On Logics of Group Belief in Structured Coalitions

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Abstract. In the study of group belief formation, groups of agents are often assumed to possess a topological structure. Here we investigate some ways in which this topological structure may provide the semantical basis for logics of group belief. We impose a partial order on a set of agents first to be able to express preferences of agents by their doxastic abilities, secondly to express the idea of a coalition (well formed group) and thirdly to give a natural semantics for the group belief operator. We define the group belief of a set of agents in two different ways and study their corresponding logics. We also study a logic where doxastic preference is expressed by a binary operator. We prove completeness and discuss correspondences between the logics.

1 Introduction

An important concept in the study of collective intentionality as well as group reasoning is that of *group belief*. The nature of group belief has been analysed by a number of scholars and is of interest in areas such as philosophy, psychology, logic, social sciences and computer science. Quinton [9] for example discussed the *summative* view whereby a group G has a group belief in a proposition p if most of the members of G believe that p; here 'most' can refer to a simple numerical majority or perhaps to a majority of members of a certain kind. More recent work in the field of social ontology has taken a non-summative view according to which individual beliefs do not play such an important role in forming the group belief [6,10]. To have a group belief that p, in this kind of a non-summative, agreement-based sense, it is neither sufficient nor even necessary that the group members individually believe p. Instead, it is required that they together agree that as a group they believe that p. Different versions of the summative and non-summative views have recently been analysed by Gaudou *et al.* [5] who develop in detail a modal logic of group belief and compare their formal system to different philosophical accounts of the group belief concept.

In the discussion of group belief an important feature is that a group should be a constituted collective. In the approach of [5] the nature of the constituted

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group is given by the logic. More precisely, the logic is equipped with a possible worlds semantics whose accessibility relation determines the nature of the group. This idea seems to work well if one assumes that each group is constituted by a unique set of agents A, but it may be problematic if a given set of individuals constitutes two or more different groups. Suppose for example that the university darts team happens to be co-extensive with the graduate admissions committee. Their group beliefs will no doubt be different in the two contexts in which they act. For instance the judgement that Phil Taylor is the greatest ever darts player might be a belief of the darts team but not of the admission committee. This difference in group beliefs will not be manifest in approaches like that of [5]. The authors are aware of this limitation. In another paper devoted to the logic of group *acceptance* [8] they have introduced the idea of an *institutional context* that enters into the semantics of group attitudes. This is a formal device that allows one to distinguish the set of agents from the group or team situation in which they are acting. It supplies an additional parameter of evaluation but doesn't impose any structure on the groups themselves.

A different kind of approach has been explored in work on judgement aggregation. For example, List and Pettit [7] discuss group agency and group beliefs by assuming that some organizational structure is associated with the groups. This structure can be understood in at least two apparently different senses. In one sense it refers to mechanisms such as voting rights and procedures that may be in place in order for group judgements to be obtained by some rational process from the beliefs and preferences of individual group members. Such mechanisms may be thought of as external to the agents themselves, since they reflect group features that may persist even if the set of agents that constitutes the group changes over time. However, [7] also discusses ways in which a group may be structured in a more internal sense. An example is when large judgmental tasks are decomposed into several smaller tasks and the corresponding group judgements for these tasks are allocated to suitable subgroups. As List and Pettit observe [7] (pp. 94–97), not all group members may have the same level of expertise, so it may be rationally justified (at least in theory) to assign judgement subtasks to say expert subgroups and then use a further aggregation mechanism to form a final collective judgement for the whole group. In such cases the chosen decomposition may reflect properties of individual members (e.g. their expertise) and hence need not persist when members leave and enter the group. Nevertheless it seems clear that such structures are group-specific in kind, since if two different groups are composed of the same set of members, the associated group structures will carve up that set in different ways.

In this paper we also study the idea of groups having a structure, but using a different approach from that of [7]. We explore the effects of imposing a topological structure directly on the set of agents and without assuming that judgmental tasks are split into subtasks for resolution by a subgroup. One effect of our approach is that even if say the university darts team and the humanities graduate committee are composed of the same individuals, their constitution *qua* groups (hence their collective beliefs) may be different. Another effect is that the topological structure may reflect a natural ordering among agents, such as their level of knowledge of a certain domain, their abilities, their degree of commitment to a certain cause, or some other relevant criterion. We will deal with finite sets of agents and therefore the topological structure will amount to a partial ordering.[1] In real life situations one observes that arbitrary subsets of agents do not form a coalition. Usually coalitions are closed under some specific properties. Having structured groups makes it possible to formalise different versions of group belief and also explore the connections and differences between different approaches. In this paper we attempt to model both ideas simultaneously by considering partial orders on the sets of agents. It is known that such orders naturally model many existing real life social commitments.[2] Moreover with partial orders we may understand coalitions as those sets of agents which have certain properties according to the given order. In particular that they are downsets.

The paper is organised as follows. In Sect. 3 we define a logic $\mathcal{GB}1$ of group belief where group belief is defined in terms of shared belief. The group belief defined in this way inherits some properties of group belief discussed in [5] although it lacks the important property that group belief p implies that it is common belief that p is a group belief. To remedy this in Sect. 4 we define a logic $\mathcal{GB}2$ where group belief is defined in terms of common belief. This logic gains the property that was missing for the logic $\mathcal{GB}1$ but it loses another property satisfied by $\mathcal{GB}1$, in particular: that group belief does not imply the common belief of group members. In both Sects. 3 and 4 we extend the logics with a modal dependency axiom which links the partial order of agents to their belief sets. In both extended logics group belief collapses to the shared belief of group members. In Sect. 5 we consider pure multi-modal logic with an additional operator $a \preceq b$ to take control over the structure of agents. We prove several completeness results. Completeness for the logics $\mathcal{GB}1$ and $\mathcal{GB}2$ is relatively simple and closely based on already existing results, while completeness for the logic $\mathcal{GB}$ from Sect. 5 is nonstandard and uses a selection method.

2 Preliminaries

We recall some basic definitions and notions which will be used throughout the paper.

Definition 1. *A partial order on a set A is a relation $\leq \subseteq A \times A$ which is reflexive $\forall a \in A)(a \leq a)$ and transitive $(\forall a, b, c \in A)(a \leq b \wedge b \leq c \rightarrow a \leq c)$.*

Every partial order has a distinguished class of subsets called *downsets*

[1] Topological structures in groups are also used to formalise group attitudes in Dunin-Keplicz and Verbrugge [3]. As they emphasise, this structure may be based on power or dependency relations that reflect different social commitments. [3] considers different group topologies but the approach is somewhat different from ours. The topologies are mainly used to model different forms of communication between agents in a group. A related, formal account of group beliefs is studied in [2] using a concept of (group) epistemic profile to model doxastic reasoning. However epistemic profiles are an additional feature, not derived from the group topological structure.

[2] See e.g. [3] and further references given there.

Definition 2. *A subset D of a partial order $(A, \leq)$ is a downset if for every $d \in D$ and every $a \in A$ if $a \leq d$ then $a \in D$. The minimal downset containing the set $J \subseteq A$ will be denoted by $\overline{J}$. In other words $\overline{J} = \{a \in A \mid \exists b \in J \text{ s.t. } a \leq b\}$.*

Throughout the paper we will be working in a standard multimodal language enriched with different operators for common belief, shared belief group belief, etc. The language $\mathcal{L}$ is defined with an infinite set of propositional letters $p, q, r..$ and connectives $\vee, \wedge, \neg, \Box_a$, for each $a \in A$, where A is a finite, partially ordered set $(A, \leq)$ of agents. Observe that the ordering of a set of agents A is common for both the syntax and semantics. Formulas are constructed in a standard way from the following recursive definition:

$$\phi := \mid p \mid \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi \mid \Box_a \phi$$

for every $a \in A$ and $G \subseteq A$. For extensions of $\mathcal{L}$ with additional operators we will use the abbreviation $\mathcal{L}(\{O_i \mid 1 \leq i \leq n\})$ where each O_i is a new operator and the set of formulas is extended in an appropriate way i.e. in the construction of formulas we will have additional clauses

$$\phi := \mid p \mid \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi \mid \Box_a \phi \mid O_i \phi$$

for every $a \in A$ and $i \in \{1, .., n\}$. For example $\mathcal{L}(\{E_J \mid J \subseteq A\}$ denotes the language $\mathcal{L}$ extended with operators E_J for each $J \subseteq A$. Throughout the paper the operators E_J, C_J and GB_J will stand for the *shared belief*, *common belief* and *group belief* operators respectively.

Shared belief is defined as the conjunction of beliefs of individual members of the group. i.e. a proposition p is a shared belief of the group J (abbreviated as $E_J p$) if every member of the group believes that p which means $\bigwedge_{i \in J} \Box_i p$. The shared belief operator is definable in the basic language and hence the languages $\mathcal{L}$ and $\mathcal{L}(\{E_J \mid J \subseteq A\})$ have the same expressive power. This is not the case for common belief. Common belief is defined as the infinite iteration of individual beliefs of group members. Formally $C_J p$ iff $\bigwedge_{n \in \omega} E_J^n p$ which is an infinite conjunction and therefore is not a formula of the language $\mathcal{L}(\{E_J \mid J \subseteq A\}$. In general it is known that $\mathcal{L}(\{C_J \mid J \subseteq A\}$ is strictly more expressive then $\mathcal{L}$.

3 Logics of Group Belief

We define a modal logic of group belief in a structured set $(A, \leq)$ of agents, where the structure $\leq \subseteq A \times A$ is a partial order. Coalitions are formed by downsets. Therefore the structure of coalitions of agents will depend on the relation $\leq$ in question.

3.1 Syntax of $\mathcal{GB}1$

The language has two operators: for shared belief and for group belief. Shared belief (analogous to shared knowledge) has been considered and studied intensively, see for example [4]. We enrich the logic with a group belief operator where group belief is defined as the shared belief of the coalition to which the group belongs. Hence the two groups J and J' of agents have the same group belief if they both belong to the same coalition.

Definition 3. *The normal modal logic* $\mathcal{GB}1$ *is defined in a modal language* $\mathcal{L}(\{E_J, GB_J \mid J \subseteq A\})$. *Operators* E_J *and* GB_J, *stand for shared belief and group belief respectively.*

The axioms of $\mathcal{GB}1$ *are all classical tautologies. Each box satisfies the* $K4$ *axioms for every* $a \in A$, *and in addition we have one axiom scheme for shared belief and one axiom scheme for the group belief,*

$$\Box_a(p \to q) \to (\Box_a p \to \Box_a q) \tag{1}$$

$$\Box_a p \to \Box_a \Box_a p \tag{2}$$

$$E_J p \leftrightarrow \bigwedge_{a \in J} \Box_a p \tag{3}$$

$$GB_J p \leftrightarrow E_{\overline{J}} p \tag{4}$$

for every $J \subseteq A$. *The rules of inference are: modus ponens, substitution and necessitation for each box modality.*

Observe that the axiom of group belief operator uses symbol $\overline{J}$ from Definition 2, hence implicitly refers to the partial order on the set of agents A. As it was mentioned in the introduction the order on agents is needed to form coalitions and coalitions are exactly downsets according to the order on agents. In these terms the group belief axiom from Definition 3 says that p is a group belief of the a group of agents J if p is a shared belief of the minimal coalition $\overline{J}$ to which the group J belongs.

Example 4 ***Every group forms a coalition.*** *Assume that* $\leq$ *is an empty relation. In this case the downset* $\overline{J} = J$. *Hence every subset of agents forms a coalition and hence group belief coincides with shared belief.* $GB_J p \leftrightarrow E_{\overline{J}} p \leftrightarrow E_J p$.

Example 5 ***The only coalition.*** *Assume that* $\leq = A \times A$. *In this case we have only one coalition as far as* $\overline{J} = A$ *for every* $J \subseteq A$. *Hence something is a group belief only if it is a shared belief of all agents.*

Example 6. *Let* $A = \{w, u, v\}$ *and* $\leq = \{(w, w), (u, u), (v, v), (w, u), (w, v)\}$. *In this case we have* 4 *different coalitions* $\{w, u\}$, $\{w, v\}$, $\{w, u, v\}$ *and* $\{w\}$. *Group belief for this case depends on the group. If* $J = \{u, w\}$, $J = \{v, w\}$ *or* $J = \{w\}$, *group belief coincides with shared belief* $GB_J p \leftrightarrow E_J p$, *while when* $J = \{u, v\}$ *we have* $GB_J p \leftrightarrow E_A p$ *and in cases when* $J = \{u\}$ *or* $J = \{v\}$ *group belief is a shared belief of a corresponding coalition* $GB_J p \leftrightarrow E_{\{u,w\}} p$ *and* $GB_J p \leftrightarrow E_{\{v,w\}} p$ *respectively.*

3.2 Semantics

Semantics for the modal logics $\mathcal{GB}1$ is provided by OUR-models.

Definition 7. *An OUR-structure for a partially ordered set of agents* $(A, \leq)$ *is a tuple* $(W, \{R_a | a \in A\})$ *where* W *is a set of worlds,* R_a *for each* $a \in A$ *is a transitive relation on* W*. An OUR-model is an OUR-structure together with a valuation function* $V : Prop \times W \to 2$*.*

Notice that the structure on a set of agent as well as the set of agents itself is common both to the syntax and semantics. It is true that the syntax does not contain any symbol for the relation $\leq$ but it interacts with this relation by the group belief axiom. The semantics, as is clear from the next definition, has a more straightforward interaction with the structure on the set of agents.

Definition 8. *For a given OUR-model* $M = (W, \{R_a | a \in A\}, V)$*, the satisfaction of a formula at a point* $w \in W$ *is defined inductively as follows:*
$w \models p$ *iff* $w \in V(p)$*;*
the boolean cases are standard;
$w \models \Box_a \phi$ *iff* $(\forall v)(wR_a v \Rightarrow v \models \phi)$*;*
$w \models E_J \phi$ *iff* $(\forall v)(wRv \Rightarrow v \models \phi)$ *where* $R = \bigcup_{a \in J} R_a$*;*
$w \models GB_J \phi$ *iff* $(\forall v)(wR'v \Rightarrow v \models \phi)$ *where* $R' = \bigcup_{a \in \overline{J}} R_a$*;*

A formula is valid in an OUR-structure if it is satisfiable at every point $w \in W$ *under every valuation* V*. A formula is valid in a class* $\mathfrak{C}$ *of OUR-structures if it is valid in every OUR-structure* $\mathfrak{F} \in \mathfrak{C}$*.*

What does the last definition imply in different examples? The idea is to think of coalitions as downsets. In such a setting each member of a group J may believe the sentence p but the coalition $\overline{J}$ may have additional members who do not share this belief and hence the group J as part of the coalition does not have p as a group belief of a coalition. In other words *only those sentences are believed by the group which are shared beliefs of the coalition to which the group belongs*. We might call this kind of belief "coalition dependent".

The group belief operator defined in this sense has the following properties discussed in [5]:

Proposition 9

1. *No combination of individual beliefs implies group belief;*
2. *Not all sets of agents form coalitions;*
3. *Group belief does not imply the common belief of the group;*

Proof. 2 follows by the definition of coalition. 3 is an easy application of the definitions of common belief and shared belief. See Sect. 2. For 1 let us consider a partial order $(\{a, b, c\}, \leq)$ of agents where $a \leq b \leq c$. Let $J = \{b, c\}$. As for the set of possible worlds and relations, let us take $W = \{w, u, v\}$, $R_a = \{(w, u), (w, v), (u, v)\}$ and $R_b = R_c = \{(v, u), (v, w), (u, w)\}$ let $w \models p$ and $v \not\models q$. See Fig. 1.

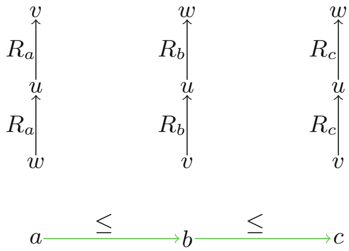

Fig. 1. .

In this case $u \models \Box_b p \wedge \Box_c p$ since the only successor of u both by R_b and R_c is w which on its own models p. This means that all members of the group J believe in p but still p is not a group belief of the group. This is because the coalition $\overline{J}$ containing the group also contains agent a. $u \not\models \Box_a p$ as there is an R_a successor v of u which does not model p. So $u \not\models E_{\overline{J}} p$.

Note that $\Box_b p \wedge \Box_c p$ is just one particular combination of individual beliefs and hence it is not enough to claim that no combination of individual beliefs implies group belief. But an easy argument shows that indeed no formula written in a restricted language which only contains $\Box_b$ and $\Box_c$ can imply group belief. The full proof of this claim needs additional definitions and properties and is given in the appendix.

The other two important properties from [5] "Goup belief does not imply individual beliefs of the group members" and "Group belief does not imply subgroup belief" are not satisfied. This is because the group is always contained in the coalition and as well every subgroup is contained in a coalition formed by a bigger group. Now what happens if we add the belief dependency axiom? Does it effect the structure. The answer is yes. The belief dependency axiom sets some constraints on the structure of frames.

3.3 Completeness

One way to prove completeness is via a standard canonical model construction. Here we use a different method and prove completeness by applying results from [11]. First we show that the axiom for the group belief modality is a relational modal definition. Secondly we will use the result that modal logic with the shared belief modality is complete, and lastly we will apply the result that extensions of complete logics with relational modal definitions yield complete logics.

Definition 10. *A modal definition* $\boxplus p \leftrightarrow \phi(p, p_1, \ldots, p_n)$ *is called a* relational modal definition *if there exists a first-order formula* $\Psi_+(x, y)$ *with two free* *variables using only symbols that occur in* $ST_x[\phi(p, p_1, \ldots, p_n)]$ *such that for every formula* ψ *in the language without* $\boxplus$ *it holds that*

$$(\forall y)(\Psi_+(x, y) \Rightarrow ST_y[\psi]) \textit{ is logically equivalent to } ST_x[\phi(\psi, p_1, \ldots, p_n)].$$

Let $\Psi_+(x, y)$ be the first-order formula corresponding to a relational modal definition. Given a model $\mathfrak{M} = (\mathfrak{F}, V)$, we uniquely construct the model $\mathfrak{M}_+ = (\mathfrak{F}_+, V)$, where the underlying frame $\mathfrak{F}_+$ is obtained from $\mathfrak{F}$ by adding the binary relation $R_+ \subseteq W^2$ defined as:

$$(x, y) \in R_+ \text{ if, and only if, } \mathfrak{M} \models \Psi_+(x, y).$$

For a class $\mathcal{C}$ of models, we denote by $\mathcal{C}_+$ the class consisting of the models $\mathfrak{M}_+$, where $\mathfrak{M}$ ranges over the models in $\mathcal{C}$.

Fact 11. *Let* $\mathcal{L}$ *be the modal language for a signature* $\langle \Pi, M \rangle$*, and let* $\mathcal{L}_+$ *be the modal language for* $\langle \Pi, M \cup \{+\} \rangle$ *for some fresh symbol '+'. Let* $L \subseteq \mathcal{L}$ *be a modal logic that is complete w.r.t. a class* $\mathcal{C}$ *of models. Let* $L_+ \subseteq \mathcal{L}_+$ *be the modal logic obtained by extending* L *with the relational modal definition* $\boxplus p \leftrightarrow \alpha(p, p_1, \ldots, p_n)$*. Then* L_+ *is complete w.r.t.* $\mathcal{C}_+$*.*

Another result which we are going to use is completeness of the modal logic obtained by eliminating the group belief operator from logic $\mathcal{GB}1$. The result as stated does not appear anywhere but an exact analog of the result is known for the shared knowledge operator, see [4]. And the distinction between the two is insignificant for these results.

Proposition 12. *The modal logic of shared belief (The logic obtained by eliminating operator* GB_J *together with the group belief axiom from* $\mathcal{GB}1$*) is sound and complete w.r.t. possible world structures (Kripke structures), where each relation is transitive.*

Lastly, to obtain the completeness for the logic $\mathcal{GB}1$ it remains to show that the group belief axiom is a relational modal definition and describe the class of frames it specifies.

Proposition 13. *The axiom* $GB_G\phi \leftrightarrow E_{\overline{G}}\phi$ *is a relational modal definition.*

Proof. Immediate if we take $\Psi_+(x, y)$ in Definition 10 to be xRy where $R = \bigcup_{a \in \overline{G}} R_a$.

Corollary 14. *The modal logic* $\mathcal{GB}1$ *is sound and complete w.r.t. OUR-structures.*

3.4 Fibered Structures

By ordering the set of agents we want to reflect the intuition that not all agents have the same belief sets. Moreover it is natural to think that the structure of agents is connected with the structure of their belief sets. Which is not the case in OUR-frames from previous section. For instance if $a \leq b$, then belief set of a is smaller then belief set of b. At this point we don't have such a requirement. One could obtain this property by adding the law $a \leq b \Rightarrow \vdash \Box_b p \to \Box_a p$, which we encode by the following axiom:

- *Belief dependency axiom*

$$\Box_a p \to GB_{\{a\}} p$$

Now the meta-rule $a \leq b \Rightarrow \vdash \Box_b p \to \Box_a p$ becomes satisfied. For, assume $a \leq b$, by the belief dependency axiom we have $\Box_a p \to GB_{\{a\}} p$, and by the axiom for the group belief operator we get $\Box_b p \to E_{\overline{\{b\}}} p$ and, as $a \leq b$, we know that $a \in \overline{\{b\}}$. By the axiom for shared belief we obtain $E_{\overline{\{b\}}} p \to \bigwedge_{i \in \overline{\{b\}}} \Box_i p$ which on its own implies $\Box_a p$. Hence we get $\Box_b p \to \Box_a p$. Thus, despite the fact that our language does not contain the symbol $\leq$, it is strong enough to express the property of belief dependency. By $\mathcal{GB}1^{\leq}$ we denote the extension of $\mathcal{GB}1$ by the belief dependency axiom.

Definition 15. *Let us call an OUR-structure* $(W, \{R_a | a \in A\})$ *a fibered frame iff* $a \leq b$ *implies* $R_a \subseteq R_b$.

Proposition 16. *The belief dependency axiom is valid in an OUR-frame* $\mathfrak{F} = (W, \{R_a | a \in A\})$ *iff* $\mathfrak{F}$ *is a fibered frame.*

Proof. Assume for the contradiction that an OUR-structure $\mathfrak{F}$ is not fibered. By definition this means that there exists a and b in the set of agent A such that $a \leq b$ while $R_a \not\subseteq R_b$, i.e. there are points $w, u \in W$ such that $wR_a u$ while not $wR_b u$. Take a valuation such that p is true everywhere in a frame except at u, then it is clear that $w \models \Box_b p$ while $w \not\models \Box_a p$ sincet $wR_a u$ and $u \not\models p$. Hence $w \not\models GB_{\{b\}} p$ which falsifies the axiom.

Now assume that $\mathfrak{F}$ is a fibered OUR-structure. Let V be an arbitrary valuation on $\mathfrak{F}$. Let us take an arbitrary point $w \in W$ and show that $w \models \Box_b p \to GB_{\{b\}} p$ for an arbitrary $b \in A$. Assume that $w \models \Box_b p$. Hence for every R_b successor v of w it holds that $v \models p$. Let us show that $w \models E_{\overline{\{b\}}} p$. By the axiom of shared belief $E_{\overline{b}} p \leftrightarrow \bigwedge_{a \in \overline{b}} \Box_a p$, it suffices to show that $w \models \Box_a p$ for every $a \leq b$. Now since $\mathfrak{F}$ is fibered, $a \leq b$ implies that $R_a \subseteq R_b$. Hence every R_a successor u of w is also an R_b successor and we already know that every such u satisfies p.

The following proposition shows that fibered frames do not preserve the property of group belief from Proposition 9. Proof of the following proposition can be found in Appendix.

Proposition 17. *In every fibered OUR-structure, the group belief of a set of agents is implied by the conjunction of the individual beliefs of those agents that have maximal belief sets from the group.*

Corollary 18. *In every fibered OUR-structure, the group belief of a set of agents is equivalent to the shared belief of the same set of agents.*

This shows that the notion of group belief as defined above does not make much sense in the class of fibered frames and the language collapses to a simple modal language with many modalities. In Sect. 5 we will consider the logic of a pure modal language of ordered agents with an additional operator reflecting the order of agents and derive the completeness of the logic w.r.t. the class of fibered structures.

4 Syntax of $\mathcal{GB}2$

An important property of group belief discussed in [5], which our definition of group belief lacks, is the following: 'If p is a group belief of a group G, then it is a common belief that p is a group belief of the group'. As we saw from the example this property is not satisfied for $\mathcal{GB}1$. In this section is modify the logic $\mathcal{GB}1$ so that the desirable properties of $\mathcal{GB}1$ are preserved but additionally group belief satisfies the above condition. We consider a modal logic $\mathcal{GB}2$ in which shared belief is replaced by common belief.

4.1 Syntax

Definition 19. *The language of the normal modal logic $\mathcal{GB}2$ is $\mathcal{L}(\{C_J, GB_J \mid J \subseteq A\})$ where the operators C_J stand for common belief. The axioms are all classical tautologies, each box satisfies $K4$ axioms $\Box_a(p \to q) \to (\Box_a p \to \Box_a q)$ and $\Box_a p \to \Box_a \Box_a p$ for every $a \in A$. In addition we have an equilibrium axiom for common belief:*

$$(equi) : C_J p \leftrightarrow \bigwedge_{a \in J} \Box_a p \wedge \bigwedge_{a \in J} \Box_a C_J p$$

And a new axiom for the group belief operator

$$GB_J p \leftrightarrow C_{\overline{J}} p$$

for every $J \subseteq A$. The rules of inference are: modus ponens, substitution and necessitation for each box modality and additionally an induction rule for the common believe operator:

$$(ind) : \frac{\vdash p \to \bigwedge_{a \in J} \Box_a (p \wedge q)}{\vdash p \to C_J q}$$

4.2 Semantics

A semantics for $\mathcal{GB}2$ is provided by OUR-models. Let us first recall the definition of the transitive closure of a binary relation.

Definition 20. *The transitive closure R^+ of the relation R is defined in the following way: $xR^+y \Leftrightarrow (\exists x_1, \exists x_2, ..., \exists x_n)(x = x_1 \wedge x_1 R x_2 \wedge x_2 R x_3 \wedge ... \wedge x_n R y)$ for some $n \in \omega$.*

Now we are ready to define the satisfaction of modal formulas on OUR-models.

Definition 21. *For a given OUR-model $M = (W, \{R_a | a \in A, V\})$, the satisfaction of a formula at a point $w \in W$ is defined inductively as follows:*
$w \models p$ *iff* $w \in V(p)$;
the boolean cases are standard;
$w \models \Box_a \phi$ *iff* $(\forall v)(wR_a v \Rightarrow v \models \phi)$;
$w \models C_J \phi$ *iff* $(\forall v)(wRv \Rightarrow v \models \phi)$ *where* $R = (\bigcup_{a \in J} R_a)^+$;
$w \models GB_J \phi$ *iff* $(\forall v)(w\overline{R}v \Rightarrow v \models \phi)$ *where* $\overline{R} = (\bigcup_{a \in \overline{J}} R_a)^+$;
A formula is valid in an OUR-structure if it is satisfiable at every point $w \in W$ under every valuation V. A formula is valid in a class $\mathfrak{C}$ of OUR-structures if it is valid in every OUR-structure $\mathfrak{F} \in \mathfrak{C}$.

The following result for $\mathcal{GB}2$ shows that some of the good properties of group belief defined the previous section are preserved for the group belief operator of $\mathcal{GB}2$ and additionally the latter has the property that 'If p is a group belief of a group J then it is a common belief that p is a group belief of the group.' Proof is given in Appendix.

Proposition 22

1. *No combination of individual beliefs imply group belief;*
2. *Not all sets of agents form coalitions;*
3. *If a sentence p is a group belief of a set of agents J then is is common belief (of the set of agents J) that p is a group belief of the set of agents J;*

4.3 Completeness

The main result for this section is that the logic $\mathcal{GB}2$ is the logic of all OUR-structures with the given semantics. Observe that completeness for this case can not be obtained by the technique of Sect. 3.3 since the axiom $GB_J p \leftrightarrow C_{\overline{J}} p$ is not a relational modal definition. The reason is that the transitive closure used for defining the semantics of the common belief operator is not first order definable. Nevertheless we are able to prove the completeness of the logic $\mathcal{GB}2$ by a slight modification of the completeness proof for the logic of common belief [4]. A proof sketch can be found in Appendix.

Theorem 23. *The logic $\mathcal{GB}2$ is sound and complete w.r.t. the class of all OUR-structures.*

5 The Logic of Fibered Structures

In this section we introduce the logic of fibered structures in a simpler language which does not contain a group belief operator. Instead we have an operator $\preceq$ which captures the partial order of agents. An analogous approach with geometric interpretations of the operator $\preceq$ has been introduced in [1]. The set FOR of all formulas (with typical members denoted ϕ, ψ, etc.) is now inductively defined as follows:

- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \Box_a \phi \mid a \preceq b.$

We define the other Boolean constructs as usual. The formula $a \not\preceq b$ is an abbreviation for: $\neg a \preceq b$. We omit parentheses if this does not lead to any ambiguity. The notion of a subformula is standard. For all sets x of formulas, let $\Box_a x = \{\phi : \Box_a\phi \in x\}$.

5.1 Semantics

For a given OUR-model $M = (W, \{R_a \mid a \in A\}, V)$, the satisfaction relation is defined as follows for formulas of the form $a \preceq b$:

- $w \models a \preceq b$ iff $(\forall v)(wR_a v \Rightarrow wR_b v)$.

Therefore, in our setting, "$a \preceq b$" means that a believes everything that b believes.

We remark that

Lemma 24. *The following formulas are satisfied in any world of any model:*

- $\Box_a\phi \to \Box_a\Box_a\phi$,
- $a \preceq a$,
- $a \preceq b \wedge b \preceq c \to a \preceq c$,
- $a \preceq b \to (\Box_b\phi \to \Box_a\phi)$,
- $\Box_a\bot \to a \preceq b$.

Proof. Since OUR-models are based on transitive relations, formulas of the form $\Box_a\phi \to \Box_a\Box_a\phi$ are valid. The validity of formulas of the form $a \preceq a$ and $a \preceq b \wedge b \preceq c \to a \preceq c$ comes from the fact that the relation of inclusion between sets is reflexive and transitive. For formulas of the form $a \preceq b \to (\Box_b\phi \to \Box_a\phi)$, they are valid because in an OUR-model $M = (W, \{R_a \mid a \in A\}, V)$, if $w \models a \preceq b$ then $R_a(w) \subseteq R_b(w)$ where $R(w)$ denotes the set of all accessible porints from w. Concerning formulas of the form $\Box_a\bot \to a \preceq b$, they are valid because in an OUR-model $M = (W, \{R_a \mid a \in A\}, V)$, if $R_a(w) = \emptyset$ then $w \models a \preceq b$.

5.2 Axiomatization/Completeness

Let L be the least normal modal logic in our language containing the formulas of Lemma 24. We want to show that L provides a sound and complete axiomatization of the set of all valid formulas. By Lemma 24, L is sound. To prove completeness, we must show that every valid formula is in L. It suffices to prove that every consistent formula is satisfiable. To reach this goal, we use a step-by-step method. We define a subordination model to be a structure $S = (W, \{R_a \mid a \in A\}, \sigma)$ where W is a nonempty subset of $\mathbb{N}$, R_a is an irreflexive transitive relation on W and σ is a function assigning to each $x \in W$ a maximal L-consistent set $\sigma(x)$ of formulas such that

- if $\Box_a\phi \in \sigma(x)$ then for all $y \in W$, if $xR_a y$ then $\phi \in \sigma(y)$,
- if $a \preceq b \in \sigma(x)$ then $R_a(x) \subseteq R_b(x)$.

For all maximal L-consistent sets Γ of formulas, let $S^\Gamma = (W^\Gamma, \{R_a^\Gamma \mid a \in A\}, \sigma^\Gamma)$ be the structure where $W^\Gamma = \{0\}$, $R_a^\Gamma = \emptyset$, $\sigma^\Gamma(0) = \Gamma$. The reader may easily verify that

Lemma 25. *S^Γ is a finite subordination model.*

Consider a finite subordination model $S' = (W', \{R'_a \mid a \in A\}, \sigma')$. We define a $\Box$-imperfection in S' to be a triple of the form (x, a, ϕ) where $x \in W'$, a is an agent and ϕ is a formula such that $\Box_a\phi \notin \sigma'(x)$ and for all $y \in W$, if $xR'_a y$ then $\phi \in \sigma'(y)$.

Lemma 26. *Let (x, a, ϕ) be a $\Box$-imperfection in S'. Let Γ be a maximal L-consistent set of formulas such that $\Box_a\sigma'(x) \subseteq \Gamma$ and $\phi \notin \Gamma$. Let y be a new nonnegative integer. Let $S = (W, \{R_a \mid a \in A\}, \sigma)$ be the structure where*

- *$W = W' \cup \{y\}$,*
- *$zR_b t$ iff one of the following conditions holds:*
  - *$z \in W'$, $t \in W'$ and $zR'_b t$,*
  - *$z \in W' \setminus \{x\}$, $t = y$, $zR'_b x$ and $a \preceq b \in \sigma'(x)$,*
  - *$z = x$, $t = y$ and $a \preceq b \in \sigma'(x)$,*
- *$\sigma(z) =$ if $z = y$ then Γ else $\sigma'(z)$.*

Then, S is a finite subordination model. We shall say that S is the local completion of S' with respect to the $\Box$-imperfection (x, a, ϕ).

We define a $\preceq$-imperfection in S' to be a triple of the form (x, a, b) where $x \in W'$ and a, b are agents such that $a \preceq b \notin \sigma'(x)$ and $R'_a(x) \subseteq R'_b(x)$.

Lemma 27. *Let (x, a, b) be a $\preceq$-imperfection in S'. Let Γ be a maximal L-consistent set of formulas such that $\Box_a\sigma'(x) \subseteq \Gamma$. Let y be a new nonnegative integer. Let $S = (W, \{R_a \mid a \in A\}, \sigma)$ be the structure where*

- *$W = W' \cup \{y\}$,*
- *$zR_c t$ iff one of the following conditions holds:*
  - *$z \in W'$, $t \in W'$ and $zR'_c t$,*
  - *$z \in W' \setminus \{x\}$, $t = y$, $zR'_c x$ and $a \preceq c \in \sigma'(x)$,*
  - *$z = x$, $t = y$ and $a \preceq c \in \sigma'(x)$,*
- *$\sigma(z) =$ if $z = y$ then Γ else $\sigma'(z)$.*

Then, S is a finite subordination model. We shall say that S is the local completion of S' with respect to the $\preceq$-imperfection (x, a, b).

Let $(x_0, a_0, \phi_0), (x_1, a_1, b_1), (x_2, a_2, \phi_2), (x_3, a_3, b_3), \ldots$ be an enumeration of $(\mathbb{N} \times A \times FOR) \cup (\mathbb{N} \times A \times A)$ in which each item appears infinitely many times. For all maximal L-consistent sets Γ of formulas, let $T^0 = (W^0, \{R_a^0 \mid a \in A\}, \sigma^0)$, $T^1 = (W^1, \{R_a^1 \mid a \in A\}, \sigma^1)$, etc., be the infinite sequence of subordination models defined as follows. Let $T^0 = S^\Gamma$. Let n be a nonnegative integer. Given $T^{2\times n}$, let $T^{2\times n+1}$ be the local completion of $T^{2\times n}$ with respect to the $\Box$-imperfection $(x_{2\times n}, a_{2\times n}, \phi_{2\times n})$ when $(x_{2\times n}, a_{2\times n}, \phi_{2\times n})$ is a $\Box$-imperfection of $T^{2\times n}$. Otherwise, let $T^{2\times n+1}$ be $T^{2\times n}$. Now, let $T^{2\times n+2}$ be the local completion of $T^{2\times n+1}$ with respect to the $\preceq$-imperfection $(x_{2\times n+1}, a_{2\times n+1}, b_{2\times n+1})$ when $(x_{2\times n+1}, a_{2\times n+1}, b_{2\times n+1})$ is a $\preceq$-imperfection of $T^{2\times n+1}$. Otherwise, let $T^{2\times n+2}$ be $T^{2\times n+1}$. Now, we put $T^\omega = (W^\omega, \{R_a^\omega \mid a \in A\}, \sigma^\omega)$ to be the subordination model defined as follows:

– $W^\omega = \bigcup\{W^n$: n is a nonnegative integer$\}$,
– if $x \in W^m$ for some nonnegative integer m and $y \in W^n$ for some nonnegative integer n then $xR_a^\omega y$ iff $xR_a^{m+n}y$,
– if $x \in W^n$ for some nonnegative integer n then $\sigma^\omega(x) = \sigma^n(x)$.

The reader may easily verify that T^ω has no imperfection. The result that emerges from the discussion above is:

Proposition 28. *The following conditions are equivalent for every formula ϕ:*

1. ϕ is in L.
2. ϕ is valid.

Proof. 1. $\Rightarrow$ 2.: By Lemma 24.
2. $\Rightarrow$ 1.: Suppose $\phi \notin L$. Let Γ be a maximal L-consistent set of formulas such that $\phi \notin \Gamma$. Let $T^\omega = (W^\omega, \{R_a^\omega \mid a \in A\}, \sigma^\omega)$ be the subordination model associated to Γ as above. Let $M = (W, \{R_a \mid a \in A\}, V)$ be the model defined as follows:

$$W = W^\omega, \;\; xR_a y \;\text{ iff }\; xR_a^\omega y, \;\; V(p) = \{x \mid p \in \sigma^\omega(x)\}.$$

By induction on ψ, the reader may easily verify that for all $x \in W$, $x \Vdash \psi$ iff $\psi \in \sigma^\omega(x)$. Since $\phi \notin \Gamma$, therefore $0 \nVdash \phi$. Consequently, ϕ is not valid.

6 Summary and Future Work

In this preliminary study we have explored different ways in which group belief might be modeled when a certain structure is imposed on the set of agent. Group belief in the resulting logics displays different properties, suggesting that the logics may have different types of application - a topic for further study in the future.

As we have seen both logics, $\mathcal{GB}1$ and $\mathcal{GB}2$, collapse to standard multi-modal languages when a belief dependency axiom is added. This shows that on a semantical level there is natural correspondence between the $\mathcal{GB}1^{\leq}$ and $\mathcal{GB}2^{\leq}$ and the logic of all fibered structures from Sect. 5. This suggests the possibility of syntactic connections between the three logics which we aim to explore in future work.

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