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Mouad Allalou, Abderrahmane Raji, Farah Balaadich

THE EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS TO OBSTACLE PROBLEMS WITH VARIABLE GROWTH AND WEAK MONOTONICITY

Abstract. In this paper, we show the existence and uniqueness of weak solutions to obstacle problem

$$\int_{\Omega} \sigma(x, Du) : D(v-u) + \left\langle u|u|^{p(x)-2}, v-u \right\rangle \mathrm{d}x \geq 0,$$

for v belonging to the convex set $\mathcal{K}_{\psi,\theta}$. The main tool used here is the Young measure theory and a theorem of Kinderlehrer and Stampacchia combined with the theory of Sobolev spaces with variable exponent.

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1 Introduction

We are interested in the following obstacle problem:

$$\int_{\Omega} \sigma(x, Du) : D(v-u) + \left\langle u|u|^{p(x)-2}, v-u \right\rangle \mathrm{d}x \geq 0, \quad v \in \mathcal{K}_{\psi,\theta}, \tag{1.1}$$

where

$$\mathcal{K}_{\psi,\theta} = \Big\{ v \in W^{1,p(x)}(\Omega; \mathbb{R}^m) : \quad v-\theta \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m), \quad v \geq \psi \text{ a.e. in } \Omega \Big\}. \tag{1.2}$$

Here, Ω is a bounded open domain in $\mathbb{R}^n (n \geq 2)$ and $u : \Omega \to \mathbb{R}^m$ is a vector-valued function and variable exponent $p(x)$ with locally log-Hölder continuity in Ω satisfies

$$1 < p^- \leq p(x) \leq p^+ < \infty \text{ for a.e. } x \in \Omega. \tag{1.3}$$

An obstacle problem is a type of partial differential equation (PDE) in which the solution must remain above a predetermined function, or obstacle. The obstacle is usually defined as a lower bound for the solution of the PDE. This type of problem is often used to model physical systems such as heat flow, or fluid dynamics, where the obstacle may represent the physical boundaries of the system.

Junxia and Yuming [19] studied the boundary regularity of weak solutions to a nonlinear obstacle problem with $C^{1,\beta}$-obstacle function and obtained the $C^{1,\alpha}_{\text{loc}}$ boundary regularity. In [27], the author has considered obstacle problems with measure data related to elliptic equations of p-Laplace type, and investigated the connections between low order regularity properties of the solutions and nonlinear potential of the data. Jacques–Louis Lions [21] studied the existence of solutions to the parabolic obstacle problems via variational inequalities. H. El Hammar et al. in [16, 17] proved the existence of a weak solution to the quasilinear elliptic system under regularity, growth and coercivity conditions for σ by using Galerkin's approximation and the theory of Young measures. A large number of papers were devoted to the study of the existence and uniqueness of a weak solution for the obstacle problem (1.1) under classical monotone methods developed by [1, 2, 29].

In [28], the author investigated the scalar version of problem (1.1) and demonstrated the existence of a weak solution with variable growth (for related topics, see [11, 15]). For the utilisation of Young measures in elliptic systems, we refer the reader to see [6, 16, 17].

E. Azroul and F. Balaadich in [8] treated the following obstacle problem:

$$\int_{\Omega} \sigma(x, Du) : D(v-u) \, \mathrm{d}x \geq 0, \quad v \in \mathcal{K}_{\psi,\theta},$$

where

$$\mathcal{K}_{\psi,\theta} = \Big\{ v \in W^{1,p}(\Omega; \mathbb{R}^m) : \quad v-\theta \in W_0^{1,p}(\Omega; \mathbb{R}^m), \quad v \geq \psi \text{ a.e. in } \Omega \Big\},$$

and proved the existence of weak solutions under some conditions on $\sigma : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$.

In this paper, we obtain the existence and uniqueness of weak solutions for the obstacle problem (1.1), inspired by the works mentioned above, and we extend the result established in [8] by considering a general source term under growth condition, constant growth and under weak monotonicity with the use of the concept of Young measure combined with the Kinderlehrer and Stampacchia theorem.

We denote by $\mathbb{M}^{m\times n}$ the set of real $m \times n$ matrices equipped with the usual inner product $S : K = S_{ij}K_{ij}$. The obstacle function $\psi : \Omega \to \mathbb{R}^m$ is defined in (1.2) and $\theta \in W^{1,p(x)}(\Omega; \mathbb{R}^m)$ is a function which provides the boundary values. We study the solution $u \in K_{\psi,\theta}$ for (1.1) under the following hypotheses:

(f_0) $\sigma : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e., the mapping $x \mapsto \sigma(x, S)$ is measurable for all $S \in \mathbb{M}^{m\times n}$, $S \mapsto \sigma(x, S)$ is continuous for a.e. $x \in \Omega$.

(f_1) There exist $N_1(x) \in L^{p'(x)}(\Omega)$, $N_2(x) \in L^1(\Omega)$ and $c_1, c_2 > 0$ such that

$$|\sigma(x, S)| \leq N_1(x) + c_1|S|^{p(x)-1},$$
$$\sigma(x, S) : S \geq -N_2(x) + c_2|S|^{p(x)}.$$

(f_2) σ satisfies one of the following conditions:

(a) The map $S \mapsto \sigma(x,S)$ is strictly quasimonotone, i.e., there exists a constant $c_3 > 0$ such that

$$\int_\Omega (\sigma(x,S) - \sigma(x,K)) : (S-K)\,\mathrm{d}x \geq c_3 \int_\Omega |S-K|^{p(x)}\,\mathrm{d}x$$

for all $x \in \Omega$ and $S, K \in \mathbb{M}^{m\times n}$.

(b) There exists a function $Z : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,S) = \frac{\partial Z}{\partial S}(x,S)$, and $S \to \sigma(x,S)$ is convex and belongs to C^1.

(c) For all $x \in \Omega$, the map $S \mapsto \sigma(x,S)$ is a C^1-function and is monotone, i.e.,

$$(\sigma(x,S) - \sigma(x,K)) : (S-K) \geq 0$$

for all $x \in \Omega$ and $S, K \in \mathbb{M}^{m\times n}$.

Let us rapidly summarize the paper's contents. In Section 2, we lay out the fundamentals of Sobolev spaces with varying exponents along with the Kinderlehrer and Stampacchia theorem and a brief explanation of Young measures. We then move on to the proof of the existence of solutions to obstacle problems in Section 3. The proof of uniqueness of solutions to obstacle problems is given in Section 4.

2 Preliminaries

We recall some necessary notations, definitions and properties for our function spaces (see [12, 23, 25, 26, 28]) and an overview about Young measures (see [10, 14, 22]). For each open bounded subset Ω of $\mathbb{R}^n$ $(n \geq 2)$, we denote $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}),\ p(x) > 1 \text{ for any } x \in \Omega\}$. For every $p \in C^+(\overline{\Omega})$, define

$$p^- = \inf_{x\in\Omega} p(x) \ \text{ and } \ p^+ = \sup_{x\in\Omega} p(x).$$

The Sobolev space $W^{1,p(x)}(\Omega;\mathbb{R}^m)$ consists of all functions u in the Lebesgue space

$$L^{p(x)}(\Omega;\mathbb{R}^m) = \left\{u:\ \Omega \to \mathbb{R}^m \ \text{ measurable}:\ \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty\right\}$$

such that $Du \in L^{p(x)}(\Omega;\mathbb{M}^{m\times n})$. The space $L^{p(x)}(\Omega;\mathbb{R}^m)$ is endowed with the norm

$$\|u\|_{p(x)} = \inf\left\{\beta > 0,\ \int_\Omega \left|\frac{u(x)}{\beta}\right|^{p(x)} \mathrm{d}x \leq 1\right\},$$

being a Banach space. Moreover, it is reflexive if and only if $1 < p^- \leq p^+ < \infty$. Its dual is defined by $L^{p'(x)}(\Omega;\mathbb{R}^m)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega;\mathbb{R}^m)$ and $v \in L^{p'(x)}(\Omega;\mathbb{R}^m)$, the generalized Hölder inequality

$$\left|\int_\Omega uv\,\mathrm{d}x\right| \leq \left(\frac{1}{p^-} + \frac{1}{p^+}\right)\|u\|_{p(x)}\|v\|_{p'(x)}$$

holds true. The space $W^{1,p(x)}(\Omega;\mathbb{R}^m)$ is endowed with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|Du\|_{p(x)}.$$

Proposition 2.1 ([18, 24])**.** *We denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\,\mathrm{d}x \ \ \forall\, u \in L^{p(x)}(\Omega;\mathbb{R}^m).$$

If $u_k, u \in L^{p(x)}(\Omega;\mathbb{R}^m)$ and $p^+ < \infty$, then:

(i) $\|u\|_{p(x)} < 1(=1;>1) \Longleftrightarrow \rho(u) < 1(=1;>1)$;

(ii) $\|u\|_{p(x)} > 1 \Longrightarrow \|u\|^{p^-}_{p(x)} \le \rho(u) \le \|u\|^{p^+}_{p(x)}$; $\|u\|_{p(x)} < 1 \Longrightarrow \|u\|^{p^+}_{p(x)} \le \rho(u) \le \|u\|^{p^-}_{p(x)}$;

(iii) $\|u_k\|_{p(x)} \to 0 \Longleftrightarrow \rho(u_k) \to 0$; $\|u_k\|_{p(x)} \to +\infty \Longleftrightarrow \rho(u_k) \to +\infty$.

We denote by $W^{1,p(x)}_0(\Omega;\mathbb{R}^m)$ the closure of $C^\infty_0(\Omega;\mathbb{R}^m)$ in $W^{1,p(x)}(\Omega;\mathbb{R}^m)$ and $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ is its dual space. We denote $p^*(x) = \frac{np(x)}{n-p(x)}$ for $p(x) < n$; $=\infty$ for $p(x) > n$.

Theorem 2.1 (see [13, 28])**.** *If $p(x)$ satisfies (1.3), then the inequality*

$$\int_\Omega \langle u(x), v(x)\rangle \,\mathrm{d}x \le C\|u(x)\|_{L^{p(x)}(\Omega,\mathbb{R}^m)}\|v(x)\|_{L^{p'(x)}(\Omega,\mathbb{R}^m)}$$

holds for every $u(x) \in L^{p(x)}(\Omega,\mathbb{R}^m), v(x) \in L^{p'(x)}(\Omega,\mathbb{R}^m)$ with constant C depending only on $p(x)$.

Theorem 2.2 (see [13, 28])**.** *If $p(x)$ satisfies (1.3), then the spaces $L^{p(x)}(\Omega,\mathbb{R}^m)$ and $W^{1,p(x)}(\Omega,\mathbb{R}^m)$ are reflexive Banach spaces.*

To establish the existence and uniqueness of a weak solution for the obstacle problem with variable growth, we first introduce a Kinderlehrer–Stampacchia theorem.

Theorem 2.3 (Kinderlehrer and Stampacchia [20])**.** *Let K be a nonempty closed convex subset of X and let $L : K \to X'$ be monotone, coercive and strong-weakly continuous on K. Then there exists an element u such that*

$$\langle L(u), v-u\rangle \ge 0 \ \text{ for all } \ v \in K.$$

A Young measure is a device to understand and to control the difficulties that arise when a weak convergence does not behave as one would desire with respect to nonlinear functional and operators.

Definition 2.1. Assume that the sequence $\{f_j\}_{j\ge1}$ is bounded in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exist a subsequence $\{f_k\}_{k\ge1} \subset \{f_j\}_{j\ge1}$ and a Borel probability measure v_x on $\mathbb{R}^m$ for a.e. $x \in \Omega$ such that for each $\varphi \in C(\mathbb{R}^m)$, we have

$$\varphi(f_k) \to^* \overline{\varphi} \ \text{ weakly}^* \text{ in } L^\infty(\Omega),$$

where

$$\overline{\varphi}(x) := \int_{\mathbb{R}^m} \varphi(\lambda)\, dv_x(\lambda) \ \text{ for a.e. } \ x \in \Omega.$$

We call $\{v_x\}_{x\in\Omega}$ a family of Young measure associated with $\{f_k\}_{k\ge1}$.

Lemma 2.1 ([14])**.** *Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable (not necessarily bounded) and $w_j : \Omega \to \mathbb{R}^m$, $j = 1, 2, \ldots,$ be a sequence of Lebesgue measurable functions. Then there exist a subsequence w_k and a family $\{v_x\}$ of nonnegative Radon measures on $\mathbb{R}^n$ such that*

(i) $\|v_x\|_{\mathcal{M}} := \int_{\mathbb{R}^m} dv_x(\lambda) \le 1$ *for almost every $x \in \Omega$.*

(ii) $\varphi(w_k) \to^* \overline{\varphi}$ *weakly* * *in $L^\infty(\Omega)$ for any $\varphi \in C_0(\mathbb{R}^m)$, where $\overline{\varphi} = \langle v_x, \varphi\rangle$ and*

$$C_0(\mathbb{R}^m) = \Big\{\varphi \in C(\mathbb{R}^m) : \ \lim_{|w|\to\infty} |\varphi(w)| = 0\Big\}.$$

(iii) *If for any $R > 0$,*

$$\lim_{L\to\infty} \sup_{k\in\mathbb{N}} \Big|\Big\{x \in \Omega \cap B_R(0) : \ |w_k(x)| \ge L\Big\}\Big| = 0,$$

then $\|v_x\|_{\mathcal{M}} = 1$ for almost every $x \in \Omega$, and for any measurable $\Omega' \subset \Omega$, we have $\varphi(w_k) \to \overline{\varphi} = \langle v_x, \varphi\rangle$ weakly in $L^1(\Omega')$ for continuous φ provided the sequence $\varphi(w_k)$ is weakly precompact in $L^1(\Omega')$.

Lemma 2.1 is the fundamental theorem for the Young measure, and the following Fatou-type lemma can be seen as its application, useful for us.

Lemma 2.2 ([14])**.** *Let $a : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function and $u_k : \Omega \to \mathbb{R}^m$ be a sequence of measurable functions such that Du_k generates the Young measure v_x. Then*

$$\liminf_{k\to\infty} \int_\Omega a(x, Du_k(x))\,\mathrm{d}x \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} a(x,\lambda)\,dv_x(\lambda)\,\mathrm{d}x,$$

provided that the negative part $a^-(x, Du_k(x))$ is equiintegrable.

3 Main results

In this section, we obtain the existence and uniqueness of the solution for the obstacle problem (1.1), (1.2).

3.1 Weak solution of obstacle problem

Through the above definitions, we utilize the concept of Young measure to demonstrate the existence of weak solutions for the obstacle problem stated in (1.1), (1.2), by defining a mapping $L : \mathcal{K}_{\psi,\theta} \to W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ by

$$\langle L(u), v\rangle = \int_\Omega \sigma(x, Du) : Dv + \left\langle u|u|^{p(x)-2}, v\right\rangle \mathrm{d}x$$

satisfying the hypothesis of Theorem 2.3.

Theorem 3.1. *Suppose $\mathcal{K}_{\psi,\theta} \neq \varnothing$ and σ satisfies conditions (f_0)–(f_2). Then there exists a weak solution $u \in \mathcal{K}_{\psi,\theta}$ to the obstacle problem (1.1), (1.2). In other words, there exists a function $u \in \mathcal{K}_{\psi,\theta}$ satisfying*

$$\int_\Omega \sigma(x, Du) : D(v-u) + \left\langle u|u|^{p(x)-2}, v-u\right\rangle \mathrm{d}x \geq 0$$

for each $v \in \mathcal{K}_{\psi,\theta}$.

3.2 Proof of of the existence of a weak solution

Now, we solve problem (1.1). Towards this end, first we show the following Lemmas.

Lemma 3.1. *Suppose σ satisfies (f_0)–(f_2) and $\mathcal{K}_{\psi,\theta} \neq \varnothing$, set in (1.2), is given for arbitrary $u \in \mathcal{K}_{\psi,\theta}$, then:*

(i) *$\mathcal{K}_{\psi,\theta}$ is a closed convex set.*

(ii) *For each $v \in \mathcal{K}_{\psi,\theta}$, $Lu \in W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$.*

Proof. (i) is immediate that $\mathcal{K}_{\psi,\theta}$ is a closed convex set.

(ii) By the Hölder growth condition in (f_1), we have

$$\begin{aligned}
|\langle Lu, v\rangle| &= \left| \int_\Omega \sigma(x, Du) : Dv + \left\langle u|u|^{p(x)-2}, v\right\rangle \mathrm{d}x \right| \\
&\leq \left| \int_\Omega \sigma(x, Du) : Dv\,\mathrm{d}x \right| + \left| \int_\Omega \langle u|u|^{p(x)-2}, v\rangle\,\mathrm{d}x \right| \\
&\leq \left(\|N_1\|_{p'(x)} + C_1 \|Du\|_{p(x)}^{p(x)-1} \right) \|Dv\|_{p(x)} + C_2 \|u\|_{p(x)} \|v\|_{p(x)} \\
&\leq \left(\|N_1\|_{p'(x)} + C_1 \|Du\|_{p(x)}^{p(x)-1} \right) \|v\|_{1,p(x)} + C_2 \|u\|_{p(x)} \|v\|_{1,p(x)}
\end{aligned}$$

$$
\begin{aligned}
&\le \Big(\|N_1\|_{p'(x)} + C_1\|Du\|_{p(x)}^{p(x)-1} + C_2\|u\|_{p(x)}\Big)\|v\|_{1,p(x)}\\
&\le \ C\|v\|_{1,p(x)}\,.
\end{aligned}
$$

So, we get $Lu \in W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$. □

Lemma 3.2. *Suppose $\mathcal{K}_{\psi,\theta} \neq \varnothing$ and σ satisfies conditions (f_0)–(f_2). Then the mapping L is monotone and coercive on $\mathcal{K}_{\psi,\theta}$.*

Proof. For fixed $v \in \mathcal{K}_{\psi,\theta}$, by the strict quasimonotonicity, we have

$$
\begin{aligned}
&\langle Lu - Lv, u-v\rangle\\
&\quad = \int_\Omega (\sigma(x,Du)-\sigma(x,Dv)) : (Du-Dv)\,\mathrm{d}x + \int_\Omega \big\langle u|u|^{p(x)-2} - v|v|^{p(x)-2}, u-v\big\rangle\,\mathrm{d}x\\
&\quad \ge C_2\int_\Omega |Du-Dv|^{p(x)}\,\mathrm{d}x + \int_\Omega |u|^{p(x)} + |v|^{p(x)}\,\mathrm{d}x - \int_\Omega \frac{p(x)-1}{p(x)}\,|u|^{p(x)}\,\mathrm{d}x\\
&\qquad - \int_\Omega \frac{1}{p(x)}|v|^{p(x)}\,\mathrm{d}x - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,\mathrm{d}x - \int_\Omega \frac{p(x)-1}{p(x)}\,|v|^{p(x)}\,\mathrm{d}x\\
&\quad \ge C_2\int_\Omega |Du-Dv|^{p(x)}\,\mathrm{d}x + \int_\Omega |u|^{p(x)}dx + \int_\Omega |v|^{p(x)}\,\mathrm{d}x - \int_\Omega |u|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}\,|u|^{p(x)}\,\mathrm{d}x\\
&\qquad - \int_\Omega \frac{1}{p(x)}|v|^{p(x)}\,\mathrm{d}x - \int_\Omega \frac{1}{p(x)}\,|u|^{p(x)}\,\mathrm{d}x - \int_\Omega |v|^{p(x)}\,\mathrm{d}x + \int_\Omega \frac{1}{p(x)}\,|v|^{p(x)}\,\mathrm{d}x\\
&\quad \ge C_2\int_\Omega |Du-Dv|^{p(x)}\,\mathrm{d}x.
\end{aligned}
$$

Then L is monotone on $\mathcal{K}_{\psi,\theta}$.

Next, we show that L is coercive. Indeed, for the fixed element $v \in \mathcal{K}_{\psi,\theta}$, in view of condition (f_1), we have

$$
\begin{aligned}
&\langle Lu - Lv, u-v\rangle\\
&\quad = \int_\Omega (\sigma(x,Du)-\sigma(x,Dv)) : (Du-Dv)\,\mathrm{d}x + \int_\Omega \big\langle u|u|^{p(x)-2} - v|v|^{p(x)-2}, u-v\big\rangle\,\mathrm{d}x\\
&\quad \ge \int_\Omega \sigma(x,Du) : Du\,\mathrm{d}x + \int_\Omega \sigma(x,Dv) : Dv\,\mathrm{d}x - \int_\Omega \sigma(x,Du) : Dv\,\mathrm{d}x\\
&\qquad - \int_\Omega \sigma(x,Dv) : Du\,\mathrm{d}x + \int_\Omega |u|^{p(x)} + |v|^{p(x)}\,\mathrm{d}x - \int_\Omega |u|^{p(x)-1}|v| + |u|\,|v|^{p(x)-1}\,\mathrm{d}x\\
&\quad \ge \int_\Omega \big(-N_2(x) + C_2|Du|^{p(x)}\big)\,\mathrm{d}x - \int_\Omega N_2(x) + C_2|Dv|^{p(x)}\,\mathrm{d}x - \int_\Omega |N_1(x)|\,|Dv|\,\mathrm{d}x\\
&\qquad - C_1\int_\Omega |Du|^{p(x)-1}|Dv|\,\mathrm{d}x - \int_\Omega |N_1(x)|\,|Du|\,\mathrm{d}x - C_1\int_\Omega |Dv|^{p(x)-1}|Du|\,\mathrm{d}x\\
&\qquad + \int_\Omega |u|^{p(x)} + |v|^{p(x)}\,\mathrm{d}x - \int_\Omega |u|^{p(x)-1}|v| + |u|\,|v|^{p(x)-1}\,\mathrm{d}x\\
&\quad \ge C_2\int_\Omega |Du|^{p(x)} + |Dv|^{p(x)}\,\mathrm{d}x - 2\int_\Omega |N_2(x)|\,\mathrm{d}x - \int_\Omega |Du|\,|N_1(x)|\,\mathrm{d}x - \int_\Omega |Dv|\,|N_1(x)|\,\mathrm{d}x\\
&\qquad - C_1\int_\Omega |Du|^{p(x)-1}|Dv| - |Dv|^{p(x)-1}|Du|\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_\Omega |u|^{p(x)} + |v|^{p(x)} \,\mathrm{d}x - \int_\Omega |u|^{p(x)-1}|v| + |u|\,|v|^{p(x)-1} \,\mathrm{d}x \\
\geq{}& \min\{C_2, 1\} \int_\Omega |Du|^{p(x)} + |u|^{p(x)} + |Dv|^{p(x)} + |v|^{p(x)} \,\mathrm{d}x - 2 \int_\Omega |N_2(x)| \,\mathrm{d}x \\
&- \int_\Omega |Dv|\,|N_1(x)| \,\mathrm{d}x - \int_\Omega \frac{\varepsilon}{p(x)} \left(|Du|^{p(x)} + |u|^{p(x)}\right) + \frac{\varepsilon^{\frac{1}{1-p(x)}}}{p'(x)} \left(|N_1(x)|^{p'(x)}\right) \mathrm{d}x \\
&- (C_1+1) \int_\Omega \frac{\varepsilon}{p'(x)} \left(|Du|^{p(x)} + |u|^{p(x)}\right) + \frac{\varepsilon^{\frac{1}{1-p'(x)}}}{p(x)} \left(|Dv|^{p(x)} + |v|^{p(x)}\right) \mathrm{d}x \\
&{}_1 + 1) \int_\Omega \frac{\varepsilon}{p(x)} \left(|Du|^{p(x)} + |u|^{p(x)}\right) + \frac{\varepsilon^{\frac{1}{1-p(x)}}}{p'(x)} \left(|Dv|^{p(x)} + |v|^{p(x)}\right) \mathrm{d}x \\
\geq{}& \left(\min\{C_2, 1\} - \left(\frac{1}{p_*} + C_1 + 1\right)\varepsilon \right) \int_\Omega \left(|Du|^{p(x)} + |u|^{p(x)}\right) \mathrm{d}x - C(N_1, N_2, \varepsilon, v, p(x)).
\end{aligned}
$$

Taking

$$
\varepsilon = \frac{\min\{C_2, 1\} p_*}{2(1 + (C_1+1)p_*)}\,,
$$

we obtain

$$
\begin{aligned}
\langle Lu - Lv, u - v\rangle &\geq C \int_\Omega |Du|^{p(x)} + |u|^{p(x)} \,\mathrm{d}x - C(N_1, N_2, \varepsilon, v, p(x)) \\
&\geq C \int_\Omega 2^{-p^*} \left(|Du - Dv|^{p(x)} + |u - v|^{p(x)}\right) - |Dv|^{p(x)} - |v|^{p(x)} \,\mathrm{d}x \\
&\qquad - C(N_1, N_2, \varepsilon, v, p(x)) \\
&\geq 2^{-p^*} C \int_\Omega |Du - Dv|^{p(x)} + |u - v|^{p(x)} \,\mathrm{d}x - C(N_1, N_2, \varepsilon, v, p(x)).
\end{aligned}
$$

For a sufficiently small constant δ, we have

$$
\begin{aligned}
\frac{\int_\Omega |Du - Dv|^{p(x)} \,\mathrm{d}x}{\|Du - Dv\|_{L^{p(x)}(\Omega,\mathbb{R}^m)}} &= \int_\Omega \left(\frac{|Du - Dv|}{\|Du - Dv\|_{L^{p(x)}(\Omega,\mathbb{R}^m)} - \delta}\right)^{p(x)} \frac{(\|Du - Dv\|_{L^{p(x)}(\Omega,\mathbb{R}^m)} - \delta)^{p(x)}}{\|Du - Dv\|_{L^{p(x)}(\Omega,\mathbb{R}^m)}} \,\mathrm{d}x \\
&\geq \frac{(\|Du - Dv\|_{L^{p(x)}(\Omega,\mathbb{R}^m)} - \delta)^{p(x)}}{\|Du - Dv\|_{L^{p(x)}(\Omega,\mathbb{R}^m)}}\,.
\end{aligned}
$$

Taking $\delta = \frac{1}{2}\|Du - Dv\|_{L^{p(x)}(\Omega,\mathbb{R}^m)}$, we arrive at

$$
\frac{\int_\Omega |Du - Dv|^{p(x)} \,\mathrm{d}x}{\|Du - Dv\|_{L^{p(x)}(\Omega,\mathbb{R}^m)}} \to \infty \ \text{ as } \ \|Du - Dv\|_{L^{p(x)}(\Omega,\mathbb{R}^m)} \to \infty.
$$

Similarly, we also obtain

$$
\frac{\int_\Omega |u - v|^{p(x)} \,\mathrm{d}x}{\|u - v\|_{L^{p(x)}(\Omega,\mathbb{R}^m)}} \to \infty \ \text{ as } \ \|u - v\|_{L^{p(x)}(\Omega,\mathbb{R}^m)} \to \infty.
$$

Then it is immediate to obtain

$$
\frac{(Lu - Lv, u - v)}{\|u - v\|_{W^{1,p(x)}(\Omega,\mathbb{R}^m)}} \to \infty \ \text{ as } \ \|u - v\|_{W^{1,p(x)}(\Omega,\mathbb{R}^m)} \to \infty.
$$

That is to say, L is coercive on $\mathcal{K}_{\psi,\theta}$. □

Lemma 3.3. *Suppose $\mathcal{K}_{\psi,\theta} \neq \varnothing$ and σ satisfies conditions (f_0)–(f_2). Then the mapping L is strongly-weakly continuous.*

Proof. We choose a sequence $u_k \in \mathcal{K}_{\psi,\theta}$ such that $u_k \to u \in \mathcal{K}_{\psi,\theta}$ in $W^{1,p(x)}(\Omega;\mathbb{R}^m)$. Then $\|u_k\|_{1,p(x)} \leq C$ for some constant C. In view of Lemma 2.1, there exists a Young measure v_x generated by $\{Du_k\}$ such that $\|v_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ and

$$Du_k \to \langle v_x, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda\, dv_x(\lambda) \ \text{ in } L^1(\Omega). \tag{3.1}$$

Since $L^{p(x)}(\Omega;\mathbb{M}^{m\times n})$ is reflexive, $Du_k \to Du$ in $L^{p(x)}(\Omega;\mathbb{M}^{m\times n}) \subset L^1(\Omega;\mathbb{M}^{m\times n})$, thus $Du(x) = \langle v_x, id\rangle$ for a.e. $x \in \Omega$ (for the uniqueness of limit, see also [3, Lemma 4.1]).

The following lemmas allow us to prove Lemma 3.3.

Lemma 3.4 (div-curl inequality)**.** *Suppose σ satisfies (f_0)–(f_2) and $\{Du_k\}$ generates the Young measure v_x, then*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} (\sigma(x,\lambda) - \sigma(x,Du)) : (\lambda - Du)\, dv_x(\lambda)\, \mathrm{d}x \leq 0.$$

Proof. Let us consider the sequence

$$I_k := (\sigma(x,Du_k) - \sigma(x,Du)) : (Du_k - Du) = \sigma(x,Du_k) : (Du_k - Du) - \sigma(x,Du) : (Du_k - Du).$$

By the growth condition in (f_1), we have

$$\begin{aligned}
\int_\Omega |\sigma(x,Du)|^{p'(x)}\, \mathrm{d}x &\leq \int_\Omega \left(|N_1(x)|^{p'(x)} + c_1|Du|^{p(x)}\right) \mathrm{d}x \\
&\leq \int_\Omega |N_1(x)|^{p'(x)} + c_1\|Du\|_{p(x)}^{p(x)}\, \mathrm{d}x \\
&\leq \int_\Omega |N_1(x)|^{p'(x)} + c_1 M^{p(x)}\, \mathrm{d}x \ \ \text{(where M is the upper bound of } \|Du\|_{p(x)}) \\
&< \infty.
\end{aligned}$$

Since $u \in W^{1,p(x)}(\Omega;\mathbb{R}^m)$, we get $\sigma \in L^{p'(x)}(\Omega,\mathbb{M}^{m\times n})$. According to the weak convergence in (3.1), we obtain

$$\begin{aligned}
&\liminf_{k\to\infty} \int_\Omega \sigma(x,Du) : (Du_k - Du)\, \mathrm{d}x \\
&\quad = \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,Du) : (\lambda - Du)\, dv_x(\lambda)\, \mathrm{d}x = \int_\Omega \sigma(x,Du) : \underbrace{\int_{\mathbb{M}^{m\times n}} \lambda\, dv_x(\lambda)}_{=:Du(x)} - Du)\, \mathrm{d}x = 0.
\end{aligned}$$

Therefore,

$$I := \liminf_{k\to\infty} \int_\Omega I_k\, \mathrm{d}x = \liminf_{k\to\infty} \int_\Omega \sigma(x,Du_k) : (Du_k - Du)\, \mathrm{d}x.$$

Let $\Omega' \subset \Omega$ be an arbitrary measurable subset. By the growth condition in (f_1) together with Hölder's inequality, we have

$$\int_{\Omega'} |\sigma(x,Du_k) : Du|\, \mathrm{d}x \leq \Big(\|N_1(x)\|_{p'(x)} + c_1 \underbrace{\|Du_k\|_{p(x)}^{p(x)-1}}_{\leq C}\Big)\Big(\int_{\Omega'} |Du|^{p(x)}\, \mathrm{d}x\Big)^{\frac{1}{p(x)}}.$$

Since $\int_{\Omega'} |Du|^{p(x)}\,\mathrm{d}x$ is arbitrarily small, if we choose the measure of Ω' small enough, it follows that the negative part $(\sigma(x,Du_k):Du)^-$ is equiintegrable. On the other hand, by the coercivity condition in (f_1), we have

$$\sigma(x,Du_k):Du_k \geq -N_2(x)+c_2|Du_k|^{p(x)} \geq -N_2(x).$$

Thus

$$\int_{\Omega'} \big(\sigma(x,Du_k):Du_k\big)^-\,\mathrm{d}x \leq \int_{\Omega'} |N_2(x)|\,\mathrm{d}x.$$

Hence $(\sigma(x,Du_k):Du_k)^-$is equiintegrable. We infer from Lemma 2.2 that

$$I=\liminf_{k\to\infty}\int_\Omega \sigma(x,Du_k):(Du_k-Du)\,\mathrm{d}x \geq \int_\Omega\int_{\mathbb{M}^{m\times n}} \sigma(x,\lambda):(\lambda-Du)\,dv_x(\lambda)\,\mathrm{d}x.$$

We prove that $I\leq 0$. In fact, according to Mazur's theorem (see, e.g., [29, Theorem 2, p. 120]), there exists $(v_k)\in W^{1,p(x)}(\Omega;\mathbb{R}^m)$, where each v_k is a convex linear combination of $\{u_1,\dots,u_k\}$ such that $v_k\to u$ in $W^{1,p(x)}(\Omega;\mathbb{R}^m)$. This implies that v_k belongs to the same space as u_k. It follows that

$$\begin{aligned}
I&=\liminf_{k\to\infty}\int_\Omega \sigma(x,Du_k):(Du_k-Du)\,\mathrm{d}x\\
&=\liminf_{k\to\infty}\bigg(\int_\Omega \sigma(x,Du_k):(Du_k-Dv_k)\,\mathrm{d}x+\int_\Omega \sigma(x,Du_k):(Dv_k-Du)\,\mathrm{d}x\bigg)\\
&\leq \liminf_{k\to\infty}\|\sigma(x,Du_k)\|_{p'(x)}\|Du_k-Dv_k\|_{p(x)}+\|\sigma(x,Du_k)\|_{p'(x)}\|Dv_k-Du\|_{p(x)}=0,
\end{aligned}$$

by the boundedness of $\sigma(x,Du_k)$ in $L^{p'(x)}(\Omega;\mathbb{M}^{m\times n})$, the construction of the sequence v_k and the following fact

$$\|u_k-v_k\|_{1,p(x)}\leq \|u_k-u\|_{1,p(x)}+\|v_k-u\|_{1,p(x)}\to 0 \text{ as } k\to\infty.$$

Since

$$\int_\Omega\int_{\mathbb{M}^{m\times n}} \sigma(x,Du):(\lambda-Du)\,dv_x(\lambda)\,\mathrm{d}x=\int_\Omega \sigma(x,Du):\bigg(\int_{\mathbb{M}^{m\times n}} \lambda\,dv_x(\lambda)-Du\bigg)\,\mathrm{d}x=0$$

together with $I\leq 0$, the inequality of Lemma 3.4 follows. □

Remark 3.1. An intermidiary result is the following inequality:

$$\liminf_{k\to\infty}\int_\Omega (\sigma(x,Du_k)-\sigma(x,Du)):(Du_k-Du)\,\mathrm{d}x\leq 0.$$

To see this, it suffices to repeat the proof of Lemma 3.4.

Lemma 3.5. *For almost every $x\in\Omega$, we have*

$$(\sigma(x,\lambda)-\sigma(x,Du)):(\lambda-Du)=0 \;\; \textit{on} \;\; \operatorname{supp} v_x.$$

Proof. By Lemma 3.4, we have

$$\int_\Omega\int_{\mathbb{M}^{m\times n}} (\sigma(x,\lambda)-\sigma(x,Du)):(\lambda-Du)\,dv_x(\lambda)\,\mathrm{d}x\leq 0.$$

By the monotonicity of σ, the above integrand is nonnegative, thus must vanish with respect to the product measure $\mathrm{d}v_x(\lambda)\otimes\mathrm{d}x$. Therefore,

$$(\sigma(x,\lambda)-\sigma(x,Du)):(\lambda-Du)=0 \text{ on } \operatorname{supp} v_x.$$ □

Now, we prove Lemma 3.3 for each case listed in (f_2).

Step 1. Suppose that σ satisfies condition (f_2)(a). We have

$$\int_\Omega |Du_k - Du|^{p(x)}\,\mathrm{d}x \le c\int_\Omega (\sigma(x,Du_k)-\sigma(x,Du)) : (Du_k - Du)\,\mathrm{d}x.$$

We remark that the inferior limit of the right-hand side of the above inequality is less than or equal to zero by Remark 3.1. It follows that

$$\liminf_{k\to\infty}\int_\Omega |Du_k - Du|^{p(x)}\,\mathrm{d}x = 0.$$

Let $E_{k,\epsilon} = \{x : |Du_k - Du| \ge \epsilon\}$. We have

$$\int_\Omega |Du_k - Du|^{p(x)}\,\mathrm{d}x \ge \int_{E_{k,e}} |Du_k - Du|^{p(x)}\,\mathrm{d}x \ge \epsilon^{p(x)}|E_{k,\epsilon}|$$

which yields

$$|E_{k,\epsilon}| \le \frac{1}{\epsilon^{p(x)}}\int_\Omega |Du_k - Du|^{p(x)}\,\mathrm{d}x \to 0 \;\text{ as } k\to\infty.$$

Since by the Fatou Lemma,

$$\int_\Omega \Big(\frac{|Du_k - Du|}{\epsilon}\Big)^{p(x)}\mathrm{d}x \leqslant \lim_{k'\to\infty}\sup\bigg(\int_\Omega \frac{|Du_{k'} - Du_k|}{\epsilon}\bigg)^{p(x)}\mathrm{d}x,$$

we have

$$\|Du_k - Du\|_{L^{p(x)}(\Omega,\mathbb{R}^m)} \leqslant \sup_{k'}\big\{\|Du_{k'} - Du_k\|_{L^{p(x)}(\Omega,\mathbb{R}^m)}\big\} < \varepsilon',$$

that is to say, $Du_k \to Du$ in $L^{p(x)}(\Omega,\mathbb{R}^m)$. Hence

$$Du_k \to Du \;\text{ in measure on } \Omega \text{ (for a subsequence)}.$$

After extracting a suitable subsequence, if necessary, we can infer that $Du_k \to Du$ for almost every $x \in \Omega$. Then $\sigma(x,Du_k) \to \sigma(x,Du)$ for almost every $x\in\Omega$, and in the measure. By the equiintegrability of $\sigma(x,Du_k) : Dv$, the Vitali theorem implies

$$\int_\Omega \sigma(x,Du_k) : Dv\,\mathrm{d}x \to \int_\Omega \sigma(x,Du) : Dv\,\mathrm{d}x \;\text{ as } k\to\infty.$$

Step 2. For the case (f_2)(b), we argue as follows: We start by proving that for almost every $x\in\Omega$,

$$\operatorname{supp} v_x \subset E_x = \Big\{\lambda \in \mathbb{M}^{m\times n} : \;\; Z(x,\lambda) = Z(x,Du) + \sigma(x,Du) : (\lambda - Du)\Big\}.$$

Let $\lambda \in \operatorname{supp} v_x$, then by Lemma 3.5, we get

$$(1-\tau)(\sigma(x,\lambda)-\sigma(x,Du)) : (\lambda - Du) = 0 \;\; \forall\,\tau\in[0,1]. \tag{3.2}$$

On the other hand, by the monotonicity, for $\tau\in[0,1]$ we have

$$(1-\tau)\big(\sigma(x,Du+\tau(\lambda-Du)) - \sigma(x,\lambda)\big) : (Du-\lambda) \ge 0. \tag{3.3}$$

Subtracting (3.2) from (3.3), we get

$$(1-\tau)\big(\sigma(x,Du+\tau(\lambda-Du)) - \sigma(x,Du)\big) : (Du-\lambda) \ge 0 \;\text{ for }\; \tau\in[0,1]. \tag{3.4}$$

By the monotonicity,

$$\big(\sigma(x, Du + \tau(\lambda - Du)) - \sigma(x, Du)\big) : \tau(\lambda - Du) \geq 0,$$

and since $\tau \in [0,1]$, we have

$$\big(\sigma(x, Du + \tau(\lambda - Du)) - \sigma(x, Du)\big) : (1-\tau)(\lambda - Du) \geq 0.$$

The above inequality together with (3.4) implies

$$\big(\sigma(x, Du + \tau(\lambda - Du)) - \sigma(x, Du)\big) : (\lambda - Du) = 0 \ \ \forall \tau \in [0,1].$$

Integrating this equality over $[0,1]$ and using the fact that

$$\sigma(x, Du + \tau(\lambda - Du)) : (\lambda - Du) = \frac{\partial Z}{\partial \tau}\,(x, Du + \tau(\lambda - Du)) : (\lambda - Du),$$

we conclude that

$$\begin{aligned} Z(x,\lambda) &= Z(x, Du) + \int_0^1 \sigma(x, Du + \tau(\lambda - Du)) : (\lambda - Du)\, d\tau \\ &= Z(x, Du) + \sigma(x, Du) : (\lambda - Du). \end{aligned}$$

Hence $\lambda \in E_x$, i.e., $\operatorname{supp} v_x \subset E_x$. In view of the convexity of Z, we have

$$Z(x,\lambda) \geq Z(x, Du) + \sigma(x, Du) : (\lambda - Du).$$

For all $\lambda \in E_x$, put $A(\lambda) = Z(x,\lambda)$ and $B(\lambda) = Z(x, Du) + \sigma(x, Du) : (\lambda - Du)$. Since $\lambda \mapsto A(\lambda)$ is continuous and differentiable, for all $S \in \mathbb{M}^{m\times n}$ and $\tau \in \mathbb{R}$ we obtain

$$\begin{aligned} \frac{A(\lambda+\tau S) - A(\lambda)}{\tau} &\geq \frac{B(\lambda+\tau F) - B(\lambda)}{\tau} \ \text{ if } \ \tau > 0, \\ \frac{A(\lambda+\tau S) - A(\lambda)}{\tau} &\leq \frac{B(\lambda+\tau S) - B(\lambda)}{\tau} \ \text{ if } \ \tau < 0. \end{aligned}$$

Thus $DA = DB$ and therefore

$$\sigma(x,\lambda) = \sigma(x, Du) \quad \forall \lambda \in E_x \supset \operatorname{supp} v_x. \tag{3.5}$$

The equiintegrability of $\sigma(x, Du_k)$ implies that its weak L^1-limit is given by

$$\overline{\sigma}(x) := \int_{\mathbb{M}^{m\times n}} \sigma(x,\lambda)\, \mathrm{d}v_x(\lambda) = \int_{\operatorname{supp} v_x} \sigma(x,\lambda)\, \mathrm{d}v_x(\lambda) = \int_{\operatorname{supp} v_x} \sigma(x, Du)\, \mathrm{d}v_x(\lambda) = \sigma(x, Du), \tag{3.6}$$

where we have used (3.5) and $\|v_x\|_{\mathcal{M}} = 1$. Now, consider the Carathéodory function

$$\omega(x,\lambda) = |\sigma(x,\lambda) - \overline{\sigma}(x)|, \ \ \lambda \in \mathbb{M}^{m\times n}.$$

The sequence $\omega_k(x) := \omega(x, Du_k(x))$ is equiintegrable by that of $\sigma(x, Du_k(x))$, hence its weak L^1-limit is given by

$$\omega_k \to \overline{\omega} \ \text{ in } L^1(\Omega),$$

where

$$\begin{aligned} \overline{\omega}(x) &= \int_{\mathbb{M}^{m\times n}} |\sigma(x,\lambda) - \overline{\sigma}(x)|\, \mathrm{d}v_x(\lambda) \\ &= \int_{\operatorname{supp} v_x} |\sigma(x,\lambda) - \overline{\sigma}(x)|\, dv_x(\lambda) = 0 \ \text{ (by (3.6) and (3.5))}. \end{aligned}$$

Since $\omega_k \geq 0$, we deduce that $\omega_k \to 0$ in $L^1(\Omega)$ as $k \to \infty$. Hence

$$\int_\Omega \sigma(x, Du_k) : Dv \, \mathrm{d}x \to \int_\Omega \sigma(x, Du) : Dv \, \mathrm{d}x \text{ as } k \to \infty.$$

Step 3. For the last case $(f_2)(c)$, we claim that for a.e. $x \in \Omega$ and every $S \in \mathbb{M}^{m\times n}$,

$$\sigma(x, \lambda) : S = \sigma(x, Du) : S + (\nabla\sigma(x, Du)) : (Du - S)$$

holds on supp v_x, where ∇ is the derivative with respect to the second variable of σ. The monotonicity of σ implies that for $\tau \in \mathbb{R}$,

$$(\sigma(x, \lambda) - \sigma(x, Du + \tau S)) : (\lambda - Du - \tau S) \geq 0,$$

which implies

$$-\sigma(x, \lambda) : \tau S \geq -\sigma(x, \lambda) : (\lambda - Du) + \sigma(x, Du + \tau S) : (\lambda - Du - \tau S).$$

By virtue of Lemma 3.5, we get

$$-\sigma(x, \lambda) : \tau S \geq -\sigma(x, Du) : (\lambda - Du) + \sigma(x, Du + \tau S) : (\lambda - Du - \tau S).$$

Note that

$$\sigma(x, Du + \tau S) = \sigma(x, Du) + \nabla\sigma(x, Du)\tau S + o(\tau),$$

thus

$$\begin{aligned}\sigma(x, Du + \tau S) : (\lambda - Du - \tau S) &= \sigma(x, Du + \tau S) : (\lambda - Du) - \sigma(x, Du + \tau S) : \tau S\\ &= \sigma(x, Du) : (\lambda - Du) + \nabla\sigma(x, Du)\tau S : (\lambda - Du) - \sigma(x, Du) : \tau S - \nabla\sigma(x, Du)\tau S : \tau S + o(\tau)\\ &= \sigma(x, Du) : (\lambda - Du) + \tau[\nabla\sigma(x, Du)S : (\lambda - Du) - \sigma(x, Du)] + o(\tau).\end{aligned}$$

Therefore,

$$-\sigma(x, \lambda) : \tau S \geq \tau\big[(\nabla\sigma(x, Du)S) : (\lambda - Du) - \sigma(x, Du) : S\big] + o(\tau).$$

Since τ is arbitrary in $\mathbb{R}$, our claim follows. By the equiintegrability of $\sigma(x, Du_k)$, its weak L^1-limit is then given by

$$\begin{aligned}\overline{\sigma}(x) &= \int_{\operatorname{supp} v_x} \sigma(x, \lambda) \, \mathrm{d}v_x(\lambda)\\ &= \int_{\operatorname{supp} v_x} \sigma(x, Du) \, \mathrm{d}v_x(\lambda) + (\nabla\sigma(x, Du))^t \int_{\operatorname{supp} v_x} (Du - \lambda) \, \mathrm{d}v_x(\lambda) = \sigma(x, Du),\end{aligned}$$

where we have used our claim and the fact that $Du(x) = \langle v_x, id\rangle$. On the other hand, since $L^{p'(x)}(\Omega; \mathbb{M}^{m\times n})$ is reflexive, the sequence $\{\sigma(x, Du_k)\}$ converges weakly in $L^{p'(x)}(\Omega; \mathbb{M}^{m\times n})$ and its weak $L^{p'(x)}$-limit is also $\sigma(x, Du)$. Thus we conclude that

$$\int_\Omega \sigma(x, Du_k) : Dv \, \mathrm{d}x \to \int_\Omega \sigma(x, Du) : Dv \, \mathrm{d}x \text{ as } k \to \infty.$$

Hence $\int_\Omega \sigma(x, Du_k) : Dv \, \mathrm{d}x \to \int_\Omega \sigma(x, Du) : Dv \, \mathrm{d}x$ as $k \to \infty$ in the cases (a), (b) and (c).

Now we show that

$$\big\langle u_k|u_k|^{p(x)-2}, v\big\rangle \to \big\langle u|u|^{p(x)-2}, v\big\rangle \text{ as } k \to \infty.$$

Since $u_k \to u$ in $W^{1,p(x)}(\Omega; \mathbb{R}^m)$, the continuity of the inner product implies

$$\langle u_k, v\rangle \to \langle u, v\rangle \text{ as } k \to \infty.$$

Therefore, by Theorem 2.1, we have that $\langle u_k|u_k|^{p(x)-2}, v\rangle$ is bounded for all $u_k \geq \psi$ a.e in Ω, $p(x) \geq 1$. Hence, by using $u_k \to u$, we get

$$\left\langle u_k|u_k|^{p(x)-2}, v\right\rangle \to \left\langle u|u|^{p(x)-2}, v\right\rangle \text{ as } k \to \infty.$$

Next, passing to the limit, we get

$$(Lu_k, v) = \int_\Omega \sigma(x, Du_k) : Dv + \langle u_k|u_k|^{p(x)-2}, v\rangle \,\mathrm{d}x$$

$$\to \int_\Omega \sigma(x, Du) : Dv + \langle u|u|^{p(x)-2}, v\rangle \,\mathrm{d}x = (Lu, v).$$

This is the strong-weakly continuity of L on $\mathcal{K}_{\psi,\theta}$. Thus the proof of Lemma 3.3 is complete. □

Now, we can apply Theorem 2.3 and the above lemmas to obtain the existence. In sum, we conclude the existence of an element $u \in \mathcal{K}_{\psi,\theta}$ such that $\langle L(u), v-u\rangle \geq 0$, i.e.,

$$\int_\Omega \sigma(x, Du) : (Dv - Du) + \left\langle u|u|^{p(x)-2}, v-u\right\rangle \mathrm{d}x \geq 0 \text{ for all } v \in \mathcal{K}_{\psi,\theta}.$$

4 Uniqueness of weak solutions to problem

In order to obtain the uniqueness of the solution, we need to prove the following theorem.

Theorem 4.1. *Suppose $\mathcal{K}_{\psi,\theta} \neq \phi$ and $p(x)$ satisfies* (1.3). *Under conditions* (f_1)–(f_2)(c)*, there exists a unique solution $u \in \mathcal{K}_{\psi,\theta}$ to the obstacle problem* (1.1). *That is to say, there exists a unique $u \in \mathcal{K}_{\psi,\theta}$ such that*

$$\int_\Omega \sigma(x, Du) : (Dv - Du) + \left\langle u|u|^{p(x)-2}, v-u\right\rangle \mathrm{d}x \geq 0 \text{ for all } v \in \mathcal{K}_{\psi,\theta}.$$

Proof. It is immediate to obtain the existence from the above lemmas. If there are two weak solutions $u_1, u_2 \in \mathcal{K}_{\psi,\theta}$ to the obstacle problem (1.1), then

$$\int_\Omega \sigma(x, Du_1) : (Du_2 - Du_1)\,\mathrm{d}x + \left\langle u_1|u_1|^{p(x)-2}, u_2 - u_1\right\rangle \mathrm{d}x \geq 0 \text{ for all } v \in \mathcal{K}_{\psi,\theta}$$

and

$$-\int_\Omega \sigma(x, Du_2) : (Du_2 - Du_1)\,\mathrm{d}x + \left\langle u_2|u_2|^{p(x)-2}, u_2 - u_1\right\rangle \mathrm{d}x$$

$$= \int_\Omega \sigma(x, Du_2) : (Du_1 - Du_2)\,\mathrm{d}x + \left\langle u_2|u_2|^{p(x)-2}, u_1 - u_2\right\rangle \mathrm{d}x \geq 0.$$

Furthermore,

$$\int_\Omega \sigma(x, Du_1) - \sigma(x, Du_2) : (Du_1 - Du_2) + \left\langle u_1|u_1|^{p(x)-2} - u_2|u_2|^{p(x)-2}, u_1 - u_2\right\rangle \mathrm{d}x \leq 0.$$

In view of (f_2)(c), we can further infer that

$$\int_\Omega \sigma(x, Du_1) - \sigma(x, Du_2) : (Du_1 - Du_2)\,\mathrm{d}x = 0 \text{ on } \Omega$$

and

$$\int_\Omega \left\langle u_1|u_1|^{p(x)-2} - u_2|u_2|^{p(x)-2}, u_1 - u_2\right\rangle \mathrm{d}x = 0,$$

that is to say, $u_1 = u_2$ a.e. on Ω, and now the proof is completed. □

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Authors' addresses:

Mouad Allalou
Laboratory LMACS, FST of Beni-Mellal, Sultan Moulay slimane University, Morocco.
E-mail: mouadallalou@gmail.com

Abderrahmane Raji
Laboratory LMACS, FST of Beni-Mellal, Sultan Moulay slimane University, Morocco.
E-mail: rajiabd2@gmail.com

Farah Balaadich
Laboratory LMACS, FST of Beni-Mellal, Sultan Moulay slimane University, Morocco.
E-mail: balaadich.edp@gmail.com