaystyle\int_{\kappa_1}^{t} \frac{f(s, y^s(\cdot))}{(s-\kappa_1)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-\kappa_1)^{\alpha} - (t-\kappa_1)^{\alpha}]}\, ds, \quad t \in [\kappa_1, t_1], \\[4pt]
\displaystyle\sum_{i=1}^{k} \Phi_i(y(t_i^-)) + \frac{1}{\alpha} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{f(s, y^s(\cdot))}{(s-t_{i-1})^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_{i-1})^{\alpha} - (t_i-t_{i-1})^{\alpha}]}\, ds \\[4pt]
\qquad + \dfrac{1}{\alpha} \displaystyle\int_{t_k}^{t} \frac{f(s, y^s(\cdot))}{(s-t_k)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_k)^{\alpha} - (t-t_k)^{\alpha}]}\, ds, \quad \text{if } t \in (t_k, t_{k+1}]; \quad k = 1, \ldots, m, \\[4pt]
\widetilde{\chi}(t) \quad \text{if } t \in [\kappa_2, \kappa_2 + \delta].
\end{cases} \tag{3.1}
$$

*Proof.* Assume that $y$ verifies (1.1)–(1.4). If $t \in [\kappa_1, t_1]$, then we have

$$ {}_{\kappa_1}^{C}\widetilde{\mathcal{T}}_\alpha y(t) = f(t, y^t(\cdot)). $$

By Lemma 2.4, we obtain

$$ y(t) = \frac{1}{\alpha}\int_{\kappa_1}^{t} \frac{f(s, y^s(\cdot))}{(s-\kappa_1)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-\kappa_1)^\alpha-(t-\kappa_1)^\alpha]}\, ds, \quad t \in [\kappa_1, t_1]. $$

If $t \in (t_1, t_2]$, then Lemma 2.4 implies

$$ \begin{aligned} y(t) &= y({t_1}^+) + \frac{1}{\alpha}\int_{t_1}^{t} \frac{f(s, y^s(\cdot))}{(s-t_1)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_1)^\alpha-(t-t_1)^\alpha]}\, ds \\ &= \Delta y\big|_{t=t_1} + y({t_1}^-) + \frac{1}{\alpha}\int_{t_1}^{t} \frac{f(s, y^s(\cdot))}{(s-t_1)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_1)^\alpha-(t-t_1)^\alpha]}\, ds \\ &= \Phi_1(y(t_1^-)) + \frac{1}{\alpha}\int_{\kappa_1}^{t_1} \frac{f(s, y^s(\cdot))}{(s-\kappa_1)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-\kappa_1)^\alpha-(t_1-\kappa_1)^\alpha]}\, ds \\ &\quad + \frac{1}{\alpha}\int_{t_1}^{t} \frac{f(s, y^s(\cdot))}{(s-t_1)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_1)^\alpha-(t-t_1)^\alpha]}\, ds. \end{aligned} $$

If $t \in (t_2, t_3]$, then Lemma 2.4 implies

$$ \begin{aligned} y(t) &= y({t_2}^+) + \frac{1}{\alpha}\int_{t_2}^{t} \frac{f(s, y^s(\cdot))}{(s-t_2)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_2)^\alpha-(t-t_2)^\alpha]}\, ds \\ &= \Delta y\big|_{t=t_2} + y({t_2}^-) + \frac{1}{\alpha}\int_{t_2}^{t} \frac{f(s, y^s(\cdot))}{(s-t_2)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_2)^\alpha-(t-t_2)^\alpha]}\, ds \\ &= \Phi_1(y(t_1^-)) + \Phi_2(y(t_2^-)) + \frac{1}{\alpha}\int_{\kappa_1}^{t_1} \frac{f(s, y^s(\cdot))}{(s-\kappa_1)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-\kappa_1)^\alpha-(t_1-\kappa_1)^\alpha]}\, ds \\ &\quad + \frac{1}{\alpha}\int_{t_1}^{t_2} \frac{f(s, y^s(\cdot))}{(s-t_1)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_1)^\alpha-(t_2-t_1)^\alpha]}\, ds \\ &\quad + \frac{1}{\alpha}\int_{t_2}^{t} \frac{f(s, y^s(\cdot))}{(s-t_2)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_2)^\alpha-(t-t_2)^\alpha]}\, ds. \end{aligned} $$

Repeating the process in this way, the solution $y(t)$ for $t \in (t_k, t_{k+1}]$, where $k = 1, \ldots, m$, can be written as

$$ \begin{aligned} y(t) = \sum_{i=1}^{k} \Phi_i(y(t_i^-)) &+ \frac{1}{\alpha}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \frac{f(s, y^s(\cdot))}{(s-t_{i-1})^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_{i-1})^\alpha-(t_i-t_{i-1})^\alpha]}\, ds \\ &+ \frac{1}{\alpha}\int_{t_k}^{t} \frac{f(s, y^s(\cdot))}{(s-t_k)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_k)^\alpha-(t-t_k)^\alpha]}\, ds. \end{aligned} $$

Now, let us assume that $y$ verifies equation (3.1). For $t \in (t_k, t_{k+1}]$, where $k = 0, \dots, m$, and by Lemma 2.1 and Lemma 2.2, we can apply the fractional operator ${}^{C}_{t_k}\widetilde{\mathcal{T}}_\alpha(\,\cdot\,)$ to obtain

$$ {}^{C}_{t_k}\widetilde{\mathcal{T}}_\alpha y(t) = f(t, y^t(\,\cdot\,)). $$

Also, we can easily show that

$$ \Delta y\big|_{t=t_k} = \Phi_k(y(t_k^-)); \quad k = 1, \dots, m. \qquad \square $$

In the sequel, the following hypotheses are used:

$(H_1)$ The function $f : \Theta \times \mathcal{PC}([-r, \delta]) \to \mathbb{R}$ is continuous.

$(H_2)$ There exists a constant $\xi_1 > 0$ such that

$$ \big|f(t, \beta_1) - f(t, \beta_2)\big| \le \xi_1 \|\beta_1 - \beta_2\|_{[-r,\delta]} $$

for $t \in \Theta$ and $\beta_1, \beta_2 \in \mathcal{PC}([-r, \delta])$.

$(H_3)$ There exists a constant $\xi_2 > 0$ such that

$$ \big|\Phi_k(\beta) - \Phi_k(\overline{\beta})\big| \le \xi_2 |\beta - \overline{\beta}| $$

for any $\beta, \overline{\beta} \in \mathbb{R}$ and $k = 1, \dots, m$.

Now, we declare and demonstrate our first existence result for problem (1.1)–(1.4) based on the Banach contraction principle [14].

**Theorem 3.1.** *Assume that $(H_1)$–$(H_3)$ hold. If*

$$ \ell := m\xi_2 + \frac{\xi_1 \big(1 - e^{\frac{\alpha-1}{\alpha^2}(\kappa_2-\kappa_1)^\alpha}\big)(1+m)}{1-\alpha} < 1, \tag{3.2} $$

*then problem (1.1)–(1.4) has a unique solution.*

*Proof.* Let $T : \mathcal{C} \mapsto \mathcal{C}$ be the operator defined by

$$ (Tx)(t) = \begin{cases} \chi(t) \;\text{ if }\; t \in [\kappa_1 - r, \kappa_1], \\ \displaystyle\sum_{\kappa_1 < t_k < t} \Phi_k(y(t_k^-)) + \frac{1}{\alpha} \sum_{\kappa_1 < t_k < t} \int_{t_{k-1}}^{t_k} \frac{f(s, y^s(\,\cdot\,))}{(s - t_{k-1})^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_{k-1})^\alpha - (t_k - t_{k-1})^\alpha]}\, ds \\ \qquad + \displaystyle\frac{1}{\alpha} \int_{t_k}^{t} \frac{f(s, y^s(\,\cdot\,))}{(s - t_k)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_k)^\alpha - (t-t_k)^\alpha]}\, ds, \quad t \in \Theta, \\ \widetilde{\chi}(t) \;\text{ if }\; t \in [\kappa_2, \kappa_2 + \delta]. \end{cases} $$

According to Lemma 3.1, the fixed points of $T$ are the solutions of problem (1.1)–(1.4).

Let $x_1, x_2 \in \mathcal{C}$. If $t \in [\kappa_1 - r, \kappa_1]$ or $t \in [\kappa_2, \kappa_2 + \delta]$, then

$$ \big|(Tx_1)(t) - (Tx_2)(t)\big| = 0. $$

For $t \in \Theta$, we have

$$ \begin{aligned} \big|(Tx_1)(t) - (Tx_2)(t)\big| \le & \sum_{\kappa_1 < t_k < t} \big|\Phi_k(x_1(t_k^-)) - \Phi_k(x_2(t_k^-))\big| \\ & + \frac{1}{\alpha} \sum_{\kappa_1 < t_k < t} \int_{t_{k-1}}^{t_k} \frac{|f(s, x_1^s(\,\cdot\,)) - f(s, x_2^s(\,\cdot\,))|}{(s - t_{k-1})^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_{k-1})^\alpha - (t_k-t_{k-1})^\alpha]}\, ds \\ & + \frac{1}{\alpha} \int_{t_k}^{t} \frac{|f(s, x_1^s(\,\cdot\,)) - f(s, x_2^s(\,\cdot\,))|}{(s - t_k)^{1-\alpha}}\, e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_k)^\alpha - (t-t_k)^\alpha]}\, ds. \end{aligned} $$

By $(H_2)$ and $(H_3)$, we have

$$\begin{aligned}\big|(Tx_1)(t)-(Tx_2)(t)\big| \leq & \sum_{\kappa_1<t_k<t}\xi_2|x_1(t_k)-x_2(t_k)| \\ & +\frac{\xi_1}{\alpha}\sum_{\kappa_1<t_k<t}\int_{t_{k-1}}^{t_k}\frac{\|x_1-x_2\|_{[-r,\delta]}}{(s-t_{k-1})^{1-\alpha}}\,e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_{k-1})^\alpha-(t_k-t_{k-1})^\alpha]}\,ds \\ & +\frac{\xi_1}{\alpha}\int_{t_k}^{t}\frac{\|x_1-x_2\|_{[-r,\delta]}}{(s-t_k)^{1-\alpha}}\,e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_k)^\alpha-(t-t_k)^\alpha]}\,ds,\end{aligned}$$

therefore,

$$\begin{aligned}\big|(Tx_1)(t)-(Tx_2)(t)\big| \leq & \sum_{k=1}^{m}\xi_2|x_1(t_k)-x_2(t_k)| \\ & +\frac{\xi_1}{\alpha}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\frac{\|x_1-x_2\|_{[-r,\delta]}}{(s-t_{k-1})^{1-\alpha}}\,e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_{k-1})^\alpha-(t_k-t_{k-1})^\alpha]}\,ds \\ & +\frac{\xi_1}{\alpha}\int_{t_k}^{t}\frac{\|x_1-x_2\|_{[-r,\delta]}}{(s-t_k)^{1-\alpha}}\,e^{\frac{(1-\alpha)}{\alpha^2}[(s-t_k)^\alpha-(t-t_k)^\alpha]}\,ds, \\ \|Tx_1-Tx_2\|_{\mathcal{C}} \leq & \left[m\xi_2+\frac{\xi_1(1-e^{\frac{\alpha-1}{\alpha^2}(\kappa_2-\kappa_1)^\alpha})(1+m)}{1-\alpha}\right]\|x_1-x_2\|_{\mathcal{C}} \\ \leq & \ \ell\|x_1-x_2\|_{\mathcal{C}}.\end{aligned}$$

Hence, by the Banach contraction principle, $T$ has a unique fixed point which is a unique solution of problem (1.1)–(1.4). □

# 4 Existence results for the second problem

We consider the following fractional differential equation:

$$ {}^{e}_{t_k}\mathcal{D}^\alpha y(t)=\varphi(t),\ \ t\in\overline{\Theta}:=\Theta\setminus\{t_1,\dots,t_m\},\ \ \Theta:=[\kappa_1,\kappa_2], \tag{4.1}$$

where $0<\alpha<1$, with the conditions

$$\Delta y\big|_{t=t_k}=\Psi_k(y(t_k^-));\ \ k=1,\dots,m, \tag{4.2}$$

$$\vartheta_1 y(\kappa_1)+\vartheta_2 y(\kappa_2)=\vartheta_3, \tag{4.3}$$

$$y(t)=\chi(t),\ \ t\in[\kappa_1-r,\kappa_1],\ \ r>0, \tag{4.4}$$

$$y(t)=\widetilde{\chi}(t),\ \ t\in[\kappa_2,\kappa_2+\delta],\ \ \delta>0, \tag{4.5}$$

where $0<\alpha<1$, $\varphi\in C(\Theta,\mathbb{R})$, $\widetilde{\chi}\in C([\kappa_2,\kappa_2+\delta],\mathbb{R})$ and $\chi\in C([\kappa_1-r,\kappa_1],\mathbb{R})$, $\vartheta_1,\vartheta_2,\vartheta_3\in\mathbb{R}$ such that $\vartheta_1+\vartheta_2\neq 0$.

The next lemma shows that problem (4.1)–(4.5) has a unique solution.

**Lemma 4.1.** *The function $y(\cdot)$ satisfies problem (4.1)–(4.5) if and only if*

$$y(t)=\begin{cases}\chi(t) \ \ \textit{if} \ \ t\in[\kappa_1-r,\kappa_1],\\ \left[\dfrac{\vartheta_3}{\vartheta_1+\vartheta_2}-\dfrac{\vartheta_2}{\vartheta_1+\vartheta_2}\displaystyle\sum_{i=1}^{m}\Psi_i(y(t_i^-))-\dfrac{\vartheta_2}{\vartheta_1+\vartheta_2}\int\limits_{\kappa_1}^{\kappa_2}\varphi(s)e^{(1-\alpha)s}\,ds\right]\\ \qquad +\displaystyle\int\limits_{\kappa_1}^{t}\varphi(s)e^{(1-\alpha)s}\,ds, \ \ t\in[\kappa_1,t_1],\\ \left[\dfrac{\vartheta_3}{\vartheta_1+\vartheta_2}-\dfrac{\vartheta_2}{\vartheta_1+\vartheta_2}\displaystyle\sum_{i=1}^{m}\Psi_i(y(t_i^-))-\dfrac{\vartheta_2}{\vartheta_1+\vartheta_2}\int\limits_{\kappa_1}^{\kappa_2}\varphi(s)e^{(1-\alpha)s}\,ds\right]\\ \qquad +\displaystyle\sum_{i=1}^{k}\Psi_i(y(t_i^-))+\int\limits_{\kappa_1}^{t}\varphi(s)e^{(1-\alpha)s}\,ds, \ \ \textit{if} \ \ t\in(t_k,t_{k+1}]; \ \ k=1,\dots,m,\\ \widetilde{\chi}(t) \ \ \textit{if} \ \ t\in[\kappa_2,\kappa_2+\delta].\end{cases}$$

*Proof.* Assume that $y$ verifies (4.1)–(4.5). If $t\in[\kappa_1,t_1]$, then we have

$${}^{e}_{\kappa_1}\mathcal{D}^{\alpha}y(t)=\varphi(t).$$

By Lemma 2.5, we obtain

$$y(t)=y(\kappa_1)+\int\limits_{\kappa_1}^{t}\varphi(s)e^{(1-\alpha)s}\,ds, \ \ t\in[\kappa_1,t_1]. \tag{4.6}$$

If $t\in(t_1,t_2]$, then Lemma 2.5 implies

$$\begin{aligned}y(t)&=y(t_1{}^{+})+\int\limits_{t_1}^{t}\varphi(s)e^{(1-\alpha)s}\,ds=\Delta y\big|_{t=t_1}+y(t_1{}^{-})+\int\limits_{t_1}^{t}\varphi(s)e^{(1-\alpha)s}\,ds\\ &=\Psi_1(y(t_1^-))+y(\kappa_1)+\int\limits_{\kappa_1}^{t_1}\varphi(s)e^{(1-\alpha)s}\,ds+\int\limits_{t_1}^{t}\varphi(s)e^{(1-\alpha)s}\,ds.\end{aligned}$$

If $t\in(t_2,t_3]$, then Lemma 2.5 implies

$$\begin{aligned}y(t)&=y(t_2{}^{+})+\int\limits_{t_2}^{t}\varphi(s)e^{(1-\alpha)s}\,ds=\Delta y\big|_{t=t_2}+y(t_2{}^{-})+\int\limits_{t_2}^{t}\varphi(s)e^{(1-\alpha)s}\,ds\\ &=y(\kappa_1)+\Psi_1(y(t_1^-))+\Psi_2(y(t_2^-))+\int\limits_{\kappa_1}^{t_1}\varphi(s)e^{(1-\alpha)s}\,ds+\int\limits_{t_1}^{t_2}\varphi(s)e^{(1-\alpha)s}\,ds+\int\limits_{t_2}^{t}\varphi(s)e^{(1-\alpha)s}\,ds.\end{aligned}$$

By repeating the process in this way, the solution $y(t)$ for $t\in(t_k,t_{k+1}]$, where $k=1,\dots,m$, can be written as

$$y(t)=y(\kappa_1)+\sum_{i=1}^{k}\Psi_i(y(t_i^-))+\int\limits_{\kappa_1}^{t}\varphi(s)e^{(1-\alpha)s}\,ds. \tag{4.7}$$

Applying the boundary condition (4.3), we obtain

$$\vartheta_1y(\kappa_1)+\vartheta_2y(\kappa_2)=\vartheta_3,$$

$$\vartheta_3=(\vartheta_1+\vartheta_2)y(\kappa_1)+\vartheta_2\sum_{i=1}^{m}\Psi_i(y(t_i^-))+\vartheta_2\int\limits_{\kappa_1}^{\kappa_2}\varphi(s)e^{(1-\alpha)s}\,ds.$$

Thus

$$y(\kappa_1) = \frac{\vartheta_3}{\vartheta_1+\vartheta_2} - \frac{\vartheta_2}{\vartheta_1+\vartheta_2}\sum_{i=1}^{m}\Psi_i(y(t_i^-)) - \frac{\vartheta_2}{\vartheta_1+\vartheta_2}\int_{\kappa_1}^{\kappa_2}\varphi(s)e^{(1-\alpha)s}\,ds.$$

Replacing the value of $y(\kappa_1)$ in equations (4.6) and (4.7), we get

$$y(t) = \begin{cases} \left[\frac{\vartheta_3}{\vartheta_1+\vartheta_2} - \frac{\vartheta_2}{\vartheta_1+\vartheta_2}\sum_{i=1}^{m}\Psi_i(y(t_i^-)) - \frac{\vartheta_2}{\vartheta_1+\vartheta_2}\int_{\kappa_1}^{\kappa_2}\varphi(s)e^{(1-\alpha)s}\,ds\right] \\ \qquad + \int_{\kappa_1}^{t}\varphi(s)e^{(1-\alpha)s}\,ds, \quad t\in[\kappa_1,t_1], \\ \left[\frac{\vartheta_3}{\vartheta_1+\vartheta_2} - \frac{\vartheta_2}{\vartheta_1+\vartheta_2}\sum_{i=1}^{m}\Psi_i(y(t_i^-)) - \frac{\vartheta_2}{\vartheta_1+\vartheta_2}\int_{\kappa_1}^{\kappa_2}\varphi(s)e^{(1-\alpha)s}\,ds\right] \\ \qquad + \sum_{i=1}^{k}\Psi_i(y(t_i^-)) + \int_{\kappa_1}^{t}\varphi(s)e^{(1-\alpha)s}\,ds \text{ if } t\in(t_k,t_{k+1}]; \quad k=1,\dots,m. \end{cases} \tag{4.8}$$

Now, let us assume that $y$ verifies equation (3.1). Using the fact that ${}^{e}_{t_k}\mathcal{D}^{\alpha}(t) = e^{(\alpha-1)t}f'(t)$ for all $t\in\Theta$ (see [16]), for $t\in(t_k,t_{k+1}]$, where $k=0,\dots,m$, and by Lemma 2.3, we can apply the fractional operator ${}^{e}_{t_k}\mathcal{D}^{\alpha}(\,\cdot\,)$ on both sides of (4.8) to obtain

$${}^{e}_{t_k}\mathcal{D}^{\alpha}y(t) = \varphi(t).$$

Also, we can easily show that

$$\Delta y\big|_{t=t_k} = \Psi_k(y(t_k^-)), \quad k=1,\dots,m. \qquad \square$$

As a consequence of Lemma 4.1, we have the following result.

**Lemma 4.2.** *Let* $0<\alpha<1$, $\widetilde{\chi}\in C([\kappa_2,\kappa_2+\delta],\mathbb{R})$ *and* $\chi\in C([\kappa_1-r,\kappa_1],\mathbb{R})$*, and* $g:\Theta\times\mathcal{PC}([-r,\delta])\times\mathbb{R}\to\mathbb{R}$ *and* $\Psi_k:\mathbb{R}\to\mathbb{R}$ *be the given continuous functions. Then* $y\in\mathcal{C}$ *verifies problem* (1.5)–(1.9) *if and only if* $y$ *is the fixed point of the operator* $S:\mathcal{C}\mapsto\mathcal{C}$ *given by*

$$(Sy)(t) = \begin{cases} \chi(t) \quad \text{if } t\in[\kappa_1-r,\kappa_1], \\ \left[\frac{\vartheta_3}{\vartheta_1+\vartheta_2} - \frac{\vartheta_2}{\vartheta_1+\vartheta_2}\sum_{i=1}^{m}\Psi_i(y(t_i^-)) - \frac{\vartheta_2}{\vartheta_1+\vartheta_2}\int_{\kappa_1}^{\kappa_2}\varphi(s)e^{(1-\alpha)s}\,ds\right] \\ \qquad + \sum_{\kappa_1<t_k<t}\Psi_k(y(t_k^-)) + \int_{\kappa_1}^{t}\varphi(s)e^{(1-\alpha)s}\,ds, \quad \text{if } t\in\Theta, \\ \widetilde{\chi}(t) \quad \text{if } t\in[\kappa_2,\kappa_2+\delta], \end{cases}$$

*where* $\varphi$ *is a function satisfying the functional equation*

$$\varphi(t) = g(t, y^t(\,\cdot\,), \varphi(t)).$$

In this section, we demonstrate our existence result for (1.5)–(1.9) by employing the Schauder fixed point theorem [14].

Assume that the following hypotheses hold:

$(B_1)$ The function $g:\Theta\times\mathcal{PC}([-r,\delta])\times\mathbb{R}\to\mathbb{R}$ is continuous.

$(B_2)$ There exist the constants $\varpi_1>0$ and $1>\varpi_2>0$ such that

$$\left|g(t,\beta_1,\overline{\beta}_1) - g(t,\beta_2,\overline{\beta}_2)\right| \le \varpi_1\|\beta_1-\beta_2\|_{[-r,\delta]} + \varpi_2|\overline{\beta}_1-\overline{\beta}_2|$$

for $t\in\Theta$, $\overline{\beta}_1,\overline{\beta}_2\in\mathbb{R}$ and $\beta_1,\beta_2\in\mathcal{PC}([-r,\delta])$.

$(B_3)$ There exist the constants $\varpi_3, \varpi_4 > 0$ such that

$$|\Psi_k(\beta)| \leq \varpi_3 + \varpi_4|\beta|$$

for any $\beta \in \mathbb{R}$ and $k = 1, \dots, m$.

**Theorem 4.1.** *Assume that $(B_1)$–$(B_3)$ hold. If*

$$\eta := \frac{m\varpi_4|\vartheta_2|}{|\vartheta_1+\vartheta_2|} + m\varpi_4 + \frac{\varpi_1|\vartheta_2|(e^{(1-\alpha)\kappa_2} - e^{(1-\alpha)\kappa_1})}{(1-\alpha)(1-\varpi_2)|\vartheta_1+\vartheta_2|} + \frac{\varpi_1(e^{(1-\alpha)\kappa_2} - e^{(1-\alpha)\kappa_1})}{(1-\alpha)(1-\varpi_2)} < 1,$$

*then problem (1.5)–(1.9) has at least one solution.*

*Proof.* We establish the proof in several steps.

***Step 1.*** $S$ is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $\mathcal{C}$. If $t \in [\kappa_1 - r, \kappa_1]$ or $t \in [\kappa_2, \kappa_2 + \delta]$, then

$$|(Sy_n)(t) - (Sy)(t)| = 0. \tag{4.9}$$

For $t \in \Theta$, we have

$$\begin{aligned} &\big|(Sy_n)(t) - (Sy)(t)\big| \\ &\quad \leq \frac{|\vartheta_2|}{|\vartheta_1+\vartheta_2|}\sum_{i=1}^{m}\big|\Psi_i(y_n(t_i^-)) - \Psi_i(y(t_i^-))\big| + \frac{|\vartheta_2|}{|\vartheta_1+\vartheta_2|}\int_{\kappa_1}^{\kappa_2}|\varphi_n(s) - \varphi(s)|e^{(1-\alpha)s}\,ds \\ &\qquad + \sum_{\kappa_1<t_k<t}\big|\Psi_k(y_n(t_k^-)) - \Psi_k(y(t_k^-))\big| + \int_{\kappa_1}^{t}|\varphi_n(s) - \varphi(s)|e^{(1-\alpha)s}\,ds, \end{aligned} \tag{4.10}$$

where

$$\varphi_n(t) = g(t, y_n^t(\,\cdot\,), \varphi_n(t))$$

and

$$\varphi(t) = g(t, y^t(\,\cdot\,), \varphi(t)).$$

Since $y_n \to y$, and by $(B_1)$, we get $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$ for each $t \in \Theta$.

Then by Lebesgue dominated convergence theorem and $(B_1)$, equations (4.9) and (4.10) imply

$$\|S(y_n) - S(y)\|_{\mathcal{C}} \to 0 \ \text{ as } \ n \to \infty.$$

Consequently, $S$ is continuous.

***Step 2.*** $S(B_R) \subset B_R$.

Let the constant $R$ be such that

$$R \geq \max\left\{\frac{\overline{\eta}}{1-\eta}, \|\chi\|_{[\kappa_1-r,\kappa_1]}, \|\widetilde{\chi}\|_{[\kappa_2,\kappa_2+\delta]}\right\}$$

with

$$\eta = \frac{m\varpi_4|\vartheta_2|}{|\vartheta_1+\vartheta_2|} + m\varpi_4 + \frac{\varpi_1|\vartheta_2|(e^{(1-\alpha)\kappa_2} - e^{(1-\alpha)\kappa_1})}{(1-\alpha)(1-\varpi_2)|\vartheta_1+\vartheta_2|} + \frac{\varpi_1(e^{(1-\alpha)\kappa_2} - e^{(1-\alpha)\kappa_1})}{(1-\alpha)(1-\varpi_2)} < 1$$

and

$$\overline{\eta} = \frac{|\vartheta_3|}{|\vartheta_1+\vartheta_2|} + \frac{m\varpi_3|\vartheta_2|}{|\vartheta_1+\vartheta_2|} + \frac{g^*|\vartheta_2|(e^{(1-\alpha)\kappa_2} - e^{(1-\alpha)\kappa_1})}{(1-\alpha)(1-\varpi_2)|\vartheta_1+\vartheta_2|} + m\varpi_3 + \frac{g^*(e^{(1-\alpha)\kappa_2} - e^{(1-\alpha)\kappa_1})}{(1-\alpha)(1-\varpi_2)}.$$

And we define the following ball:

$$B_R = \big\{y \in \mathcal{C} : \ \ \|y\|_{\mathcal{C}} \leq R\big\}.$$

Then $B_R$ is a closed, convex and bounded subset of $\mathcal{C}$.

Let $y \in B_R$. We show that $Sy \in B_R$.

If $t \in [\kappa_1 - r, \kappa_1]$, then

$$|Sy(t)| \leq \|\chi\|_{[\kappa_1 - r, \kappa_1]} \leq R,$$

and if $t \in [\kappa_2, \kappa_2 + \delta]$, then

$$|Sy(t)| \leq \|\widetilde{\chi}\|_{[\kappa_2, \kappa_2 + \delta]} \leq R.$$

For $t \in \Theta$, we get

$$\begin{aligned}|Sy(t)| \leq & \frac{|\vartheta_3|}{|\vartheta_1 + \vartheta_2|} + \frac{|\vartheta_2|}{|\vartheta_1 + \vartheta_2|} \sum_{i=1}^{m} |\Psi_i(y(t_i^-))| \\ & + \frac{|\vartheta_2|}{|\vartheta_1 + \vartheta_2|} \int_{\kappa_1}^{\kappa_2} |\varphi(s)| e^{(1-\alpha)s}\, ds + \sum_{\kappa_1 < t_k < t} |\Psi_k(y(t_k^-))| + \int_{\kappa_1}^{t} |\varphi(s)| e^{(1-\alpha)s}\, ds. \end{aligned} \tag{4.11}$$

By the hypothesis $(B_2)$, for $t \in \Theta$, we have

$$\begin{aligned}|\varphi(t)| &= \left|g(t, y^t(\,\cdot\,), \varphi(t)) - g(t,0,0) + g(t,0,0)\right| \\ &\leq \left|g(t, y^t(\,\cdot\,), \varphi(t)) - g(t,0,0)\right| + |g(t,0,0)| \\ &\leq g^* + \varpi_1 \|y^t\|_{[-r,\delta]} + \varpi_2 |\varphi(t)|,\end{aligned}$$

where $g^* = \sup\limits_{t \in \Theta} g(t,0,0)$, which implies that

$$|\varphi(t)| \leq \frac{g^* + \varpi_1 R}{1 - \varpi_2} := \Lambda.$$

Thus for $t \in \Theta$, from (4.11) and by $(B_3)$, we obtain

$$\begin{aligned}|Sy(t)| \leq & \frac{|\vartheta_3|}{|\vartheta_1 + \vartheta_2|} + \frac{m(\varpi_3 + \varpi_4 R)|\vartheta_2|}{|\vartheta_1 + \vartheta_2|} + \frac{\Lambda |\vartheta_2|}{|\vartheta_1 + \vartheta_2|} \int_{\kappa_1}^{\kappa_2} e^{(1-\alpha)s}\, ds \\ & + m(\varpi_3 + \varpi_4 R) + \Lambda \int_{\kappa_1}^{t} e^{(1-\alpha)s}\, ds \\ \leq & \frac{|\vartheta_3|}{|\vartheta_1 + \vartheta_2|} + \frac{m(\varpi_3 + \varpi_4 R)|\vartheta_2|}{|\vartheta_1 + \vartheta_2|} + \frac{\Lambda |\vartheta_2| (e^{(1-\alpha)\kappa_2} - e^{(1-\alpha)\kappa_1})}{(1-\alpha)|\vartheta_1 + \vartheta_2|} \\ & + m(\varpi_3 + \varpi_4 R) + \frac{\Lambda}{1-\alpha}(e^{(1-\alpha)\kappa_2} - e^{(1-\alpha)\kappa_1}) \\ \leq & R.\end{aligned}$$

Then for $t \in [\kappa_1 - r, \kappa_2 + \delta]$, we have $|Sy(t)| \leq R$, which implies that $\|Sy\|_{\mathcal{C}} \leq R$. Consequently,

$$S(B_R) \subset B_R.$$

***Step 3.*** $S(B_R)$ is equicontinuous and bounded.

By Step 2, we have $S(B_R)$ is bounded.

Let $\tau_1, \tau_2 \in \Theta$, $\tau_1 < \tau_2$, and $y \in B_R$, then

$$\begin{aligned}\left|(Sy)(\tau_2) - (Sy)(\tau_1)\right| &\leq \sum_{\tau_1 < t_k < \tau_2} |\Psi_k(y(t_k^-))| + \left| \int_{\kappa_1}^{\tau_2} \varphi(s) e^{(1-\alpha)s}\, ds - \int_{\kappa_1}^{\tau_1} \varphi(s) e^{(1-\alpha)s}\, ds \right| \\ &\leq \sum_{\tau_1 < t_k < \tau_2} |\Psi_k(y(t_k^-))| + \int_{\tau_1}^{\tau_2} |\varphi(s)| e^{(1-\alpha)s}\, ds.\end{aligned}$$

By condition $(B_2)$, we obtain

$$\begin{aligned}\big|(Sy)(\tau_2)-(Sy)(\tau_1)\big| &\le \sum_{\tau_1<t_k<\tau_2} |\Psi_k(y(t_k^-))| + \Lambda\int_{\tau_1}^{\tau_2} e^{(1-\alpha)s}\,ds\\ &\le \sum_{\tau_1<t_k<\tau_2} |\Psi_k(y(t_k^-))| + \frac{\Lambda}{1-\alpha}\left[e^{(1-\alpha)\tau_2}-e^{(1-\alpha)\tau_1}\right].\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the other cases is obvious, thus we omit the details. As a consequence of Step 1 to Step 3, together with the Arzelà–Ascoli theorem, we can conclude that $S$ is continuous and completely continuous. From Schauder's theorem [14], we conclude that $S$ has a fixed point which is a solution of problem (1.5)–(1.9). □

## 5 Examples

**Example.** We consider the following example of problem (1.1)–(1.4):

$$\begin{aligned} y(t) &= e^t, \quad t\in\Big[1,\frac{5}{4}\Big],\\ {}^{C}_{t_k}\widetilde{\mathcal{T}}_{\frac12} y(t) &= \frac{\sin(t)+1}{151^{t+4}(1+\|y\|_{[-r,\delta]})}, \quad t\in[0,1]\setminus\Big\{\frac13,\frac23\Big\},\\ \Delta y\big|_{t=t_k} &= \frac{|y(t_k^-)|}{151+|y(t_k^-)|}, \quad k=1,2,\\ y(t) &= t^2, \quad t\in\Big[-\frac14,0\Big],\end{aligned} \tag{5.1}$$

where $\alpha=\frac12$, $r=\delta=\frac12$, $\kappa_1=0$, $\kappa_2=1$, $t_1=\frac13$, $t_2=\frac23$ and $m=2$.

Set

$$f(t,y^t(\cdot)) = \frac{\sin(t)+1}{151e^{t+4}(1+\|y\|_{[-r,\delta]})}$$

and

$$\Phi_k(y(t_k^-)) = \frac{|y(t_k^-)|}{151+|y(t_k^-)|}, \quad k=1,2.$$

For each $\beta_1,\overline{\beta}_1\in\mathcal{PC}([-r,\delta])$ and $t\in[0,1]$, we have

$$|f(t,\beta_1)-f(t,\overline{\beta}_1)| \le \frac{\sin(t)+1}{151e^{t+4}}\,\|\beta_1-\overline{\beta}_1\|_{[-r,\delta]} \le \frac{1}{151e^4}\,\|\beta_1-\overline{\beta}_1\|_{[-r,\delta]},$$

and for each $\beta_2,\overline{\beta}_2\in\mathbb{R}$, we have

$$\big|\Phi_k(\beta_2)-\Phi_k(\overline{\beta}_2)\big| \le \frac{1}{151}\,|\beta_2-\overline{\beta}_2|.$$

Therefore, $(H_2)$ and $(H_3)$ are verified with

$$\xi_1=\frac{1}{151e^4} \quad\text{and}\quad \xi_2=\frac{1}{151}.$$

Also, for $t\in\Theta$, we have

$$\ell = \frac{2}{151}+\frac{6(1-e^{-2})}{151e^4} \approx 0.0138743133792875 < 1.$$

Then condition (3.2) is satisfied. Hence, since all conditions of Theorem 3.1 are satisfied, problem (5.1) has a unique solution.

**Example.** Let us now consider an example of problem (1.5)–(1.9). Let

$$\begin{aligned} {}^{e}_{t_k}\mathcal{D}^{\frac{1}{2}} y(t) &= g(t, y^t(\,\cdot\,), {}^{e}_{t_k}\mathcal{D}^{\frac{1}{2}} y(t)), \quad t \in [e,\pi] \setminus \{3\}, \\ \Delta y\big|_{t=t_1} &= \frac{1+|y(t_1^-)|}{212}, \\ y(e) + y(\pi) &= 1, \\ y(t) &= \chi(t), \quad t \in [0,e], \\ y(t) &= \widetilde{\chi}(t), \quad t \in [\pi, \pi+e], \end{aligned} \tag{5.2}$$

where $\alpha = \frac{1}{2}$, $r = \delta = e$, $\kappa_1 = e$, $\kappa_2 = \pi$, $t_1 = 3$, $\vartheta_1 = \vartheta_2 = \vartheta_3 = 1$ and $m = 1$.

Set

$$g(t, y, \overline{y}) = \frac{1 + \|y\|_{[-r,\delta]} + |\overline{y}|}{312 + 312e^{\pi - t}}$$

and

$$\Psi_1(\overline{y}) = \frac{1+|\overline{y}|}{212}$$

for $t \in [e,\pi]$, $y \in \mathcal{PC}([-r,\delta])$, $\overline{y} \in \mathbb{R}$, $\alpha = \frac{1}{4}$ and $r = \delta = 1$.

All conditions of Theorem 4.1 are satisfied with $\varpi_1 = \varpi_2 = \frac{1}{624}$, $\varpi_3 = \varpi_4 = \frac{1}{212}$, and

$$\eta = \frac{3}{424} + \frac{3(e^{\frac{\pi}{2}} - e^{\frac{e}{2}})}{623} \approx 0.0114942348091363 < 1.$$

Then it follows that problem (5.2) admits at least one solution.

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