Memoirs on Differential Equations and Mathematical Physics

Volume ??, 2023, 1–28

Mohamed Badr Benboubker, Stanislas Ouaro, Urbain Traore

NEUMANN PROBLEMS FOR NONLINEAR ELLIPTIC EQUATIONS INVOLVING VARIABLE EXPONENT AND MEASURE DATA

Abstract. This paper deals with the question of the existence of entropy solutions for the problem $-\operatorname{div}(a(x,u,\nabla u)+\phi(u))+g(x,u,\nabla u)=\mu$ posed in an open bounded subset Ω of $\mathbb{R}^N$ with the homogeneous Neumann boundary condition $(a(x,u,\nabla u)+\phi(u))\cdot\eta=0$. The functional setting involves Lebesgue and Sobolev spaces with variable exponent.

2020 Mathematics Subject Classification. 35J25,35J60,35Dxx.

Key words and phrases. Nonlinear elliptic problem, variable exponents, entropy solution, Neumann boundary conditions, Radon measure.

რეზიუმე. ნაშრომში განხილულია ენტროპიული ამონახსნების არსებობის საკითხი $\mathbb{R}^N$-ის ღია შემოსაზღვრულ Ω ქვესიმრავლეში დასმული $-\operatorname{div}\left(a\left(x,u,\nabla u\right)+\phi\left(u\right)\right)+g\left(x,u,\nabla u\right)=\mu$ ამოცანისთვის ერთგვაროვანი ნეიმანის საზღვრით პირობით $\left(a\left(x,u,\nabla u\right)+\phi\left(u\right)\right)\cdot\eta=0$. ამოცანის ფუნქციონალური დასმა მოიცავს ლებეგის და სოლოლევის სივრცეებს ცვლადი მაჩვენებლით.

1 Introduction

The purpose of this paper is to study the existence of entropy solutions to the following nonlinear elliptic problem:

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)+\phi(u))+g(x,u,\nabla u)=\mu & \text{in } \Omega,\\ (a(x,u,\nabla u)+\phi(u))\cdot\eta=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where η is the outer unit normal vector on $\partial\Omega$, a is a Leray–Lions type operator, $\phi\in\mathcal{C}^0(\mathbb{R},\mathbb{R}^N)$ and μ is a diffuse measure such that $\mu=\mu\lfloor\Omega$. The function $g(x,u,\nabla u)$ is a nonlinear order term with natural growth with respect to $|\nabla u|$ satisfying the sign condition, that is, $g(x,u,\nabla u)u\geq 0$.

The study of PDEs with variable exponents experienced a revival of interest over the past few years (see [6,11,12,18,34]) due to the fact that they can model various phenomena which arise in the study of elastic mechanics (see [5]), electrorheological fluids(see [19,21,29]) or image restoration (see [18]). The interest of the study of problem (1.1) is due to the fact that it can model various phenomena in elasticity, non-Newtonian fluids (sometimes referred to as smart fluids), the flow through porous medias and image processing. On the other hand, the introduction of the Neumann boundary condition brings us to introduce new ideas for the survey of this problem.

It is important to remember that problem like (1.1) was studied by many authors in the case of homogenous Dirichlet boundary condition (see [1,7,10,32]). More recently, Benboubker *et al.* [10] established the existence of entropy and renormalized solutions for the problem

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)+\phi(u))+g(x,u,\nabla u)=\mu & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\mu\in L^1(\Omega)+W^{-1,p'(\cdot)}(\Omega)$. Zhang and Zhou [32] have proved the existence of entropy and renormalized solutions for problem (**??**) in the particular case, where $a(x,s,\xi)=|\xi|^{p(x)-2}\xi$, $g\equiv 0$ and $\phi\equiv 0$.

In the last years, increasing attention has been devoted towards the study of elliptic problems with measure data and Neumann boundary condition. The study of these problems is also based on the decomposition of the measure in the context of constant exponent (cf. [3,14,15,23]) and in the variable exponent setting (see [12,25–27]).

In this paper, our aim is to prove the existence of entropy solutions for the nonlinear boundary value problem (1.1) in order to extend the results of [10] to the case of Neumann boundary condition and general measure data. Let us recall that, when the boundary value condition is a Neumann boundary condition in the context of a variable exponent, we must work with the space $W^{1,p(\cdot)}(\Omega)$ instead of the common space $W_0^{1,p(\cdot)}(\Omega)$ (the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$). The main difficulty which appears in this case is that for the proofs of some a priori estimates, the famous Poincaré inequality doesn't apply, even for the Poincaré–Sobolev inequality (since we have a homogeneous Neumann condition).

The plan of this paper is the following. In Section 2, we recall some basic notations and properties about Sobolev spaces with variable exponents. In Section 3, we give our basic assumptions and some fundamental lemmas. In Section 4, the definition of entropy solution as well as the main result are given.

2 Preliminaries

For each open bounded subset Ω of $\mathbb{R}^N$ $(N\geq 2)$, we denote

$$\mathcal{C}_+(\overline{\Omega})=\big\{p:\ \ p\in\mathcal{C}(\overline{\Omega}),\ \ p(x)>1\ \text{ for any }\ x\in\overline{\Omega}\big\}.$$

For every $p\in\mathcal{C}_+(\overline{\Omega})$, we define

$$p_+=\sup_{x\in\Omega}p(x)\ \text{ and }\ p_-=\inf_{x\in\Omega}p(x).$$

We denote the Lebesgue spaces with variable exponents $L^{p(\,\cdot\,)}(\Omega)$ (see [19]) as the set of all measurable functions $u:\Omega\to\mathbb{R}$ for which the convex modular

$$\rho_{p(\,\cdot\,)}(u) := \int_\Omega |u|^{p(x)}\,dx$$

is finite.

If the exponent is bounded, i.e., if $p_+<+\infty$, then the expression

$$\|u\|_{p(\,\cdot\,)} := \inf\Big\{\lambda>0:\ \ \rho_{p(\,\cdot\,)}\Big(\frac{u}{\lambda}\Big)\le 1\Big\}$$

defines a norm in $L^{p(\,\cdot\,)}(\Omega)$, called the Luxemburg norm.

The space $(L^{p(\,\cdot\,)}(\Omega),\|\cdot\|_{p(\,\cdot\,)})$ is a separable Banach space. Moreover, if $1<p_-\le p_+<+\infty$, then $L^{p(\,\cdot\,)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(\,\cdot\,)}(\Omega)$, where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$.

Finally, we have the Hölder type inequality

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{p_-}+\frac{1}{(p')_-}\Big)\|u\|_{p(\,\cdot\,)}\|v\|_{p'(\,\cdot\,)}$$

for all $u\in L^{p(\,\cdot\,)}(\Omega)$ and $v\in L^{p'(\,\cdot\,)}(\Omega)$.

Let

$$W^{1,p(\,\cdot\,)}(\Omega) := \big\{u\in L^{p(\,\cdot\,)}(\Omega):\ \ |\nabla u|\in L^{p(\,\cdot\,)}(\Omega)\big\},$$

which is a Banach space equipped with the following norm:

$$\|u\|_{1,p(\,\cdot\,)} := \|u\|_{p(\,\cdot\,)} + \|\nabla u\|_{p(\,\cdot\,)}.$$

The space $(W^{1,p(\,\cdot\,)}(\Omega),\|\cdot\|_{1,p(\,\cdot\,)})$ is a separable and reflexive Banach space.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(\,\cdot\,)}$ of the space $L^{p(\,\cdot\,)}(\Omega)$. We have the following result.

Proposition 2.1 (see [20,34])**.** *If $u_n,u\in L^{p(\,\cdot\,)}(\Omega)$ and $p_+<\infty$, the following properties hold true:*

(i) $\|u\|_{p(\,\cdot\,)}>1 \Longrightarrow \|u\|_{p(\,\cdot\,)}^{p_-} < \rho_{p(\,\cdot\,)}(u) < \|u\|_{p(\,\cdot\,)}^{p_+}$;

(ii) $\|u\|_{p(\,\cdot\,)}<1 \Longrightarrow \|u\|_{p(\,\cdot\,)}^{p_+} < \rho_{p(\,\cdot\,)}(u) < \|u\|_{p(\,\cdot\,)}^{p_-}$;

(iii) $\|u\|_{p(\,\cdot\,)}<1$ *(resp.* $=1;>1$*)* $\Longleftrightarrow \rho_{p(\,\cdot\,)}(u)<1$ *(resp.* $=1;>1$*)*;

(iv) $\|u_n\|_{p(\,\cdot\,)}\to 0$ *(resp.* $\to+\infty$*)* $\Longleftrightarrow \rho_{p(\,\cdot\,)}(u_n)<1$ *(resp.* $\to+\infty$*)*;

(v) $\rho_{p(\,\cdot\,)}\Big(\dfrac{u}{\|u\|_{p(\,\cdot\,)}}\Big)=1.$

Proposition 2.2 (see [20,33])**.** *If $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying*

$$|f(x,s)|\le a(x)+b|s|^{\frac{p_1(x)}{p_2(x)}}\ \ \textit{for any } x\in\Omega,\ s\in\mathbb{R},$$

where $p_1,p_2\in\mathcal{C}_+(\overline{\Omega})$, $a\in L^{p_2(\,\cdot\,)}(\Omega)$ is a positive function and $b\ge 0$ is a constant, then the Nemytskii operator from $L^{p_1(\,\cdot\,)}(\Omega)$ to $L^{p_2(\,\cdot\,)}(\Omega)$ defined by $(N_f(u))(x)=f(x,u(x))$ is a continuous and bounded operator.

For a measurable function $u:\Omega\to\mathbb{R}$, we introduce the following notation:

$$\rho_{1,p(\,\cdot\,)}(u) = \int_\Omega |u|^{p(x)}\,dx + \int_\Omega |\nabla u|^{p(x)}\,dx.$$

Proposition 2.3 (see [30,31])**.** *If $u \in W^{1,p(\cdot)}(\Omega)$, the following properties hold true:*

(i) $\|u\|_{1,p(\cdot)} > 1 \Longrightarrow \|u\|_{1,p(\cdot)}^{p_-} < \rho_{1,p(\cdot)}(u) < \|u\|_{1,p(\cdot)}^{p_+}$;

(ii) $\|u\|_{1,p(\cdot)} < 1 \Longrightarrow \|u\|_{1,p(\cdot)}^{p_+} < \rho_{1,p(\cdot)}(u) < \|u\|_{1,p(\cdot)}^{p_-}$;

(iii) $\|u\|_{1,p(\cdot)} < 1$ (*resp.* $= 1; > 1$) $\Longleftrightarrow \rho_{1,p(\cdot)}(u) < 1$ (*resp.* $= 1; > 1$).

Put

$$p^{\partial}(x) := (p(x))^{\partial} \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)} & \text{if } \ p(x) < N, \\ \infty & \text{if } \ p(x) \geq N. \end{cases}$$

Proposition 2.4 (see [31])**.** *Let $p \in \mathcal{C}(\overline{\Omega})$ and $p_- > 1$. If $q \in \mathcal{C}(\partial\Omega)$ satisfies the condition*

$$1 < q(x) < p^{\partial}(x) \ \ \forall\, x \in \partial\Omega,$$

then there is a compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial\Omega)$. In particular, there is a compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\partial\Omega)$.

Proposition 2.5 (see [22])**.** *Let $p \in \mathcal{C}_+(\overline{\Omega})$ be such that $1 < p_- \leq p_+ < +\infty$. Assume that p satisfies the log-Hölder continuity condition, that is, there is a constant C such that*

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|} \tag{2.1}$$

for every $x, y \in \Omega$ with $0 < |x-y| \leq 1/2$. Then the inequality

$$\|u - u_\Omega\|_{p(\cdot)} \leq C\, \mathrm{diam}(\Omega)\Big(1 + \max\Big\{|\Omega|^{(1/p_+)-(1/p_-)}, |\Omega|^{(1/p_-)-(1/p_+)}\Big\}\Big)\|\nabla u\|_{p(\cdot)}$$

holds for every $u \in W^{1,p(\cdot)}(\Omega)$, where $u_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,dx$. Here, the constant C depends on the dimension N, Ω and p.

Throughout this paper, we assume that $p \in \mathcal{C}_+(\overline{\Omega})$ satisfies the log-Hölder continuity condition (2.1). For any given $k > 0$, we define the truncation function T_k by

$$T_k(s) := \max\{-k, \min\{k, s\}\} = \begin{cases} -k & \text{if } \ s \leq -k, \\ s & \text{if } \ |s| < k, \\ k & \text{if } \ s \geq k. \end{cases}$$

For all $u \in W^{1,p(\cdot)}(\Omega)$, we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense.

In the sequel, we will identify at the boundary, u and $\tau(u)$.

Set

$$\mathcal{T}^{1,p(\cdot)}(\Omega) = \Big\{u: \ \ \Omega \to \mathbb{R}, \ \text{ measurable such that } \ T_k(u) \in W^{1,p(\cdot)}(\Omega) \ \text{ for any } \ k > 0\Big\}.$$

Proposition 2.6 (see [13])**.** *Let $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$. Then there exists a unique measurable function $v: \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$ for all $k > 0$. The function v is denoted by ∇u. Moreover, if $u \in W^{1,p(\cdot)}(\Omega)$, then $v \in (L^{p(\cdot)}(\Omega))^N$ and $v = \nabla u$ in the usual sense.*

We denote by $\mathcal{T}_{tr}^{1,p(\cdot)}(\Omega)$ (cf. [27,28]) the set of functions $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p(\cdot)}(\Omega)$ satisfying the following conditions:

(C1) $u_n \to u$ a.e. in Ω.

(C2) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $(L^1(\Omega))^N$ for any $k > 0$.

(C3) There exists a measurable function v on $\partial\Omega$ such that $u_n \to v$ a.e. on $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [2,4]. In the sequel, the trace of $u \in \mathcal{T}_{tr}^{1,p(\cdot)}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$. If $u \in W^{1,p(\cdot)}(\Omega)$, then $tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover, $u \in \mathcal{T}_{tr}^{1,p(\cdot)}(\Omega)$ and for every $k > 0$, $\tau(T_k(u)) = T_k(tr(u))$, and if $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, then $(u-\varphi) \in \mathcal{T}_{tr}^{1,p(\cdot)}(\Omega)$ and $tr(u-\varphi) = tr(u) - tr(\varphi)$.

We define $\mathcal{M}_b(X)$ as the space of bounded Radon measure in X equipped with its standard norm $\|\cdot\|_{\mathcal{M}_b(X)}$.

In the context of a variable exponent, the $p(\cdot)$-capacity of any subset $B \subset X$ is defined by

$$\mathrm{Cap}_{p(\cdot)}(B,X) = \inf_{u \in S_{p(\cdot)}(B)} \left\{ \int_X \left(|u|^{p(x)} + |\nabla u|^{p(x)}\right) dx \right\}$$

with

$$S_{p(\cdot)}(B) = \left\{ u \in W_0^{1,p(\cdot)}(X): \ u \geq 1 \text{ in an open set containing } B \text{ and } u \geq 0 \text{ in } X \right\}.$$

If $S_{p(\cdot)}(B) = \varnothing$, we set $\mathrm{Cap}_{p(\cdot)}(B,X) = +\infty$.

For $\mu \in \mathcal{M}_b(X)$, we say that μ is diffuse with respect to the capacity $W^{1,p(\cdot)}(X)$ ($p(\cdot)$-capacity, for short) if $\mu(B) = 0$ for every set B such that $\mathrm{Cap}_{p(\cdot)}(B,X) = 0$.

The set of bounded Radon diffuse measure in a variable exponent setting is denoted by $\mathcal{M}_b^{p(\cdot)}(X)$.

3 Basic assumptions and some fundamental lemmas

We assume that Ω is a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$) with boundary $\partial\Omega$ of class $\mathcal{C}^1$. Then it has an extension domain (cf. [17]). So, for any fixed open bounded subset U_Ω of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U_\Omega$, there exists a bounded linear operator

$$E: W^{1,p(\cdot)}(\Omega) \to W_0^{1,p(\cdot)}(U_\Omega),$$

for which

(i) $E(u) = u$ a.e. in Ω for each $u \in W^{1,p(\cdot)}(\Omega)$;

(ii) $\|E(u)\|_{W_0^{1,p(\cdot)}(U_\Omega)} \leq C\|u\|_{W^{1,p(\cdot)}(\Omega)}$, where C is a constant depending only on Ω.

We introduce the set

$$\mathfrak{M}_b^{p(\cdot)}(\Omega) := \left\{ \mu \in \mathcal{M}_b^{p(\cdot)}(U_\Omega): \ \mu \text{ is concentrated on } \Omega \right\}.$$

This definition is independent of the open set U_Ω. Note that for $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $\mu \in \mathfrak{M}_b^{p(\cdot)}(\Omega)$, we have

$$\langle \mu, E(u) \rangle = \int_\Omega u \, d\mu.$$

On the other hand, as μ is diffuse, there exist $f \in L^1(U_\Omega)$ and $F \in (L^{p(\cdot)}(U_\Omega))^N$ such that $\mu = f - \mathrm{div}(F)$ in $\mathcal{D}'(U_\Omega)$ (see [25]).

Therefore, we can also write

$$\langle \mu, E(u) \rangle = \int_{U_\Omega} f E(u) \, dx + \int_{U_\Omega} F \cdot \nabla E(u) \, dx.$$

We consider a Leray–Lions operator from $W^{1,p(\cdot)}(\Omega)$ into its dual $(W^{1,p(\cdot)}(\Omega))'$ defined by the formula

$$\mathrm{A}u = -\operatorname{div} a(x,u,\nabla u),$$

where $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions:

$$|a(x,s,\xi)|\le\beta\big[k(x)+|s|^{p(x)-1}+|\xi|^{p(x)-1}\big],\tag{3.1}$$

$$a(x,s,\xi)\xi\ge\alpha|\xi|^{p(x)},\tag{3.2}$$

$$\big[a(x,s,\xi)-a(x,s,\eta)\big](\xi-\eta)>0\ \text{ for all }\ \xi\ne\eta\in\mathbb{R}^N,\tag{3.3}$$

for a.e. $x\in\Omega$ and all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where $k(x)$ is a positive function lying in $L^{p'(\,\cdot\,)}(\Omega)$ and $\alpha,\beta>0$.

The nonlinear term $g:\Omega\times\mathbb{R}\times\mathbb{R}^N$ is a Carathéodory function satisfying.

$$|g(x,s,\xi)|\le b(|s|)\big(c(x)+|\xi|^{p(x)}\big),\tag{3.4}$$

$$g(x,s,\xi)s\ge0,\tag{3.5}$$

where $b:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous, nondecreasing function and $c:\Omega\to\mathbb{R}^+$ with $c\in L^1(\Omega)$.

Moreover, assume that ϕ is a continuous function defined from $\mathbb{R}$ into $\mathbb{R}^N$ and there exists a positive real number M_0 such that

$$|\phi(s)|\le M_0\ \text{ for all }\ s\in\mathbb{R}.\tag{3.6}$$

Lemma 3.1 (see [9])**.** *Let $g\in L^{p(\,\cdot\,)}(\Omega)$ and $g_n\in L^{p(\,\cdot\,)}(\Omega)$ with $\|g_n\|_{L^{p(\,\cdot\,)}(\Omega)}\le C$ for $1<p(x)<\infty$. If $g_n(x)\to g(x)$ a.e. in Ω, then $g_n(x)\rightharpoonup g(x)$ in $L^{p(\,\cdot\,)}(\Omega)$.*

Lemma 3.2. *Assume that (3.1)–(3.3) hold, let u_n be a sequence in $W^{1,p(\,\cdot\,)}(\Omega)$ such that $u_n\rightharpoonup u$ in $W^{1,p(\,\cdot\,)}(\Omega)$ and*

$$\int_\Omega\big[a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u)\big]\nabla(u_n-u)\to0.\tag{3.7}$$

Then $u_n\to u$ in $W^{1,p(\,\cdot\,)}(\Omega)$.

Proof. Let

$$D_n=[a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u)]\nabla(u_n-u).$$

Then by (3.3), D_n is a positive function and by (3.7), $D_n\to0$ in $L^1(\Omega)$. Extracting a subsequence, still denoted by u_n, we can write $u_n\rightharpoonup u$ in $W^{1,p(\,\cdot\,)}(\Omega)$ which implies $u_n\to u$ a.e. in Ω. Similarly, $D_n\to0$ a.e. in Ω. Then there exists a subset B of Ω of zero measure such that for $x\in\Omega\setminus B$,

$$|u(x)|<\infty,\ \ |\nabla u(x)|<\infty,\ \ k(x)<\infty,\ \ u_n(x)\to u(x),\ \ D_n(x)\to0.$$

Defining $\xi_n=\nabla u_n(x)$, $\xi=\nabla u(x)$, we have

$$\begin{aligned}D_n(x)&=\big[a(x,u_n,\xi_n)-a(x,u_n,\xi)\big](\xi_n-\xi)\\&=a(x,u_n,\xi_n)\xi_n+a(x,u_n,\xi)\xi-a(x,u_n,\xi_n)\xi-a(x,u_n,\xi)\xi_n\\&\ge\alpha|\xi_n|^{p(x)}+\alpha|\xi|^{p(x)}-\beta(k(x)+|u_n|^{p(x)-1}+|\xi_n|^{p(x)-1})|\xi|\\&\qquad-\beta(k(x)+|u_n|^{p(x)-1}+|\xi|^{p(x)-1})|\xi_n|\\&\ge\alpha|\xi_n|^{p(x)}-C_x\big[1+|\xi_n|^{p(x)-1}+|\xi_n|\big],\end{aligned}\tag{3.8}$$

where C_x is a constant which depends on x, but does not depend on n. Since $u_n(x)\to u(x)$, we have $|u_n(x)|\le M_x$, where M_x is some positive constant. Then, by a standard argument, ξ_n is bounded uniformly with respect to n, indeed, (3.8) becomes

$$D_n(x)\ge|\xi_n|^{p(x)}\Big(\alpha-\frac{C_x}{|\xi_n|^{p(x)}}-\frac{C_x}{|\xi_n|}-\frac{C_x}{|\xi_n|^{p(x)-1}}\Big).$$

If $|\xi_n|\to\infty$ (for a subsequence), then $D_n(x)\to\infty$, which gives a contradiction. Let now ξ^* be a cluster point of ξ_n. We have $|\xi^*|<\infty$ and by the continuity of a, we obtain

$$\big[a(x,u(x),\xi^*)-a(x,u(x),\xi)\big](\xi^*-\xi)=0.$$

In view of (3.3), we have $\xi^* = \xi$. The uniqueness of the cluster point implies

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega.$$

Since the sequence $a(x, u_n, \nabla u_n)$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$ and $a(x, u_n, \nabla u_n) \to a(x, u, \nabla u)$ a.e. in Ω, Lemma 3.1 implies that

$$a(x, u_n, \nabla u_n) \to a(x, u, \nabla u) \text{ in } (L^{p'(\cdot)}(\Omega))^N \text{ a.e. in } \Omega.$$

We set $\overline{y}_n = a(x, u_n, \nabla u_n)\nabla u_n$ and $\overline{y} = a(x, u, \nabla u)\nabla u$.

As in [16], we can write

$$\overline{y}_n \to y \text{ in } L^1(\Omega).$$

By (3.2), we have

$$\alpha|\nabla u_n| \le a(x, u_n, \nabla u_n)\nabla u_n.$$

Let $z_n = |\nabla u_n|^{p(x)}$, $z = |\nabla u|^{p(x)}$, $y_n = \frac{\overline{y}_n}{\alpha}$ and $y = \frac{\overline{y}}{\alpha}$. Then, by Fatou's lemma,

$$\int_\Omega 2y\, dx \le \liminf_{n\to\infty} \int_\Omega \big(y + y_n - |z_n - z|\big)\, dx,$$

i.e.,

$$0 \le -\limsup_{n\to\infty} \int_\Omega |z_n - z|\, dx.$$

Therefore,

$$0 \le \liminf_{n\to\infty} \int_\Omega |z_n - z|\, dx \le \limsup_{n\to\infty} \int_\Omega |z_n - z|\, dx \le 0,$$

which implies that

$$\nabla u_n \to \nabla u \text{ in } (L^{p(\cdot)}(\Omega))^N. \tag{3.9}$$

It remains to prove that $u_n \to u$ in $L^{p(\cdot)}(\Omega)$. Since $u_n \rightharpoonup u$ in $W^{1,p(\cdot)}(\Omega)$, by the compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p_-}(\Omega)$, we have $u_n \to u$ in $L^{p_-}(\Omega)$ and a.e. in Ω. Owing to Proposition 2.5, we have

$$\begin{aligned}\Big\| u_n - \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega u_n\, dx - \Big(u - \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega u\, dx\Big)\Big\|_{p(\cdot)} \\ = \Big\| (u_n - u) - \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega (u_n - u)\, dx \Big\|_{p(\cdot)} \le C\|\nabla(u_n - u)\|_{p(\cdot)},\end{aligned}$$

where C is a positive constant which does not depend on n. Therefore, letting $n \to +\infty$ and using the fact that ∇u_n converges strongly to ∇u in $(L^{p(\cdot)}(\Omega))^N$, we deduce that

$$u_n - \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega u_n\, dx \longrightarrow u - \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega u\, dx \text{ in } L^{p(\cdot)}(\Omega), \tag{3.10}$$

$$
\begin{aligned}
\|u_n - u\|_{p(\cdot)} &= \Bigg\| \Big(u_n - \frac{1}{\text{meas}\,(\Omega)} \int_\Omega u_n \, dx \Big) - \Big(u - \frac{1}{\text{meas}\,(\Omega)} \int_\Omega u \, dx \Big) \\
&\qquad + \Big(\frac{1}{\text{meas}\,(\Omega)} \int_\Omega u_n \, dx - \frac{1}{\text{meas}\,(\Omega)} \int_\Omega u \, dx \Big) \Bigg\|_{p(\cdot)} \\
&\le \Big\| \Big(u_n - \frac{1}{\text{meas}\,(\Omega)} \int_\Omega u_n \, dx \Big) - \Big(u - \frac{1}{\text{meas}\,(\Omega)} \int_\Omega u \, dx \Big) \Big\|_{p(\cdot)} \\
&\qquad + \Big\| \frac{1}{\text{meas}\,(\Omega)} \int_\Omega u_n \, dx - \frac{1}{\text{meas}\,(\Omega)} \int_\Omega u \, dx \Big\|_{p(\cdot)} \\
&\le \Big\| \Big(u_n - \frac{1}{\text{meas}\,(\Omega)} \int_\Omega u_n \, dx \Big) - \Big(u - \frac{1}{\text{meas}\,(\Omega)} \int_\Omega u \, dx \Big) \Big\|_{p(\cdot)} \\
&\qquad + \Big\| \Big(\frac{1}{\text{meas}\,(\Omega)} \int_\Omega (u_n - u) \, dx \Big) \times 1 \Big\|_{p(\cdot)} \\
&\le \Big\| \Big(u_n - \frac{1}{\text{meas}\,(\Omega)} \int_\Omega u_n \, dx \Big) - \Big(u - \frac{1}{\text{meas}\,(\Omega)} \int_\Omega u \, dx) \Big\|_{p(\cdot)} \\
&\qquad + \Big| \frac{1}{\text{meas}\,(\Omega)} \int_\Omega (u_n - u) \, dx \Big| \, \|1\|_{p(\cdot)} \\
&\le \Big\| \Big(u_n - \frac{1}{\text{meas}\,(\Omega)} \int_\Omega u_n \, dx \Big) - \Big(u - \frac{1}{\text{meas}\,(\Omega)} \int_\Omega u \, dx \Big) \Big\|_{p(\cdot)} \\
&\qquad + \frac{1}{\text{meas}\,(\Omega)^{1/p_-}} \|u_n - u\|_{L^{p_-}(\Omega)} \|1\|_{p(\cdot)}. \qquad (3.11)
\end{aligned}
$$

From (**??**) and the fact that $u_n \to u$ in $L^{p_-}(\Omega)$, we pass to the limit as n tends to infinity in (**??**) to obtain

$$
u_n \to u \ \text{ in } L^{p(\cdot)}(\Omega). \qquad (3.12)
$$

Therefore, by (3.9) and (**??**), we conclude that $u_n \to u$ in $W^{1,p(\cdot)}(\Omega)$. □

4 Entropy Solutions

This section is devoted to the proof of the existence of an entropy solution for problem (1.1). Now, we announce the concept of entropy solution for problem (1.1).

Definition 4.1. A measurable function $u : \Omega \to \mathbb{R}$ is called entropy solution of the elliptic problem (1.1) if $u \in \mathcal{T}_{tr}^{1,p(\cdot)}(\Omega)$, $g(x, u, \nabla u) \in L^1(\Omega)$ and for every $k > 0$,

$$
\begin{aligned}
\int_\Omega a(x, u, \nabla u) \nabla T_k(u - v) \, dx + \int_\Omega g(x, u, \nabla u) T_k(u - v) \, dx + \int_\Omega \phi(u) \nabla T_k(u - v) \, dx \\
\le \int_\Omega T_k(u - v) \, d\mu
\end{aligned}
$$

for all $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

The main result of this section is the following theorem.

Theorem 4.1. *Assume that* (3.1)–(3.6) *hold true. Then there exists at least one entropy solution u of problem* (1.1).

Proof. Step 1. The approximate problems.

Since $\mu \in \mathcal{M}_b^{p(\cdot)}(U_\Omega)$, we have $\mu = f - \operatorname{div}(F)$ in $\mathcal{D}'(U_\Omega)$ with $f \in L^1(U_\Omega)$ and $F \in (L^{p'(\cdot)}(U_\Omega))^N$, where U_Ω is the open bounded subset of $\mathbb{R}^N$ which extends Ω via the operator E (see [25]).

We regularize μ as follows: $\forall\, \epsilon > 0$, $\forall\, x \in U_\Omega$, we define

$$f_n(x) = T_n(f(x))\chi_\Omega(x).$$

We consider $F_R = \chi_\Omega F$ and $\mu_n = f_n - \operatorname{div}(F_R)$.

For any $n \in \mathbb{N}$, one has $\mu_n \in \mathfrak{M}_b^{p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $\mu_n \rightharpoonup \mu$ in $\mathcal{M}_b^{p(\cdot)}(U_\Omega)$. Furthermore, for any $k > 0$ and any $\xi \in \mathcal{T}^{1,p(\cdot)}(\Omega)$,

$$\Big| \int_\Omega T_k(\xi)\, d\mu_n \Big| \le k\, C(\mu, \Omega).$$

Let us define

$$\phi_n(s) = \phi(T_n(s)),$$
$$h_n(x,s) = \frac{1}{n}\, |s|^{p(x)-2} s$$

and

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1 + \frac{1}{n}\,|g(x,s,\xi)|}\,.$$

Now, we consider the approximated problem

$$\begin{cases} -\operatorname{div}(a(x,u_n,\nabla u_n) + \phi_n(u_n)) + g_n(x,u_n,\nabla u_n) + h_n(x,u_n) = \mu_n & \text{in } \Omega, \\ (a(x,u_n,\nabla u_n) + \phi_n(u_n)) \cdot \eta = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

In the rest of the paper, we denote

$$p_0 := \inf_{x\in\Omega} p'(x) \Big(\frac{\alpha}{2}\, p(x) \Big)^{\frac{p'(x)}{p(x)}}.$$

Let us prove the following result.

Lemma 4.1. *There exists at least one weak solution u_n for problem (4.1) in the sense that $u_n \in W^{1,p(\cdot)}(\Omega)$ and for all $v \in W^{1,p(\cdot)}(\Omega)$,*

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla v\, dx + \int_\Omega g_n(x,u_n,\nabla u_n) v\, dx + \int_\Omega h_n(x,u_n) v\, dx + \int_\Omega \phi_n(u_n)\nabla v\, dx = \int_\Omega v\, d\mu_n. \tag{4.2}$$

Proof. We define the operators $A, G_n, R_n : W^{1,p(\cdot)}(\Omega) \to (W^{1,p(\cdot)}(\Omega))'$ by

$$\langle A u_n, v\rangle = \int_\Omega a(x,u_n,\nabla u_n)\nabla v\, dx, \quad \langle R_n u_n, v\rangle = \int_\Omega \phi_n(u_n) v\, dx \ \ \forall\, v \in W^{1,p(\cdot)}(\Omega)$$

and

$$\langle G_n u_n, v\rangle = \int_\Omega g_n(x,u_n,\nabla u_n) v\, dx + \int_\Omega h_n(x,u_n) v\, dx \ \ \forall\, v \in W^{1,p(\cdot)}(\Omega).$$

Using [8, Lemma 4.5] and Lemma 3.2, one shows that the operator $B_n = A + G_n + R_n$ is bounded and pseudo-monotone from $W^{1,p(\cdot)}(\Omega)$ into $(W^{1,p(\cdot)}(\Omega))'$.

For all $u \in W^{1,p(\cdot)}(\Omega)$, we have

$$\begin{aligned} \langle B_n u, u\rangle &= \langle A u, u\rangle + \langle G_n u, u\rangle + \langle R_n u, u\rangle \\ &= \int_\Omega a(x,u,\nabla u)\nabla u\, dx + \int_\Omega g(x,u,\nabla u) u\, dx + \int_\Omega h_n(x,u) u\, dx + \int_\Omega \phi_n(u)\nabla u\, dx \\ &\ge \alpha \int_\Omega |\nabla u|^{p(x)}\, dx + \frac{1}{n} \int_\Omega |u|^{p(x)}\, dx - \int_\Omega (-\phi_n(u))\nabla u\, dx. \end{aligned} \tag{4.3}$$

We use Young's inequality to obtain

$$\begin{aligned}\int_\Omega (-\phi_n(u))\nabla u\,dx &\le \int_\Omega \frac{|\phi_n(u)|}{(\frac{\alpha}{2}\,p(x))^{\frac{1}{p(x)}}}\Big(\Big(\frac{\alpha}{2}\,p(x)\Big)^{\frac{1}{p(x)}}|\nabla u|\Big)\,dx\\
&\le \int_\Omega \frac{|\phi_n(u)|^{p'(x)}}{p'(x)(\frac{\alpha}{2}\,p(x))^{\frac{p'(x)}{p(x)}}}\,dx + \int_\Omega \frac{\alpha}{2}\,|\nabla u|^{p(x)}\,dx\\
&\le \frac{1}{p_0}\int_\Omega |\phi_n(u)|^{p'(x)} + \int_\Omega \frac{\alpha}{2}\,|\nabla u|^{p(x)}\,dx\\
&\le \frac{1}{p_0}\int_\Omega \sup_{\{|s|\le n\}}(|\phi(s)|+1)^{p'_+}\,dx + \int_\Omega \frac{\alpha}{2}\,|\nabla u|^{p(x)}\,dx.\end{aligned}$$

Therefore,

$$-\int_\Omega (-\phi_n(u))\nabla u\,dx \ge -\frac{1}{p_0}\,\text{meas}\,(\Omega)\sup_{\{|s|\le n\}}(|\phi(s)|+1)^{p'_+} - \int_\Omega \frac{\alpha}{2}\,|\nabla u|^{p(x)}\,dx. \tag{4.4}$$

Combining (4.3) and (4.4), we get

$$\begin{aligned}\langle B_n u, u\rangle &\ge \frac{\alpha}{2}\int_\Omega |\nabla u|^{p(x)}\,dx + \frac{1}{n}\int_\Omega |u|^{p(x)}\,dx + C_1\\
&\ge \min\Big\{\frac{\alpha}{2},\frac{1}{n}\Big\}\Big(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |u|^{p(x)}\,dx\Big) + C_1\\
&\ge \min\Big\{\frac{\alpha}{2},\frac{1}{n}\Big\}\rho_{1,p(\,\cdot\,)}(u) + C_1\\
&\ge \min\Big\{\frac{\alpha}{2},\frac{1}{n}\Big\}\|u\|^{\gamma}_{1,p(\,\cdot\,)} + C_1,\end{aligned}$$

i.e.,

$$\frac{\langle B_n u, u\rangle}{\|u\|_{1,p(\,\cdot\,)}} \ge \min\Big\{\frac{\alpha}{2},\frac{1}{n}\Big\}\|u\|^{\gamma-1}_{1,p(\,\cdot\,)} + \frac{C_1}{\|u\|_{1,p(\,\cdot\,)}}$$

with

$$\gamma = \begin{cases} p_+ & \text{if } \ \|u\|_{1,p(\,\cdot\,)} \le 1,\\ p_- & \text{if } \ \|u\|_{1,p(\,\cdot\,)} > 1.\end{cases}$$

Then it follows that

$$\frac{\langle B_n u, u\rangle}{\|u\|_{1,p(\,\cdot\,)}} \to +\infty \ \text{ as } \ \|u\|_{1,p(\,\cdot\,)} \to \infty,$$

which is equivalent to the operator B_n being coercive.

Since we have proved that the operator B_n is bounded, pseudo-monotone and coercive, then there exists at least one weak solution $u_n \in W^{1,p(\,\cdot\,)}(\Omega)$ of problem (4.1) (cf. [24]). □

Step 2. A priori estimates.

Assertion 1. $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ is bounded in $(L^{p_-}(\Omega))^N$.

We take $T_k(u_n)$ as test function in (4.2) to get

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)\,dx + \frac{1}{n}\int_\Omega |u_n|^{p(x)-2}u_n T_k(u_n)\,dx + \int_\Omega \phi_n(u_n)\nabla T_k(u_n)\,dx = \int_\Omega T_k(u_n)\,d\mu_n.$$

Since the second and the third terms on the left-hand side of the above equality is non-negative, from (3.6) we have

$$\begin{aligned}\alpha\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx &\le \int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx\\ &\le -\int_\Omega \phi_n(u_n)\nabla T_k(u_n)\,dx+\int_\Omega T_k(u_n)\,d\mu_n\\ &\le \int_\Omega |\phi(T_n(u_n))|\,|\nabla T_k(u_n)|\,dx+\Big|\int_\Omega T_k(u_n)\,d\mu_n\Big|\\ &\le \int_\Omega M_0|\nabla T_k(u_n)|\,dx+\Big|\int_\Omega T_k(u_n)\,d\mu_n\Big|. \end{aligned}\tag{4.5}$$

Now, we use Young's inequality to get

$$\begin{aligned}\int_\Omega M_0|\nabla T_k(u_n)|\,dx &= \int_\Omega \frac{M_0}{(\frac{\alpha}{2}\,p(x))^{\frac{1}{p(x)}}}\Big(\Big(\frac{\alpha}{2}\,p(x)\Big)^{\frac{1}{p(x)}}|\nabla T_k(u_n)|\Big)\,dx\\ &\le \int_\Omega \frac{M_0^{p'(x)}}{p'(x)(\frac{\alpha}{2}\,p(x))^{\frac{p'(x)}{p(x)}}}\,dx+\int_\Omega \frac{\frac{\alpha}{2}\,p(x)|\nabla T_k(u_n)|^{p(x)}}{p(x)}\,dx\\ &\le \frac{(M_0+1)^{p'_+}}{p_0}\;\mathrm{meas}\,(\Omega)+\frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx. \end{aligned}\tag{4.6}$$

Moreover, we know that

$$\Big|\int_\Omega T_k(u_n)\,d\mu_n\Big|\le k\,C(\mu,\Omega).\tag{4.7}$$

Therefore, using (4.5)–(4.7), we get

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx\le \frac{2}{\alpha}\Big(\frac{(M_0+1)^{p'_+}}{p_0}\;\mathrm{meas}\,(\Omega)+k\,C(\mu,\Omega)\Big).\tag{4.8}$$

We have

$$\int_\Omega |T_k(u_n)|^{p(x)}\,dx=\int_{\{|u_n|\le k\}}|T_k(u_n)|^{p(x)}\,dx+\int_{\{|u_n|>k\}}|T_k(u_n)|^{p(x)}\,dx.$$

Then it follows that

$$\int_{\{|u_n|\le k\}}|T_k(u_n)|^{p(x)}\,dx\le \int_{\{|u_n|\le k\}}k^{p(x)}\,dx=\begin{cases}k^{p_+}\,\mathrm{meas}\,(\Omega) & \text{if } k\ge 1,\\ \mathrm{meas}\,(\Omega) & \text{if } k<1\end{cases}$$

and

$$\int_{\{|u_n|>k\}}|T_k(u_n)|^{p(x)}\,dx=\int_{\{|u_n|>k\}}k^{p(x)}\,dx=\begin{cases}k^{p_+}\,\mathrm{meas}\,(\Omega) & \text{if } k\ge 1,\\ \mathrm{meas}\,(\Omega) & \text{if } k<1.\end{cases}$$

This allows us to write

$$\int_\Omega |T_k(u_n)|^{p(x)}\,dx\le 2(1+k^{p_+})\,\mathrm{meas}\,(\Omega).\tag{4.9}$$

Hence adding (4.8) and (4.9) yields

$$\rho_{1,p(\,\cdot\,)}(T_k(u_n))\le \frac{2}{\alpha}\Big(\frac{(M_0+1)^{p'_+}}{p_0}\,\mathrm{meas}\,(\Omega)+k\,C(\mu,\Omega)\Big)+2(1+k^{p_+})\,\mathrm{meas}\,(\Omega).$$

For $\|T_k(u_n)\|_{1,p(\,\cdot\,)} \geq 1$, we have

$$\|T_k(u_n)\|_{1,p(\,\cdot\,)}^{p_-} \leq \rho_{1,p(\,\cdot\,)}(T_k(u_n)),$$

which implies that

$$\|T_k(u_n)\|_{1,p(\,\cdot\,)} \leq \rho_{1,p(\,\cdot\,)}(T_k(u_n))^{\frac{1}{p_-}}.$$

The above inequality gives

$$\|T_k(u_n)\|_{1,p(\,\cdot\,)} \leq 1 + C(k, \alpha, \Omega, p_+, p_-, p'_+, p'_-).$$

We deduce that for any $k > 0$, the sequence $(T_k(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,p(\,\cdot\,)}(\Omega)$ and so, in $W^{1,p_-}(\Omega)$. Then, up to a subsequence, we can assume that for any $k > 0$,

$$T_k(u_n) \rightharpoonup v_k \text{ in } W^{1,p_-}(\Omega)$$

and by a compact embedding, we have

$$T_k(u_n) \to v_k \text{ in } L^{p_-}(\Omega) \text{ and a.e. in } \Omega.$$

Assertion 2. $(u_n)_{n\in\mathbb{N}}$ converges in measure to some function u.

Note that for $k > 1$ large enough, we have

$$\begin{aligned}
\int_\Omega |\nabla T_k(u_n)|^{p_-}\,dx &= \int_{\{|\nabla T_k(u_n)|\leq 1\}} |\nabla T_k(u_n)|^{p_-}\,dx + \int_{\{|\nabla T_k(u_n)|>1\}} |\nabla T_k(u_n)|^{p_-}\,dx \\
&\leq \operatorname{meas}(\Omega) + \int_{\{|\nabla T_k(u_n)|>1\}} |\nabla T_k(u_n)|^{p(x)}\,dx \\
&\leq k \operatorname{meas}(\Omega) + \int_\Omega |\nabla T_k(u_n)|^{p(x)}\,dx.
\end{aligned}$$

Due to inequality (4.8), we have

$$\begin{aligned}
\int_\Omega |\nabla T_k(u_n)|^{p_-}\,dx &\leq \frac{2}{\alpha}\Big(\frac{(M_0+1)^{p'_+}}{p_0}\operatorname{meas}(\Omega) + k\,C(\mu,\Omega)\Big) + k\operatorname{meas}(\Omega) \\
&\leq k\Big(\frac{2}{\alpha}\Big(\frac{(M_0+1)^{p'_+}}{p_0}\operatorname{meas}(\Omega) + C(\mu,\Omega)\Big) + \operatorname{meas}(\Omega)\Big) \\
&\leq k\,const\,(\alpha, \Omega, \mu, p_-, p_+, p'_-, p'_+).
\end{aligned} \tag{4.10}$$

Next, we use the Poincaré–Wirtinger inequality to obtain

$$\begin{aligned}
k^{p_-}\operatorname{meas}\{u_n > k\} &= \int_{\{u_n>k\}} |T_k(u_n)|^{p_-}\,dx \\
&= \int_{\{u_n>k\}} \Big|T_k(u_n) - \frac{1}{|\Omega|}\int_\Omega T_k(u_n)\,dx + \frac{1}{|\Omega|}\int_\Omega T_k(u_n)\,dx\Big|^{p_-}\,dx \\
&\leq 2^{p_--1}\int_\Omega \Big(\Big|T_k(u_n) - \frac{1}{|\Omega|}\int_\Omega T_k(u_n)\Big|^{p_-} + \Big|\frac{1}{|\Omega|}\int_\Omega T_k(u_n)\,dx\Big|^{p_-}\Big)\,dx \\
&\leq C\int_\Omega |\nabla T_k(u_n)|^{p_-}\,dx + \frac{2^{p_--1}}{|\Omega|^{p_-}}\Big(\int_\Omega |T_k(u_n)|\,dx\Big)^{p_-} \\
&\leq C\int_\Omega |\nabla T_k(u_n)|^{p_-}\,dx + \frac{2^{p_--1}}{|\Omega|^{p_-}}\Big(\int_\Omega |u_n|\,dx\Big)^{p_-}
\end{aligned}$$

$$\leq C\int_\Omega |\nabla T_k(u_n)|^{p_-}\,dx + \frac{2^{p_--1}}{|\Omega|^{p_-}}\Big(\int_\Omega \frac{1}{p(x)}\,|u_n|^{p(x)}\,dx + \frac{1}{p'(x)}\Big)^{p_-}$$
$$\leq C\int_\Omega |\nabla T_k(u_n)|^{p_-}\,dx + \frac{2^{p_--1}}{|\Omega|^{p_-}}\Big(\frac{1}{p_-}\,\rho_{p(\,\cdot\,)}(u_n) + \frac{1}{(p')_-}\Big)^{p_-}.$$

The above inequality and (4.10) imply that

$$\text{meas}\,\{u_n > k\} \leq \frac{1}{k^{p_--1}}\,const\,(\alpha,\Omega,\mu,p_-,p_+,(p')_-,(p')_+) + \frac{2^{p_--1}}{k^{p_-}|\Omega|^{p_-}}\Big(\frac{1}{p_-}\,\rho_{p(\,\cdot\,)}(u_n) + \frac{1}{(p')_-}\Big)^{p_-}. \tag{4.11}$$

Let $s > 0$ and $k > 0$ be fixed. We denote

$$E_n := \big\{|u_n| > k\big\},\quad E_m := \big\{|u_m| > k\big\},\quad E_{n,m} := \big\{|T_k(u_n) - T_k(u_m)| > s\big\}.$$

We have

$$\big\{|u_n - u_m| > s\big\} \subset E_n \cup E_m \cup E_{n,m},$$

which implies that

$$\text{meas}\ \big\{|u_n - u_m| > s\big\} \leq \text{meas}\,(E_n) + \text{meas}\,(E_m) + \text{meas}\,(E_{n,m}). \tag{4.12}$$

By (4.11), for any $n > 0$, $k > 0$, we have

$$\text{meas}\,(E_n) \leq \frac{1}{k^{p_--1}}\,const\,(\alpha,\Omega,\mu,p_-,p_+,(p')_-,(p')_+) + \frac{2^{p_--1}}{k^{p_-}|\Omega|^{p_-}}\Big(\frac{1}{p_-}\,\rho_{p(\,\cdot\,)}(u_n) + \frac{1}{(p')_-}\Big)^{p_-}.$$

Since the quantity $\rho_{p(\,\cdot\,)}(u_n)$ is finite, we have

$$\lim_{k\to\infty} \frac{1}{k^{p_-}}\Big(\frac{1}{p_-}\,\rho_{p(\,\cdot\,)}(u_n) + \frac{1}{(p')_-}\Big)^{p_-} = 0,$$

and, moreover, since $p_- > 1$, we get

$$\lim_{k\to\infty} \frac{1}{k^{p_--1}}\,const\,(\alpha,\Omega,\mu,p_-,p_+,(p')_-,(p')_+) = 0.$$

We deduce that

$$\lim_{k\to\infty} \text{meas}\,(E_n) = 0 \quad\text{and}\quad \lim_{k\to\infty} \text{meas}\,(E_m) = 0. \tag{4.13}$$

So, we can write: $\forall\, n > 0,\ m > 0, \forall\,\epsilon > 0, \exists\, k_0 = k_0(\epsilon)$ such that $k > k_0$,

$$\text{meas}\,(E_n) \leq \frac{\epsilon}{3} \quad\text{and}\quad \text{meas}\,(E_m) \leq \frac{\epsilon}{3}\,. \tag{4.14}$$

Since $(T_k(u_n))_{n\in\mathbb{N}}$ converges strongly in $L^{p_-}(\Omega)$, it is a Cauchy sequence in $L^{p_-}(\Omega)$. Thus,

$$\text{meas}\,(E_n) = \frac{1}{s^{p_-}}\int_\Omega |T_k(u_n) - T_k(u_m)|^{p_-} \leq \frac{\epsilon}{3} \tag{4.15}$$

for all $n, m \geq n_0(s,\epsilon)$. Finally, from (4.12), (4.14) and (4.15), it follows that

$$\text{meas}\ \big\{|u_n - u_m| > s\big\} \leq \epsilon\ \text{ for all }\ n, m \geq n_0(s,\epsilon).$$

This prove that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure and then converges almost everywhere to some measurable function u.

As for $k > 0$, T_k is continuous, then $T_k(u_n) \to T_k(u)$ a.e. in Ω and $v_k = T_k(u)$ a.e. in Ω. Therefore,

$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } W^{1,p_-}(\Omega),$$
$$T_k(u_n) \to T_k(u) \ \text{ in } L^{p_-}(\Omega) \text{ and a.e. in } \Omega.$$

Step 3. Strong convergence of truncations.

Let $k > 0$ be fixed and $h > k$. We define the function v_n by

$$\begin{cases} v_n = \varphi(\omega_n), \\ \omega_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)), \end{cases}$$

with

$$\varphi(s) = s\exp(\gamma s^2), \quad \gamma = \Big(\frac{b(k)}{2\alpha}\Big)^2.$$

Thanks to [16], we have

$$\varphi'(s) - \frac{b(k)}{\alpha}|\varphi(s)| \geq \frac{1}{2} \ \ \forall\, s \in \mathbb{R}.$$

Now, we take v_n as a test function in (4.2) to obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\varphi'(\omega_n)\nabla\omega_n\, dx + \int_\Omega g(x, u_n\nabla u_n)\varphi(\omega_n)\, dx + \int_\Omega \phi_n(u_n)\varphi'(\omega_n)\nabla\omega_n\, dx = \int_\Omega \varphi(\omega_n)\, d\mu_n. \quad (4.16)$$

Taking $M = 4k + h$, using the facts that $\nabla\omega_n = 0$ on the set $\{|u_n| > M\}$ and $g(x, u_n, \nabla u_n)\varphi(\omega_n) \geq 0$ on the subset $\{|u_n| > k\}$ (because they have the same sign on this subset), then by (4.16), we deduce that

$$\int_\Omega a\big(x, T_M(u_n), \nabla T_M(u_n)\big)\varphi'(\omega_n)\nabla\omega_n\, dx + \int_{\{|u_n| \leq k\}} g(x, u_n\nabla u_n)\varphi(\omega_n)\, dx + \int_\Omega \phi_n(T_M(u_n))\varphi'(\omega_n)\nabla\omega_n\, dx = \int_\Omega \varphi(\omega_n)\, d\mu_n. \quad (4.17)$$

In the sequel, we denote by $\varepsilon_i(n)$, $i = 1, 2, \ldots$, various functions of real numbers which converge to 0 as n tends to infinity.

We will deal with each term of (4.17). We rewrite the first term as follows:

$$\int_\Omega a\big(x, T_M(u_n), \nabla T_M(u_n)\big)\varphi'(\omega_n)\nabla\omega_n\, dx = \int_{\{|u_n| \leq k\}} a\big(x, T_M(u_n), \nabla T_M(u_n)\big)\varphi'(\omega_n)\nabla T_{2k}(u_n - T_h(u))\, dx + \int_{\{|u_n| > k\}} a\big(x, T_M(u_n), \nabla T_M(u_n)\big)\varphi'(\omega_n)\nabla\omega_n\, dx. \quad (4.18)$$

Since $a(x, s, 0) = 0\ \forall\, s \in \mathbb{R}$ and $|u_n - T_k(u)| \leq 2k$ on $\{|u_n| \leq k\}$, the first term of the right-hand side of the last equality can be written as follows:

$$\int_{\{|u_n| \leq k\}} a\big(x, T_M(u_n), \nabla T_M(u_n)\big)\varphi'(\omega_n)\nabla T_{2k}(u_n - T_h(u))\, dx$$

$$= \int_\Omega a\big(x, T_M(u_n), \nabla T_M(u_n)\big)\varphi'(\omega_n)\big[\nabla T_k(u_n) - \nabla T_k(u)\big]\,dx. \quad (4.19)$$

Concerning the second term of the right-hand side of (4.18), we use (3.2) to get

$$\int_{\{|u_n|>k\}} a\big(x, T_M(u_n), \nabla T_M(u_n)\big)\varphi'(\omega_n)\nabla\omega_n\,dx$$
$$\geq -\varphi'(2k)\int_{\{|u_n|>k\}} \big|a(x, T_M(u_n), \nabla T_M(u_n))\big|\,|\nabla T_k(u)|\,dx. \quad (4.20)$$

Hence from (4.19) and (4.20), we deduce that

$$\int_\Omega a\big(x, T_M(u_n), \nabla T_M(u_n)\big)\varphi'(\omega_n)\nabla\omega_n\,dx$$
$$\geq \int_\Omega a\big(x, T_M(u_n), \nabla T_M(u_n)\big)\varphi'(\omega_n)\big[\nabla T_k(u_n) - \nabla T_k(u)\big]\,dx$$
$$- \varphi'(2k)\int_{\{|u_n|>k\}} \big|a\big(x, T_M(u_n), \nabla T_M(u_n)\big)\big|\,|\nabla T_k(u)|\,dx.$$

Since the sequence $(a(x, T_M(u_n), \nabla T_M(u_n)))$ is bounded in $(L^{p'(\cdot)}(\Omega))^N$, and the sequence $\nabla T_k(u)\chi_{\{|u_n|>k\}}$ converges to 0 in $(L^{p(\cdot)}(\Omega))^N$, we find that the second term on the right-hand side of the above inequality tends to 0 as n tends to infinity, therefore, we can write

$$\int_\Omega a(x, u_n, \nabla u_n)\varphi'(\omega_n)\nabla\omega_n\,dx$$
$$\geq \int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\varphi'(\omega_n)\big[\nabla T_k(u_n) - \nabla T_k(u)\big]\,dx + \varepsilon_1(n). \quad (4.21)$$

On the other hand, the first term on the right-hand side of (4.21) can be written as

$$\int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\varphi'(\omega_n)\big[\nabla T_k(u_n) - \nabla T_k(u)\big]\,dx$$
$$= \int_\Omega \big[a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))\big]\,\big[\nabla T_k(u_n) - \nabla T_k(u)\big]\varphi'(\omega_n)\,dx$$
$$+ \int_\Omega a\big(x, T_k(u_n), \nabla T_k(u)\big)\nabla T_k(u_n)\varphi'(T_k(u_n) - T_k(u))\,dx$$
$$- \int_\Omega a\big(x, T_k(u_n), \nabla T_k(u)\big)\nabla T_k(u)\varphi'(\omega_n)\,dx. \quad (4.22)$$

Using the continuity of the Nemytskii operator (see [20,33]), we have

$$a\big(x, T_k(u_n), \nabla T_k(u)\big)\varphi'(T_k(u_n) - T_k(u)) \longrightarrow a\big(x, T_k(u), \nabla T_k(u)\big)\varphi'(0)$$

strongly in $(L^{p'(\cdot)}(\Omega))^N$, while $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L^{p(\cdot)}(\Omega))^N$, thus we obtain

$$\int_\Omega a\big(x, T_k(u_n), \nabla T_k(u)\big)\nabla T_k(u_n)\varphi'(T_k(u_n) - T_k(u))\,dx$$

$$= \int_\Omega a\big(x, T_k(u), \nabla T_k(u)\big)\nabla T_k(u)\varphi'(0)\, dx + \varepsilon_2(n). \quad (4.23)$$

Similarly, we have

$$-\int_\Omega a\big(x, T_k(u_n), \nabla T_k(u)\big)\nabla T_k(u)\varphi'(\omega_n)\, dx = -\int_\Omega a\big(x, T_k(u), \nabla T_k(u)\big)\nabla T_k(u)\varphi'(0)\, dx + \varepsilon_3(n). \quad (4.24)$$

Therefore, using (4.21)–(4.24), we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\varphi'(\omega_n)\nabla\omega_n\, dx$$
$$\geq \int_\Omega \Big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(u)\big)\Big]\big[\nabla T_k(u_n) - \nabla T_k(u)\big]\varphi'(\omega_n)\, dx + \varepsilon_4(n). \quad (4.25)$$

By virtue of (3.2) and (3.4), we can treat the second term on the left-hand side of (4.17) as follows:

$$\Bigg|\int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\varphi(\omega_n)\, dx\Bigg| \leq \int_{\{|u_n|\leq k\}} b(k)\big(c(x) + |\nabla T_k(u_n)|^{p(x)}\big)|\varphi(\omega_n)|\, dx$$
$$\leq b(k)\int_{\{|u_n|\leq k\}} c(x)|\varphi(\omega_n)|\, dx + \frac{b(k)}{\alpha}\int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\nabla T_k(u_n)|\varphi(\omega_n)|\, dx. \quad (4.26)$$

Using the fact that $c \in L^1(\Omega)$, one shows that

$$b(k)\int_{\{|u_n|\leq k\}} c(x)|\varphi(\omega_n)|\, dx = \varepsilon_5(n). \quad (4.27)$$

We also have

$$\int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\nabla T_k(u_n)|\varphi(\omega_n)|\, dx$$
$$= \int_\Omega \Big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(u)\big)\Big]\big[\nabla T_k(u_n) - \nabla T_k(u)\big]|\varphi(\omega_n)|\, dx$$
$$+ \int_\Omega a\big(x, T_k(u_n), \nabla T_k(u_n)\big)\nabla T_k(u)|\varphi(\omega_n)|\, dx$$
$$+ \int_\Omega a\big(x, T_k(u_n), \nabla T_k(u)\big)\big[\nabla T_k(u_n) - \nabla T_k(u)\big]|\varphi(\omega_n)|\, dx. \quad (4.28)$$

Combining (4.26)–(4.28), we deduce that

$$\Bigg|\int_{\{|u_n|\leq k\}} g_n(x, u_n, \nabla u_n)\varphi(\omega_n)\, dx\Bigg|\, dx \leq \frac{b(k)}{\alpha}$$
$$\times \int_\Omega \Big[a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(u)\big)\Big]\big[\nabla T_k(u_n) - \nabla T_k(u)\big]|\varphi(\omega_n)|\, dx + \varepsilon_6(n). \quad (4.29)$$

Consequently, from inequalities (4.17), (4.25) and (4.29), it follows that

$$\int_\Omega \Big(a\big(x, T_k(u_n), \nabla T_k(u_n)\big) - a\big(x, T_k(u_n), \nabla T_k(u)\big)\Big)\big(\nabla T_k(u_n) - \nabla T_k(u)\big)\Big(\varphi'(\omega_n) - \frac{b(k)}{\alpha}|\varphi(\omega_n)|\Big)\, dx$$
$$\leq -\int_\Omega \phi_n(T_M(u_n))\varphi'(\omega_n)\nabla\omega_n\, dx + \int_\Omega \varphi\omega_n\, d\mu_n. \quad (4.30)$$

We deal with the second term of the left hand-side of (4.30) as follows:

$$\begin{aligned}\int_\Omega \varphi(\omega_n)\,d\mu_n &= \int_\Omega E(\varphi(\omega_n))\,d\mu_n = \langle \mu_n, E(\varphi(\omega_n))\rangle\\ &= \int_{U_\Omega} f_n E(\varphi(\omega_n))\,dx + \int_{U_\Omega} F_R\cdot\nabla E(\varphi(\omega_n))\,dx\\ &= \int_{U_\Omega} T_n(f)\chi_\Omega E(\varphi(\omega_n))\,dx + \int_{U_\Omega} (\chi_\Omega F)\cdot\nabla E(\varphi(\omega_n))\,dx\\ &= \int_\Omega T_n(f)\varphi(\omega_n)\,dx + \int_{U_\Omega} F\cdot\nabla E(\chi_\Omega\varphi(\omega_n))\,dx. \end{aligned}\tag{4.31}$$

Note that

$$\begin{aligned}\left|\int_\Omega T_n(f)\varphi(\omega_n)\,dx\right| &\le \int_\Omega |T_n(f)-f|\,|\varphi(\omega_n)|\,dx + \int_\Omega |f|\,|\varphi(\omega_n)|\,dx\\ &\le \varphi(2k)\int_\Omega |T_n(f)-f|\,dx + \int_\Omega |f|\,|\varphi(\omega_n)|\,dx.\end{aligned}$$

We have $T_n(f)\to f$ in $L^1(\Omega)$ and $\varphi_k(\omega_n)\rightharpoonup\varphi_k(T_{2k}(u-T_h(u)))$ weakly-$*$ in $L^\infty(\Omega)$, then

$$\int_\Omega T_n(f)\varphi(\omega_n)\,dx = \int_\Omega f\varphi(T_{2k}(u-T_h(u)))\,dx + \varepsilon_8(n).\tag{4.32}$$

The sequence $(E(\chi_\Omega\varphi(\omega_n)))_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(\cdot)}(U_\Omega)$. Indeed, $(\chi_\Omega\varphi(\omega_n))_{n\in\mathbb{N}}$ is bounded in $W^{1,p(\cdot)}(\Omega)$, and we have the inequality

$$\|E(v)\|_{W_0^{1,p(\cdot)}(U_\Omega)} \le C\|v\|_{W^{1,p(\cdot)}(\Omega)} \ \ \forall\, v\in W^{1,p(\cdot)}(\Omega).$$

We also have

$$E(\chi_\Omega\varphi(\omega_n)) = \chi_\Omega\varphi(\omega_n) \ \text{ a.e. in } U_\Omega$$

and

$$\chi_\Omega\varphi(\omega_n)\to\chi_\Omega\varphi(T_{2k}(u-T_h(u))) \ \text{ a.e. in } U_\Omega \text{ as } n\to\infty.$$

This implies that

$$E(\chi_\Omega\varphi(\omega_n))\to E\big(\chi_\Omega\varphi(T_{2k}(u-T_h(u)))\big) \ \text{ a.e. in } U_\Omega \text{ as } n\to\infty.$$

Consequently, we have

$$\nabla E(\chi_\Omega\varphi(\omega_n))\rightharpoonup\nabla E\big(\chi_\Omega\varphi(T_{2k}(u-T_h(u)))\big) \ \text{ in } (L^{p(\cdot)}(U_\Omega))^N.$$

Finally, using the fact that $F\in(L^{p'(\cdot)}(U_\Omega))^N$, we deduce that

$$\lim_{n\to+\infty}\int_{U_\Omega} F\cdot\nabla E(\chi_\Omega\varphi(\omega_n))\,dx = \int_{U_\Omega} F\cdot\nabla E\big(\chi_\Omega\varphi(T_{2k}(u-T_h(u)))\big)\,dx.\tag{4.33}$$

For n large enough (for example $n\ge M$), we can write

$$\int_\Omega \phi_n(T_M(u_n))\varphi'(\omega_n)\nabla\omega_n\,dx = \int_{\{|u_n|\le M\}} \phi_n(T_M(u_n))\varphi'(\omega_n)\nabla\omega_n\,dx,$$

which yields

$$\int_{\{|u_n|\le M\}} \phi_n(T_M(u_n))\varphi'(\omega_n)\nabla\omega_n\,dx = \int_\Omega \phi_n(T_M(u_n))\varphi'(T_{2k}(u-T_h(u)))\nabla T_{2k}(u-T_h(u))\,dx + \varepsilon_9(n). \quad (4.34)$$

Combining (4.30)–(4.34), we are able to pass to the limit as $n\to\infty$ to obtain

$$\limsup_{n\to\infty}\int_\Omega \Big[a\big(x,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,T_k(u_n),\nabla T_k(u)\big)\Big]\big[\nabla T_k(u_n)-\nabla T_k(u)\big]\,dx$$
$$\le 2\int_\Omega f\varphi(T_{2k}(u-T_h(u)))\,dx - 2\int_\Omega \phi_n(T_M(u_n))\varphi'(T_{2k}(u-T_h(u)))\nabla T_{2k}(u-T_h(u))\,dx$$
$$+ 2\int_{U_\Omega} F\cdot\nabla E\big(\chi_\Omega\varphi(T_{2k}(u-T_h(u)))\big)\,dx. \quad (4.35)$$

Now, we prove that the three terms on the right-hand side of (4.35) converges to 0 when $h\to\infty$. Indeed, for the first term, it suffices to apply Lebesgue's theorem.

For the last term, we take $\varphi(T_{2k}(u_n-T_h(u_n)))$ as a test function in (4.2) to obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla\varphi(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$+\int_\Omega g_n(x,u_n,\nabla u_n)\varphi(T_{2k}(u_n-T_h(u_n)))\,dx + \int_\Omega \phi(u_n)\nabla\varphi(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$\le \int_\Omega T_n(f)\varphi(T_{2k}(u_n-T_h(u_n)))\,dx + \int_{U_\Omega} F\cdot\nabla E\big(\chi_\Omega\varphi(T_{2k}(u_n-T_h(u_n)))\big)\,dx.$$

Using assumptions (3.2), (3.5) and (3.6), we get

$$\int_{\{h\le|u_n|\le 2k+h\}} |\nabla u_n|^{p(x)}\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx \le \int_{\{h\le|u_n|\le 2k+h\}} M_0|\nabla u_n|\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$+\int_\Omega T_n(f)\varphi(T_{2k}(u_n-T_h(u_n)))\,dx + \int_{U_\Omega} F\cdot\nabla E\big(\chi_\Omega\varphi(T_{2k}(u_n-T_h(u_n)))\big)\,dx. \quad (4.36)$$

From Young's inequality, we obtain

$$\int_{\{h\le|u_n|\le 2k+h\}} M_0|\nabla u_n|\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$\le \frac{\alpha}{4}\int_{\{h\le|u_n|\le 2k+h\}} |\nabla u_n|^{p(x)}\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx + C\int_{\{h\le|u_n|\}} M_0^{p'(x)}\,dx \quad (4.37)$$

and

$$\int_{U_\Omega} F\cdot\nabla E\big(\chi_\Omega\varphi(T_{2k}(u_n-T_h(u_n)))\big)\,dx \le \int_{\{h\le|u_n|\le 2k+h\}} F\cdot|\nabla u_n|\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$\le C\int_{\{h\le|u_n|\}}|F|^{p'(x)}\,dx+\frac{\alpha}{4}\int_{\{h\le|u_n|\le 2k+h\}}|\nabla u_n|^{p(x)}\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx. \quad (4.38)$$

Combining (4.36)–(4.38), we deduce

$$\frac{\alpha}{2}\int_{\{h\le|u_n|\le 2k+h\}}|\nabla u_n|^{p(x)}\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$\le\int_\Omega T_n(f)\varphi(T_{2k}(u_n-T_h(u_n)))\,dx+C\,\mathrm{meas}\,\big(\{|u_n|\ge h\}\big)(M_0+1)^{p'_+}+C\int_{\{h\le|u_n|\}}|F|^{p'(x)}\,dx. \quad (4.39)$$

Since the modular $\rho_{p(\cdot)}$ is weakly lower semi-continuous (see [19, Theorem 3.29]) and $\varphi'\ge 1$, from (4.39) we have

$$\int_{\{h\le|u|\le 2k+h\}}|\nabla u|^{p(x)}\varphi'(T_{2k}(u-T_h(u)))\,dx=\int_\Omega\big|\nabla T_{2k}(u-T_h(u))\big|^{p(x)}\varphi'(T_{2k}(u-T_h(u)))\,dx$$
$$\le C\int_\Omega\big|\nabla T_{2k}(u-T_h(u))\big|^{p(x)}\,dx\le C\liminf_{n\to\infty}\int_\Omega\big|\nabla T_{2k}(u_n-T_h(u_n))\big|^{p(x)}\,dx$$
$$\le C\liminf_{n\to\infty}\int_\Omega\big|\nabla T_{2k}(u_n-T_h(u_n))\big|^{p(x)}\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$\le C\liminf_{n\to\infty}\int_{\{h\le|u_n|\le 2k+h\}}|\nabla u_n|^{p(x)}\varphi'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$\le C\liminf_{n\to\infty}\frac{2}{\alpha}\int_\Omega T_n(f)\varphi(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$+\frac{2}{\alpha}C\liminf_{n\to\infty}\mathrm{meas}\,\big(\{|u_n|\ge h\}\big)(M+1)^{(p')_+}+\frac{2}{\alpha}C\liminf_{n\to\infty}\int_{\{h\le|u_n|\}}|F|^{p'(x)}\,dx,$$

i.e.,

$$\int_{\{h\le|u|\le 2k+h\}}|\nabla u|^{p(x)}\varphi'(T_{2k}(u-T_h(u)))\,dx$$
$$\le C\frac{2}{\alpha}\int_\Omega f\varphi(T_{2k}(u-T_h(u)))\,dx+\frac{2}{\alpha}C\,\mathrm{meas}\,\big(\{|u|\ge h\}\big)(M+1)^{p'_+}+\frac{2}{\alpha}C\int_{\{h\le|u|\}}|F|^{p'(x)}\,dx.$$

Using inequality (4.13) and the fact that u_n converges almost everywhere to u, we obtain meas $\{|u|\ge h\}\to 0$ as $h\to\infty$. As $|F|\in L^{p'(\cdot)}(\Omega)$, we get

$$\int_{\{h\le|u|\}}|F|^{p'(x)}\,dx\to 0\ \text{ as }\ h\to\infty.$$

Moreover, from the Lebesgue dominated convergence theorem, we have

$$\int_\Omega f\varphi(T_{2k}(u-T_h(u)))\,dx\to 0\ \text{ as }\ h\to\infty.$$

From the above convergence result, we deduce that

$$\int_{\{h\leq|u|\leq 2k+h\}} |\nabla u|^{p(x)}\varphi'(T_{2k}(u-T_h(u)))\,dx \to 0 \text{ as } h\to\infty \text{ for any fixed number } k>0. \tag{4.40}$$

Hence from (4.38), we obtain

$$\int_{U_\Omega} F\cdot\nabla E\big(\chi_\Omega\varphi(T_{2k}(u_n-T_h(u_n)))\big)\,dx \to 0 \text{ as } h\to\infty \text{ for any fixed number } k>0.$$

Concerning the second term on the right-hand side of (4.35), we first observe that

$$0\leq\varphi'(T_{2k}(u-T_h(u)))\leq\max\big\{\varphi'(-2k),\varphi'(2k)\big\}.$$

Then, using (3.6) and Young's inequality, we have

$$\begin{aligned}
\int_\Omega \phi(T_M(u))\varphi'(T_{2k}(u-T_h(u)))\nabla T_{2k}(u-T_h(u))\,dx &\leq M_0\int_{\{h\leq|u|\leq 2k+h\}}\varphi'(T_{2k}(u-T_h(u)))|\nabla u|\,dx\\
&\leq M_0\int_{\{h\leq|u|\leq 2k+h\}}\varphi'(T_{2k}(u-T_h(u)))\,dx+M_0\int_{\{h\leq|u|\leq 2k+h\}}\varphi'(T_{2k}(u-T_h(u)))|\nabla u|^{p(x)}\,dx\\
&\leq M_0\max\big\{\varphi'(-2k),\varphi'(2k)\big\}\text{ meas }\big\{|u|\geq h\big\}+M_0\int_{\{h\leq|u|\leq 2k+h\}}\varphi'(T_{2k}(u-T_h(u)))|\nabla u|^{p(x)}\,dx.
\end{aligned}$$

Therefore, by (4.40) and the fact that meas $(\{|u|\geq h\})\to 0$ as h tends to infinity, we get

$$\int_\Omega \phi(T_M(u))\varphi'(T_{2k}(u-T_h(u)))\nabla T_{2k}(u-T_h(u))\,dx\to 0 \text{ as } h\to\infty.$$

Hence by (4.35) and letting h tends to infinity, we deduce that

$$\lim_{n\to\infty}\int_\Omega\Big[a\big(x,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,T_k(u_n),\nabla T_k(u)\big)\Big]\big[\nabla T_k(u_n)-\nabla T_k(u)\big]\,dx=0.$$

Then, according to Lemma 3.2, we conclude that

$$T_k(u_n)\to T_k(u) \text{ in } W^{1,p(\cdot)}(\Omega)\ \ \forall\, k>0.$$

Step 4. Compactness of the nonlinearities g_n.

In this part, we use Vitali's theorem to prove that

$$g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u) \text{ strongly in } L^1(\Omega).$$

Since $g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u)$ a.e. in Ω, from (3.4) it suffices to prove that the sequence $(|g_n(x,u_n,\nabla u_n)|)_{n\in\mathbb{N}}$ is uniformly equi-integrable.

Let us observe that for any measurable subset $\Omega'\subset\Omega$ and $m\geq 0$, we have

$$\begin{aligned}
\int_\Omega|g_n(x,u_n,\nabla u_n)|\,dx &= \int_{\Omega'\cap\{|u_n|\leq m\}}\big|g_n(x,u_n,\nabla u_n)\big|\,dx+\int_{\Omega'\cap\{|u_n|>m\}}\big|g_n(x,u_n,\nabla u_n)\big|\,dx\\
&\leq b(m)\int_{\Omega'\cap\{|u_n|\leq m\}}\big[c(x)+|\nabla u_n|^{p(x)}\big]\,dx+\int_{\Omega'\cap\{|u_n|>m\}}|g_n(x,u_n,\nabla u_n)|\,dx\\
&\leq b(m)\int_{\Omega'\cap\{|u_n|\leq m\}}\big[c(x)+|\nabla T_m(u_n)|^{p(x)}\big]\,dx+\int_{\Omega'\cap\{|u_n|>m\}}\big|g_n(x,u_n,\nabla u_n)\big|\,dx\\
&= K_1+K_2.
\end{aligned}$$

For any fixed m, we get

$$K_1 \leq b(m) \int\limits_{\Omega'} \left[c(x) + |\nabla T_m(u_n)|^{p(x)}\right] dx.$$

Since $T_m(u_n)$ converges strongly to $T_m(u)$ in $W^{1,p(\cdot)}(\Omega)$, we conclude that K_1 is small uniformly in n, for m fixed as meas (E) is small. For the case of K_2, we consider the function ψ_n defined by

$$\begin{cases} \psi_m(s) = 0 & \text{if } |s| \leq m-1, \\ \psi_m(s) = \operatorname{sign}(s) & \text{if } |s| \geq m, \\ \psi_m'(s) = 1 & \text{if } m-1 < |s| < m. \end{cases}$$

For $m > 1$, we take $\psi_m(u_n)$ as a test function in (4.2) to obtain

$$\int\limits_{\Omega} a(x,u_n,\nabla u_n)\nabla u_n \psi_m'(u_n)\,dx + \int\limits_{\Omega} g(x,u_n,\nabla u_n)\psi_m(u_n)\,dx + \int\limits_{\Omega} \phi(u_n)\nabla u_n \psi_m'(u_n)\,dx$$
$$= \int\limits_{\Omega} T_n(f)\psi_m(u_n)\,dx + \int\limits_{U_\Omega} F\cdot\nabla E(\chi_\Omega \psi(u_n))\,dx.$$

Then from (3.5) we get

$$\int\limits_{\{m-1<|u_n|\leq m\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx + \int\limits_{\{|u_n|>m-1\}} |g(x,u_n,\nabla u_n)|\,dx$$
$$\leq \int\limits_{\{m-1<|u_n|\leq m\}} |\phi(u_n)|\,|\nabla u_n|\,dx + \int\limits_{\{|u_n|>m-1\}} |T_n(f)|\,dx + \int\limits_{\{m-1<|u_n|\leq m\}} F\nabla u_n\,dx.$$

Hence, using assumptions (3.2) and (3.6) and Young's inequality, we obtain

$$\alpha \int\limits_{\{m-1<|u_n|\leq m\}} |\nabla u_n|^{p(x)}\,dx + \int\limits_{\{|u_n|>m-1\}} |g(x,u_n,\nabla u_n)|\,dx$$
$$\leq \int\limits_{\Omega} \frac{M_0}{(\frac{\alpha}{4}\,p(x))^{\frac{1}{p(x)}}} \left(\left(\frac{\alpha}{4}\,p(x)\right)^{\frac{1}{p(x)}} |\nabla(u_n)|\right) dx + \int\limits_{\{|u_n|>m-1\}} |T_n(f)|\,dx + \int\limits_{\{m-1<|u_n|\leq m\}} F\nabla u_n\,dx$$
$$\leq \int\limits_{\Omega} \frac{M_0^{p'(x)}}{p'(x)(\frac{\alpha}{4}\,p(x))^{\frac{p'(x)}{p(x)}}}\,dx + \int\limits_{\Omega} \frac{\frac{\alpha}{4}\,p(x)|\nabla T_k(u_n)|^{p(x)}}{p(x)}\,dx + \int\limits_{\{|u_n|>m-1\}} |F|^{p'(x)}\,dx$$
$$+ \frac{\alpha}{4} \int\limits_{\{|u_n|>m-1\}} |\nabla u_n|^{p(x)}\,dx + \int\limits_{\{|u_n|>m-1\}} |f|\,dx$$
$$\leq \frac{(M_0+1)^{p'_+}}{p_1} \operatorname{meas}(\Omega) + \frac{\alpha}{2} \int\limits_{\Omega} |\nabla T_k(u_n)|^{p(x)}\,dx + \int\limits_{\{|u_n|>m-1\}} |F|^{p'(x)}\,dx + \int\limits_{\{|u_n|>m-1\}} |f|\,dx,$$

where

$$p_1 = \inf_{x\in\Omega} p'(x)\left(\frac{\alpha}{4}\,p(x)\right)^{\frac{p'(x)}{p(x)}}.$$

This implies that

$$\int\limits_{\{|u_n|>m-1\}} |g(x,u_n,\nabla u_n)|\,dx \leq \frac{(M_0+1)^{p'_+}}{p_1} \operatorname{meas}(\Omega) + \int\limits_{\{|u_n|>m-1\}} |F|^{p'(x)}\,dx + \int\limits_{\{|u_n|>m-1\}} |f|\,dx.$$

Therefore,

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}}\int\limits_{\{|u_n|>m-1\}}|g(x,u_n,\nabla u_n)|\,dx=0,$$

which is equivalent to K_2 being small, uniformly in n and in Ω' when m is sufficiently large.

Therefore, the sequence $(|g(x,u_n,\nabla u_n)|)_{n\in\mathbb{N}}$ is uniformly equi-integrable in Ω. We conclude that

$$g(x,u_n,\nabla u_n)\to g(x,u,\nabla u)\ \text{ strongly in } L^1(\Omega).$$

Step 5. $(u_n)_{n\in\mathbb{N}}$ *converges a.e. on* $\partial\Omega$ *to some function* v.

We know that the trace operator is compact from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$, then there exists a constant C such that

$$\|T_k(u_n)-T_k(u)\|_{L^1(\partial\Omega)}\le C\|T_k(u_n)-T_k(u)\|_{W^{1,1}(\Omega)}.$$

Then

$$T_k(u_n)\to T_k(u)\ \text{ in } L^1(\partial\Omega) \text{ and a.e. on } \partial\Omega.$$

Therefore, there exists $A\subset\partial\Omega$ such that $T_k(u_n)$ converges to $T_k(u)$ on $\partial\Omega\setminus A$ with $\sigma(A)=0$, where σ is the area measure on $\partial\Omega$.

For every $k>0$, let $A_k=\{x\in\partial\Omega:\ |T_k(u(x))|<k\}$ and $B=\partial\Omega\setminus\bigcup\limits_{k>0}A_k$.

We have

$$\begin{aligned}\sigma(B)&=\frac{1}{k}\int\limits_B|T_k(u)|\,d\sigma\le\frac{1}{k}\,\|T_k(u)\|_{L^1(\partial\Omega)}\\&\le\frac{C}{k}\,\|T_k(u)\|_{W^{1,1}(\Omega)}\le\frac{C}{k}\,\|T_k(u)\|_{W^{1,p(\cdot)}(\Omega)}\le\frac{C}{k}\Big(\|T_k(u)\|_{p(\cdot)}+\|\nabla T_k(u)\|_{p(\cdot)}\Big).\end{aligned}\tag{4.41}$$

By (4.8) and Proposition 2.1, for all $k>1$, there exists a positive constant M which doesn't depend on n such that

$$\|\nabla T_k(u_n)\|_{p(\cdot)}\le M(k^{\frac{1}{p_-}}+k^{\frac{1}{p_+}}).$$

Then, by Proposition 2.5, it follows that

$$\begin{aligned}\|T_k(u_n)\|_{p(\cdot)}&=\left\|T_k(u_n)-\frac{1}{\text{meas}\,(\Omega)}\int\limits_\Omega T_k(u_n)\,dx+\frac{1}{\text{meas}\,(\Omega)}\int\limits_\Omega T_k(u_n)\,dx\right\|_{p(\cdot)}\\&\le\left\|T_k(u_n)-\frac{1}{\text{meas}\,(\Omega)}\int\limits_\Omega T_k(u_n)\,dx\right\|_{p(\cdot)}+\left\|\frac{1}{\text{meas}\,(\Omega)}\int\limits_\Omega T_k(u_n)\,dx\right\|_{p(\cdot)}\\&\le\|\nabla T_k(u_n)\|_{p(\cdot)}+\|1\|_{p(\cdot)}\left|\frac{1}{\text{meas}\,(\Omega)}\int\limits_\Omega T_k(u_n)\,dx\right|\\&\le M(k^{\frac{1}{p_-}}+k^{\frac{1}{p_+}})+\frac{\|1\|_{p(\cdot)}}{\text{meas}\,(\Omega)}\int\limits_\Omega|T_k(u_n)|\,dx.\end{aligned}\tag{4.42}$$

Using the fact that $T_k(u_n)$ converges strongly to $T_k(u)$ in $W^{1,p(\cdot)}(\Omega)$ and the inequality

$$\|T_k(u)\|_{p(\cdot)}\le\|T_k(u_n)-T_k(u)\|_{p(\cdot)}+\|T_k(u_n)\|_{p(\cdot)},$$

we obtain from (**??**) with the help of the Lebesgue dominated convergence theorem that

$$\|T_k(u)\|_{p(\cdot)}\le\lim_{n\to+\infty}\|T_k(u_n)\|_{p(\cdot)}\le M(k^{\frac{1}{p_-}}+k^{\frac{1}{p_+}})+\frac{\|1\|_{p(\cdot)}}{\text{meas}\,(\Omega)}\int\limits_\Omega|T_k(u)|\,dx.$$

According to (4.41), we deduce that

$$\sigma(B) \leq C\Big(\frac{1}{k^{1-\frac{1}{p_-}}} + \frac{1}{k^{1-\frac{1}{p_+}}}\Big) + \frac{\|1\|_{p(\,\cdot\,)}}{\text{meas}\,(\Omega)} \int\limits_{\Omega} \frac{|T_k(u)|}{k}\,dx. \tag{4.43}$$

Therefore, using the Lebesgue dominated convergence theorem and the fact that $p_- > 1$, by letting $k \to \infty$ in (**??**) we get that $\sigma(B) = 0$.

Let us now define on $\partial\Omega$ the function v by

$$v(x) = T_k(u(x)) \;\text{ if }\; x \in \mathrm{A}_k.$$

We take $x \in \partial\Omega \setminus (A \cup B)$, then there exists $k > 0$ such that $x \in A_k$ and we have

$$u_n(x) - v(x) = (u_n(x) - T_k(u_n(x))) + (T_k(u_n(x)) - T_k(u(x))).$$

Since $x \in A_k$, we have $|T_k(u_n(x))| < k$ from which we deduce that $|u_n(x)| < k$.

Therefore,

$$u_n(x) - v(x) = (T_k(u_n(x)) - T_k(u(x))) \to 0 \;\text{ as }\; n \to +\infty.$$

This means that u_n converges to v a.e. on $\partial\Omega$.

Step 6. u is an entropy solution of problem (1.1).

Since the sequence $(T_k(u_n))_{n\in\mathbb{N}}$ converges in $W^{1,p(\,\cdot\,)}(\Omega)$ to $T_k(u)$, it follows that $\nabla T_k(u_n) \to \nabla T_k(u)$, and using the fact that $p_- > 1$, we get

$$\nabla T_k(u_n) \to \nabla T_k(u) \;\text{ in }\; (L^1(\Omega))^N \;\; \forall\, k > 0. \tag{4.44}$$

Consequently, from Steps 2, 5 and (4.42) it follows that $u \in \mathcal{T}_{tr}^{1,p(\,\cdot\,)}(\Omega)$.

Let $\varphi \in W^{1,p(\,\cdot\,)}(\Omega) \cap L^\infty(\Omega)$, we take $T_k(u_n - \varphi)$ as a test function in (4.2) and put $M = k + \|\varphi\|_\infty$ to get

$$\int\limits_{\Omega} a(x, u_n, \nabla u_n)\nabla T_k(u_n - \varphi)\,dx + \int\limits_{\Omega} g_n(x, u_n, \nabla u_n) T_k(u_n - \varphi)\,dx + \int\limits_{\Omega} h_n(x, u_n) T_k(u_n - \varphi)\,dx + \int\limits_{\Omega} \phi(u_n)\nabla T_k(u_n - \varphi)\,dx = \int\limits_{\Omega} T_k(u_n - \varphi)\,d\mu_n. \tag{4.45}$$

First of all, if $|u_n| > M$, then $|u_n - \varphi| \geq |u_n| - \|\varphi\|_\infty$, then $\{|u_n - \varphi| \leq k\} \subseteq \{|u_n| \leq M\}$, so we can rewrite the first term in (4.43) as follows:

$$\begin{aligned}
\int\limits_{\Omega} a(x, u_n, \nabla u_n)\nabla T_k(u_n - \varphi)\,dx &= \int\limits_{\Omega} a\big(x, T_M(u_n), \nabla T_M(u_n)\big)(\nabla T_k(u_n) - \nabla\varphi)\chi_{\{|u_n - \varphi| \leq k\}}\,dx \\
&= \int\limits_{\Omega} \Big(a\big(x, T_M(u_n), \nabla T_M(u_n)\big) - a(x, T_M(u_n), \nabla\varphi)\Big)(\nabla T_M(u_n) - \nabla\varphi)\chi_{\{|u_n - \varphi| \leq k\}}\,dx \\
&\quad + \int\limits_{\Omega} a(x, T_M(u_n), \nabla\varphi)(\nabla T_M(u_n) - \nabla\varphi)\chi_{\{|u_n - \varphi| \leq k\}}\,dx.
\end{aligned}$$

Using Fatou's lemma, we get

$$\begin{aligned}
\liminf_{n\to+\infty} \int\limits_{\Omega} a(x, u_n, \nabla u_n)\nabla T_k(u_n - \varphi)\,dx
&\geq \int\limits_{\Omega} \Big(a\big(x, T_M(u), \nabla T_M(u)\big) - a(x, T_M(u), \nabla\varphi)\Big)(\nabla T_M(u) - \nabla\varphi)\chi_{\{|u - \varphi| \leq k\}}\,dx \\
&\quad + \lim_{n\to+\infty} \int\limits_{\Omega} a(x, T_M(u_n), \nabla\varphi)(\nabla T_M(u_n) - \nabla\varphi)\chi_{\{|u_n - \varphi| \leq k\}}\,dx.
\end{aligned} \tag{4.46}$$

The second limit in (4.44) is equal to

$$\int_\Omega a(x,T_M(u),\nabla\varphi)(\nabla T_M(u)-\nabla\varphi)\chi_{\{|u-\varphi|\le k\}}\,dx$$

and from (4.44) we obtain

$$\begin{aligned}\liminf_{n\to+\infty}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\varphi)\,dx &\ge \int_\Omega a\big(x,T_M(u),\nabla T_M(u)\big)(\nabla T_k(u)-\nabla\varphi)\chi_{\{|u-\varphi|\le k\}}\,dx\\ &=\int_\Omega a(x,u,\nabla u)(\nabla u-\nabla\varphi)\chi_{\{|u-\varphi|\le k\}}\,dx=\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)\,dx.\end{aligned}$$

We have $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ weakly-$*$ in $L^\infty(\Omega)$ and $g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u)$ in $L^1(\Omega)$, then it follows that

$$\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-\varphi)\,dx\longrightarrow\int_\Omega g(x,u,\nabla u)T_k(u-\varphi)\,dx.$$

We know that $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ in $W^{1,p(\cdot)}(\Omega)$ and $\phi_n(u_n)=\phi(T_M(u_n))$ in $\{|u-\varphi|\le k\}$ for $\{n\ge M\}$, then

$$\int_\Omega \phi_n(u_n)\nabla T_k(u_n-\varphi)\,dx\longrightarrow\int_\Omega\phi(u)\nabla T_k(u-\varphi)\,dx.$$

For the third term of (4.43), we have

$$\begin{aligned}&\frac1n\int_\Omega |u_n|^{p(x)-2}u_nT_k(u_n-\varphi)\,dx\\ &\qquad=\frac1n\int_\Omega\big(|u_n|^{p(x)-2}u_n-|\varphi|^{p(x)-2}\varphi\big)T_k(u_n-\varphi)\,dx+\frac1n\int_\Omega|\varphi|^{p(x)-2}\varphi T_k(u_n-\varphi)\,dx.\end{aligned}$$

The quantity $(|u_n|^{p(x)-2}u_n-|\varphi|^{p(x)-2}\varphi)T_k(u_n-\varphi)$ is nonnegative, and we get

$$\frac1n\int_\Omega|\varphi|^{p(x)-2}\varphi T_k(u_n-\varphi)\,dx\le\frac1n\int_\Omega|u_n|^{p(x)-2}u_nT_k(u_n-\varphi)\,dx.\tag{4.47}$$

Since $T_k(u_n-\varphi)$ converges weakly-$*$ to $T_k(u_n-\varphi)$ in $L^\infty(\Omega)$ and $|\varphi|^{p(\cdot)-2}\varphi\in L^1(\Omega)$, it follows that

$$\int_\Omega|\varphi|^{p(x)-2}\varphi T_k(u_n-\varphi)\,dx\longrightarrow\int_\Omega|\varphi|^{p(x)-2}\varphi T_k(u-\varphi)\,dx.\tag{4.48}$$

Therefore, using(4.45) and (4.46), we have

$$\lim_{n\to\infty}\frac1n\int_\Omega|u_n|^{p(x)-2}u_nT_k(u_n-\varphi)\,dx\ge0.\tag{4.49}$$

It remains to prove that

$$\int_\Omega T_k(u_n-\varphi)\,d\mu_n\longrightarrow\int_\Omega T_k(u-\varphi)\,d\mu.$$

We have

$$\int_\Omega T_k(u_n-\varphi)\,d\mu_n=\int_\Omega E(T_k(u_n-\varphi))\,d\mu_n=\int_\Omega T_n(f)(T_k(u_\epsilon-\varphi))\,dx+\int_{U_\Omega}F\cdot\nabla E(\chi_\Omega T_k(u_n-\varphi))\,dx.$$

Due to the Lebesgue dominated convergence theorem, we have

$$\int_\Omega T_n(f)T_k(u_n-\varphi)\,dx \longrightarrow \int_\Omega fT_k(u-\varphi)\,dx. \tag{4.50}$$

The sequence $(E(\chi_\Omega T_k(u_n-\varphi)))_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(\cdot)}(U_\Omega)$. Moreover, we have

$$E(\chi_\Omega T_k(u_n-\varphi)) = \chi_\Omega T_k(u_n-\varphi) \ \text{ a.e. in } U_\Omega$$

and

$$\chi_\Omega T_k(u_n-\varphi) \to \chi_\Omega T_k(u-\varphi) \ \text{ a.e. in } U_\Omega \text{ as } n\to\infty,$$

which implies that

$$E(\chi_\Omega T_k(u_n-\varphi)) \to E(\chi_\Omega T_k(u-\varphi)) \ \text{ a.e. in } U_\Omega \text{ as } n\to\infty.$$

Therefore, we have

$$\nabla E(\chi_\Omega T_k(u_n-\varphi)) \rightharpoonup \nabla E(\chi_\Omega T_k(u-\varphi)) \ \text{ in } (L^{p(\cdot)}(U_\Omega))^N.$$

Then, using the fact that $F\in(L^{p'(\cdot)}(U_\Omega))^N$, we deduce that

$$\lim_{n\to+\infty}\int_{U_\Omega} F\cdot\nabla E(\chi_\Omega T_k(u_n-\varphi))\,dx \longrightarrow \int_{U_\Omega} F\cdot\nabla E(\chi_\Omega T_k(u-\varphi)). \tag{4.51}$$

Consequently, from (4.48) and (4.49), we get

$$\lim_{n\to+\infty}\int_\Omega T_k(u_n-\varphi)\,d\mu_n = \int_\Omega f(T_k(u-\varphi))\,dx + \int_{U_\Omega} F\cdot\nabla E(\chi_\Omega T_k(u-\varphi))\,dx$$
$$= \int_{U_\Omega} fE(\chi_\Omega(T_k(u-\varphi)))\,dx + \int_{U_\Omega} F\cdot\nabla E(\chi_\Omega T_k(u-\varphi))\,dx = \langle \mu, E(T_k(u-\varphi))\rangle = \int_\Omega T_k(u-\varphi)\,d\mu.$$

Gathering the results, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-\varphi)\,dx + \int_\Omega \phi(u)\nabla T_k(u-\varphi)\,dx \le \int_\Omega T_k(u-\varphi)\,d\mu.$$

We conclude that u is an entropy solution of problem (1.1). □

References

[1] L. Aharouch, J. Bennouna and A. Touzani, Existence of renormalized solution of some elliptic problems in Orlicz spaces. *Rev. Mat. Complut.* **22** (2009), no. 1, 91–110.

[2] F. Andreu, N. Igbida, J. M. Mazón and J. Toledo, L^1 existence and uniqueness results for quasi-linear elliptic equations with nonlinear boundary conditions. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **24** (2007), no. 1, 61–89.

[3] F. Andreu, N. Igbida, J. M. Mazón and J. Toledo, Degenerate elliptic equations with nonlinear boundary conditions and measures data. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **8** (2009), no. 4, 767–803.

[4] F. Andreu, J. M. Mazón, S. Segura de León and J. Toledo, Quasi-linear elliptic and parabolic equations in L^1 with nonlinear boundary conditions. *Adv. Math. Sci. Appl.* **7** (1997), no. 1, 183–213.

[5] S. N. Antontsev and J. F. Rodrigues, On stationary thermo-rheological viscous flows. *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **52** (2006), no. 1, 19–36.

[6] E. Azroul, M. B. Benboubker and S. Ouaro, Entropy solutions for nonlinear nonhomogeneous Neumann problems involving the generalized $p(x)$-Laplace operator. *J. Appl. Anal. Comput.* **3** (2013), no. 2, 105–121.

[7] E. Azroul, M. B. Benboubker and M. Rhoudaf, On some $p(x)$-quasilinear problem with right-hand side measure. *Math. Comput. Simulation* **102** (2014), 117–130.

[8] E. Azroul, H. Hjiaj and A. Touzani, Existence and regularity of entropy solutions for strongly nonlinear $p(x)$-elliptic equations. *Electron. J. Differential Equations* **2013**, No. 68, 27 pp.

[9] M. B. Benboubker, E. Azroul and A. Barbara, Quasilinear elliptic problems with nonstandard growth. *Electron. J. Differential Equations* **2011**, No. 62, 16 pp.

[10] M. B. Benboubker, H. Chrayteh, M. El Moumni and H. Hjiaj, Entropy and renormalized solutions for nonlinear elliptic problem involving variable exponent and measure data. *Acta Math. Sin. (Engl. Ser.)* **31** (2015), no. 1, 151–169.

[11] M. B. Benboubker, H. Hjiaj, I. Ibrango and S. Ouaro, Existence of renormalized solutions for some quasilinear elliptic Neumann problems. *Nonauton. Dyn. Syst.* **8** (2021), no. 1, 180–206.

[12] M. B. Benboubker, S. Ouaro and U. Traore, Entropy solutions for nonlinear nonhomogeneous Neumann problems involving the generalized $p(x)$-Laplace operator and measure data. *J. Nonlinear Evol. Equ. Appl.* **2014**, 53–76.

[13] Ph. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L. Vázquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **22** (1995), no. 2, 241–273.

[14] L. Boccardo and T. Gallouët, Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.* **87** (1989), no. 1, 149–169.

[15] L. Boccardo and T. Gallouët, Nonlinear elliptic equations with right-hand side measures. *Comm. Partial Differential Equations* **17** (1992), no. 3-4, 641–655.

[16] L. Boccardo, F. Murat and J.-P. Puel, Existence of bounded solutions for nonlinear elliptic unilateral problems. *Ann. Mat. Pura Appl. (4)* **152** (1988), 183–196.

[17] H. Brezis, Analyse fonctionnelle. Théorie et applications. (French) Collection Mathématiques Appliquées pour la Maîtrise. Masson, Paris, 1983.

[18] Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.* **66** (2006), no. 4, 1383–1406.

[19] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*. Lecture Notes in Mathematics, 2017. Springer, Heidelberg, 2011.

[20] X. L. Fan and D. Zhao, On the generalized Orlicz-Sobolev space $W^{k,p(x)}(\Omega)$. *J. Gansu Educ. College* **12** (1998), no. 1, 1–6.

[21] M. Ghergu and V. D. Rădulescu, *Singular Elliptic Problems: Bifurcation and Asymptotic Analysis*. Oxford Lecture Series in Mathematics and its Applications, 37. The Clarendon Press, Oxford University Press, Oxford, 2008.

[22] P. Harjulehto and P. Hästö, A capacity approach to the Poincaré inequality and Sobolev imbeddings in variable exponent Sobolev spaces. *Rev. Mat. Complut.* **17** (2004), no. 1, 129–146.

[23] N. Igbida, S. Ouaro and S. Soma, Elliptic problem involving diffuse measure data. *J. Differential Equations* **253** (2012), no. 12, 3159–3183.

[24] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires.* (French) Dunod, Paris; Gauthier-Villars, Paris, 1969.

[25] I. Nyanquini, S. Ouaro and S. Soma, Entropy solution to nonlinear multivalued elliptic problem with variable exponents and measure data. *An. Univ. Craiova Ser. Mat. Inform.* **40** (2013), no. 2, 174–198.

[26] S. Ouaro, A. Ouedraogo and S. Soma, Multivalued problem with Robin boundary condition involving diffuse measure data and variable exponent. *Adv. Nonlinear Anal.* **3** (2014), no. 4, 209–235.

[27] S. Ouaro and S. Soma, Weak and entropy solutions to nonlinear Neumann boundary-value problems with variable exponents. *Complex Var. Elliptic Equ.* **56** (2011), no. 7-9, 829–851.

[28] S. Ouaro and A. Tchousso, Well-posedness result for a nonlinear elliptic problem involving variable exponent and Robin type boundary condition. *Afr. Diaspora J. Math.* **11** (2011), no. 2, 36–64.

[29] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory.* Lecture Notes in Mathematics, 1748. Springer-Verlag, Berlin, 2000.

[30] L.-L. Wang, Y.-H. Fan and W.-G. Ge, Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator. *Nonlinear Anal.* **71** (2009), no. 9, 4259–4270.

[31] J. Yao, Solutions for Neumann boundary value problems involving $p(x)$-Laplace operators. *Nonlinear Anal.* **68** (2008), no. 5, 1271–1283.

[32] C. Zhang and S. Zhou, Entropy and renormalized solutions for the $p(x)$-Laplacian equation with measure data. *Bull. Aust. Math. Soc.* **82** (2010), no. 3, 459–479.

[33] D. Zhao and X. L. Fan, On the Nemytskii operators from $L^{p_1(x)}$ to $L^{p_2(x)}$. *J. Lanzhou Univ.* **34** (1998), no. 1, 1–5.

[34] D. Zhao, W. J. Qiang and X. L. Fan, On generalized Orlicz spaces $L^{p(x)}(\Omega)$. *J. Gansu Sci.* **9** (1997), no. 2, 1–7.

(Received 14.06.2022; accepted 28.11.2022)

Authors' addresses:

Mohamed Badr Benboubker
Higher School of Technology, Sidi Mohamed Ben Abdellah University, Fez, Morocco.
E-mail: simo.ben@hotmail.com

Stanislas Ouaro
Laboratoire d'Analyse Mathématique des Equations (LAME), UFR, Sciences Exactes et Appliquées, Université de Ouagadougou, 03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso.
E-mails: uaro@yaohoo.fr, souaro@univ-ouaga.bf

Urbain Traore
Laboratoire de Mathématiques et Informatique (LAMI), Université Joseph KI-ZERBO, Ouagadougou, Burkina Faso.
E-mail: urbain.traore@yahoo.fr