The Topology of Common Belief

David Pearce and Levan Uridia

Abstract We study a modal logic $\mathbf{K4_2^C}$ of common belief for normal agents. We discuss Kripke completeness and show that the logic has tree model property. A main result is to prove that $\mathbf{K4_2^C}$ is the modal logic of all T_D-intersection closed, bi-topological spaces with derived set interpretation of modalities. Based on the splitting translation we also discuss connections with $\mathbf{S4_2^C}$, the logic of common knowledge.

Keywords Modal logic • Epistemic logic • Common belief • Common knowledge

1 Introduction

In logics for knowledge representation and reasoning, the study of epistemic and doxastic properties of agents with certain, intuitively acceptable, restrictions on their knowledge and belief is a well-developed area. Smullyan (1986) discusses various types of agents based on properties of belief. In his terminology, an agent whose belief satisfies the modal axiom (4) : $\Box p \to \Box\Box p$, translated as 'If the agent believes p, then he believes that he believes p', is called a *normal agent*. **K4** is the modal logic which formalizes the belief behavior of normal agents. This generalizes the classical doxastic system **KD45** in the same way as **S4** generalizes the epistemic logic **S5**, by dropping some restrictions on the properties of an agent.

The study of *group attitudes* is already well-established in several fields where collective opinion and reasoning are important. Also in newly emerging areas such as agreement technologies, and 'social intelligence', iterative concepts of agent belief and knowledge are of special interest. To achieve successful communication and agreement it is important for agents to reason about themselves and what others

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know or believe. Among the more fundamental concepts are the notions of *common knowledge* and *common belief*. We denote the operators for common knowledge and common belief by C_K and C_B respectively. We have: $C_K\varphi$ iff φ is common knowledge in the group K and $C_B\varphi$ iff φ is a common belief in the group B.

Following the analysis of common knowledge as originally defined by Lewis (1969), this concept has been extensively studied from various perspectives in philosophy (Barwise 1988; Aumann 1976), game theory (van Benthem 2007), artificial intelligence (Herzig et al. 2009), modal logic (Baltag et al. 1998; Baltag and Smets 2009; Bezhanishvili and van der Hoek 2014) etc. Theories of common belief are less well-developed though some approaches can be found in Stalnaker (2001), Herzig et al. (2009), and Lismont and Mongin (1994). The present chapter is devoted to a study of the common belief of 'normal' agents in the sense mentioned above. We want to extend and bring together two previous lines of work. One direction is our own study of several extensions of the modal logic **wK4** that form interesting doxastic logics different from **KD45**; see in particular Pearce and Uridia (2010, 2011a,b). **wK4** is the normal modal logic based on the axioms

(K): $\Box(p \to q) \to \Box p \to \Box q$
$(w4)$: $\Box p \wedge p \to \Box\Box p$

In previous work we showed that different extensions of **wK4** may be useful in certain doxastic contexts, for instance in modeling a notion of *minimal* belief, and more generally for *non-monotonic* reasoning about beliefs. They provide alternatives to the more familiar system **KD45** and its non-monotonic extension, *autoepistemic* logic. We also considered topological interpretations and embedding relations between epistemic and doxastic logics, i.e. translations between knowledge and belief operators. However in our earlier studies only single agent systems are treated. In our view, these different extensions of **wK4** can all be considered types of *doxastic* logics, even if they omit or weaken some of the stronger epistemic axioms.[1]

Our second point of departure is provided by the work of van Benthem and Sarenac (2004), who showed how a topological semantics for logics of common knowledge may be useful for modeling and distinguishing different concepts. A key idea here is that the knowledge of different agents is represented by different topologies over a set X. Various ways to merge that knowledge can be obtained via different modes of combining logics and topological models. van Benthem and Sarenac (2004) considers for example the fusion logic $\mathbf{S4}\circ\mathbf{S4}$ and product topologies that are complete for the common knowledge logic $\mathbf{S4}_2^{\mathbf{C}}$ of Fagin et al. (1995).

In light of Fagin et al. (1995) and van Benthem and Sarenac (2004) and our previous work several natural questions emerge that we want to address. In summary the main tasks of this chapter are:

[1]Lismont and Mongin (1994), treating common belief, and Steinsvold (2008), treating topological models for belief, are related works that also study weaker extensions of **K4**.

1. Define a logic $\mathbf{K4_2^C}$ of common belief for normal agents and prove its completeness for a Kripke, relational semantics. Show it has the finite model property and the tree model property.
2. Study a topological semantics for $\mathbf{K4_2^C}$ and prove completeness for intersection topologies. Specifically show that $\mathbf{K4_2^C}$ is the modal logic of all T_D-intersection closed, bi-topological spaces with a derived set interpretation of modalities.
3. Belief under the topological interpretation of $\mathbf{K4_2^C}$ is understood via colimits and common belief in terms of colimits in the intersection topology. From 2 we aim to derive a topological condition for common belief in terms of colimits that is very similar to the corresponding condition that defines common knowledge in the modal μ-calculus and is discussed at some length in van Benthem and Sarenac (2004).
4. Show how the common knowledge logic $\mathbf{S4_2^C}$ can be embedded in $\mathbf{K4_2^C}$ via the splitting translation that maps $C_K p$ into $p \wedge C_B p$.

1.1 Common Belief and the Topological Interpretation

As stated, we focus on the common belief of normal agents, and for ease of exposition we restrict ourselves to the two agent case. We thus consider two agents whose individual beliefs satisfy the axioms of **K4**. In other respects we adopt the main principles of the logic of common knowledge, $\mathbf{S4_2^C}$. This can be seen as a formalization of the idea that common knowledge is equivalent to an infinite conjunction of iterated individual knowledge: $\varphi \wedge \Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1\Box_1\varphi \wedge \Box_1\Box_2\varphi \wedge \Box_2\Box_1\varphi \wedge \Box_2\Box_2\varphi \wedge \Box_1\Box_1\Box_1\varphi \wedge \Box_1\Box_1\Box_2\varphi \ldots$. Later we shall see that a variation of this formula is 'true' for common belief under the relational semantics. We shall also show that the topological semantics for $\mathbf{K4_2^C}$ is compatible with the idea of common belief as a fixpoint *equilibrium*, a notion used by Barwise (1988) to describe common knowledge that can be captured by an expression of the modal μ-calculus.

Our approach to providing a topological semantics follows the work of Esakia (2001). Notice that under the topological interpretation of $\Box$ as a knowledge operator, e.g. in van Benthem and Sarenac (2004), $\Box\varphi$ refers to the topological *interior* of the points assigned to φ. In the case of a doxastic logic like **K4** our topological interpretation is different. It is perhaps simpler to state it for the $\Diamond$ operator. Following McKinsey and Tarski (1944), the idea is to treat $\Diamond\varphi$ as the *derivative* of the set φ in the topological space. Esakia showed that under this interpretation **wK4** is the modal logic of all topological spaces. **K4** is an extension of **wK4** and is characterized in this semantics by the class of all T_D-spaces (Bezhanishvil et al. 2005). Steinsvold was one of the first to look at derived set semantics from a doxastic point of view (Steinsvold 2008, 2009). By combining the ideas and results from van Benthem and Sarenac (2004), Steinsvold (2009) and Esakia (2001), we can obtain a derived set semantics for the logic of common belief based on bi-topological spaces, where the modality for common belief operates on the intersection of the two topologies. As a main result, we can prove that $\mathbf{K4_2^C}$ is sound and complete with respect to the special subclass of all bi-topological T_D-spaces.

2 Logic of Common Belief

We turn to the syntax and Kripke semantics of the logic $\mathbf{K4_2^C}$. The interpretation of common belief operator C_B on bi-relational Kripke frames is similar to the interpretation of the common knowledge operator C_K, and is based on the notion of transitive closure of a relation. In this section we show that the logic $\mathbf{K4_2^C}$ is sound and complete with respect to the class of all bi-relational transitive Kripke structures. The proof is a slight modification of the completeness proof for the logic $\mathbf{S4_2^C}$ given in Fagin et al. (1995) therefore we only sketch the essential parts where the difference shows up. Additionally we show that every non-theorem of $\mathbf{K4_2^C}$ can be falsified on an infinite, irreflexive, bi-transitive tree.

2.1 Iterative Common Belief

There are different notions of common belief (Barwise 1988). Let us mention common belief as an infinite conjunction of nested beliefs and common belief as an equilibrium. Under the former idea, a proposition p is a common belief of two agents if: agent-1 believes that p and agent-2 believes that p and agent-1 believes that agent-2 believes that p and agent-2 believes that agent-1 believes that p etc., where all possible finite mixtures occur. If we formalize this idea in a modal language with belief operators $\Box_1$ and $\Box_2$ for each agent respectively, then we arrive at the following concept of a common belief operator C_B^ω.

$$C_B^0 p = \Box_1 p \wedge \Box_2 p;$$

$$C_B^{n+1} p = \Box_1 C_B^n p \wedge \Box_2 C_B^n p;$$

$$C_B^\omega p = \bigwedge\nolimits_{n\in\omega} C_B^n p.$$

C_B^ω exactly formalizes the intuition behind the former idea of common belief. However, since C_B^ω is an infinite intersection, it cannot be expressed as an ordinary formula of modal logic and hence studied in the usual approaches to standard modal logic. Nevertheless it turns out that we can capture the infinitary behavior of C_B^ω in a finitary sense. This idea is made more precise via the modal logic $\mathbf{K4_2^C}$.

2.2 Syntax

Throughout we work in the modal language $\mathcal{L}_C$ with an infinite set *Prop* of propositional letters and symbols $\wedge, \neg, \Box_1, \Box_2, C_B$. The set of formulas *Form* is constructed in a standard way: $Prop \subseteq Form$. If $\alpha, \beta \in Form$ then $\neg\alpha, \alpha \wedge \beta, \Box_1\alpha, \Box_2\alpha, C_B\alpha \in Form$. We will use standard abbreviations for disjunction and implication, $\alpha \vee \beta \equiv \neg(\neg\alpha \wedge \neg\beta)$ and $\alpha \to \beta \equiv \neg\alpha \vee \beta$.

- The axioms of the logic $\mathbf{K4_2^C}$ are all classical tautologies, each box satisfies all **K4** axioms, i.e. we have:

$$(K): \ \Box_i(p \to q) \to (\Box_i p \to \Box_i q)$$
$$(4): \Box_i p \to \Box_i \Box_i p$$

  for each $i \in \{1, 2\}$ and in addition we have the equilibrium axiom for the common belief operator:

$$(equi): C_B p \leftrightarrow \Box_1 p \wedge \Box_2 p \wedge \Box_1 C_B p \wedge \Box_2 C_B p.$$

- The rules of inference are: Modus-Ponens, Substitution, Necessitation for $\Box_1$ and $\Box_2$ and the induction rule for the common belief operator:

$$(ind): \frac{\vdash \varphi \to \Box_1(\varphi \wedge \psi) \wedge \Box_2(\varphi \wedge \psi)}{\vdash \varphi \to C_B\psi}$$

  where φ and ψ are arbitrary formulas of the language.

2.3 Kripke Semantics

The Kripke semantics for the modal logic $\mathbf{K4_2^C}$ is provided by transitive, bi-relational Kripke frames. The triple (W, R_1, R_2), with W an arbitrary set and $R_i \subseteq W \times W$ where $i \in \{1, 2\}$, is a *bi-transitive Kripke frame* if both R_1 and R_2 are transitive relations. A quadruple (W, R_1, R_2, V) is a bi-transitive Kripke model if (W, R_1, R_2) is a bi-transitive Kripke frame and $V : Prop \to P(W)$ is a valuation function. Observe that we only have two relations, which give a semantics for $\Box_1$ and $\Box_2$. To interpret the common belief operator, C_B, we construct a new relation, which is a transitive closure of the union of R_1 and R_2.

Definition 1. The transitive closure R^+ of a relation R is defined as the least transitive relation containing the relation R.

Two points x and y are related by the transitive closure of the relation if there exists a finite path $\langle x_1, \ldots, x_n \rangle$ starting at x and ending at y.

Definition 2. For a given bi-relational Kripke model $\mathcal{M} = (W, R_1, R_2, V)$ the satisfaction of a formula at a point $w \in W$ is defined inductively as follows:

$w \Vdash p$ iff $w \in V(p)$,
$w \Vdash \alpha \wedge \beta$ iff $w \Vdash \alpha$ and $w \Vdash \beta$,
$w \Vdash \neg\alpha$ iff $w \nVdash \alpha$,
$w \Vdash \Box_i\varphi$ iff $(\forall v)(wR_iv \Rightarrow v \Vdash \varphi)$,
$w \Vdash C_B\varphi$ iff $(\forall v)(w(R_1 \cup R_2)^+v \Rightarrow v \Vdash \varphi)$.

A formula α is valid in a model $\mathcal{M}$, in symbols $\mathcal{M} \Vdash \alpha$, if for every point $w \in W$ we have $w \Vdash \alpha$. α is valid in a bi-relational frame $\mathcal{F} = (W, R_1, R_2)$, in symbols $\mathcal{F} \Vdash \alpha$, iff α is valid in every model $\mathcal{M} = (\mathcal{F}, V)$ based on the frame. α is valid in a class of bi-relational frames K if for every frame $\mathcal{F} \in K$ we have $\mathcal{F} \Vdash \alpha$.

2.4 Soundness and Completeness

Proposition 1 (Soundness). *Modal logic* $\mathbf{K4_2^C}$ *is sound with respect to the class of all bi-transitive Kripke frames.*

Proof. The only non-trivial cases are to show that the equilibrium axiom and the induction rule hold in the class of all bi-transitive models. Let $\mathcal{M} = (W, R_1, R_2, V)$ be an arbitrary bi-transitive Kripke model. And let $w \in W$. Assume $w \Vdash C_B\varphi$. Let us first show that $w \Vdash \Box_1\varphi$. Take an arbitrary $v \in W$ such that wR_1v. This implies that $w(R_1 \cup R_2)^+v$ hence $v \Vdash \varphi$. Let us show that $w \Vdash \Box_1C_B\varphi$. Take an arbitrary v and v' such that wR_1v and $v(R_1 \cup R_2)^+v'$. By Definition 1 this means that there exists a finite path $\langle v_1, \ldots, v_n\rangle$ such that each $v_i(R_1 \cup R_2)v_{i+1}$ and $v_1 = v$ and $v_n = v'$. Then the new path $\langle w, v_1, \ldots, v_n\rangle$ is also finite going from w to v'. Hence $w(R_1 \cup R_2)^+v$ which implies that $v \Vdash \varphi$. In the same way we prove that $w \Vdash \Box_2\varphi \wedge \Box_2C_B\varphi$.

For the other direction assume $w \nVdash C_B\varphi$. By Definition 2 this means that there is a finite path $\langle v_1, \ldots, v_n\rangle$ such that each $v_i(R_1 \cup R_2)v_{i+1}$ and $v_1 = w$ and $v_n \nVdash \varphi$. Without loss of generality we can assume that $v_1R_1v_2$. In case $n = 2$ we have that $w \nVdash \Box_1\varphi$. In case $n > 2$ we have that $v_2 \nVdash C_B\varphi$, hence $w \nVdash \Box_1C_B\varphi$.

Now let us show that the induction rule preserves the validity of formulas in a model. We show this by contraposition. Assume for some $\mathcal{M} = (W, R_1, R_2, V)$ we have $\mathcal{M} \nVdash p \rightarrow C_Bq$. This means that there is a point $w \in W$ with $w \Vdash p$ and $w \nVdash C_Bq$. This implies that there is a finite path $\langle w, v_1, \ldots, v_n\rangle$ starting from w with $v_n \nVdash q$. Now we look at v_{n-1}. As far as $v_{n-1}(R_1 \cup R_2)v_n$ we have that $v_{n-1} \nVdash \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$. Now in case $v_{n-1} \Vdash p$ we get that $v_{n-1} \nVdash p \rightarrow \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$ hence $\mathcal{M} \nVdash p \rightarrow \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$. In case $v_{n-1} \nVdash p$ we repeat the procedure and move to v_{n-2}. By repeating this $n - 1$ times at most, either we find the point which falsifies $p \rightarrow \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$ or obtain that $v_1 \nVdash p$. The latter implies that $w \nVdash p \rightarrow \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$.

Before starting the completeness proof we introduce the special closure of a set of subformulas of a given formula. This set will serve as the carrier set for the Kripke model we construct to falsify a formula which is not a theorem of $\mathbf{K4_2^C}$. Assume φ is an arbitrary formula. Let $Sub(\varphi)$ be the set of all sub-formulas of φ. Let $Sub^+(\varphi)$ denote the closure of $Sub(\varphi)$ in the following way: if $C_B\alpha \in Sub^+(\varphi)$ then the formulas $\Box_1\alpha$, $\Box_2\alpha$, $\Box_1 C_B\alpha$ and $\Box_2 C_B\alpha$ are also in $Sub^+(\varphi)$. Let $\sim Sub^+(\varphi)$ denote the closure of $Sub^+(\varphi)$ under a single negation. For readability reasons let us denote this set by $FL(\varphi)$ (another motivation for $FL(\varphi)$ is that this construction is very much alike to the Fisher-Ladner closure used in completeness proofs for propositional dynamic logic PDL (Fischer and Ladner 1979)).

Proposition 2 (Completeness). *Modal logic* $\mathbf{K4_2^C}$ *is complete with respect to the class of all finite, bi-transitive Kripke frames.*

Proof. Assume $\mathbf{K4_2^C} \nvdash \varphi$. Let W be the set of all maximally consistent subsets of $FL(\varphi)$. Let us define the relations R_1 and R_2 on W in the following way: For every $\Gamma, \Gamma' \in W$ we define $\Gamma R_x \Gamma'$ iff $(\forall\alpha)(\Box_x\alpha \in \Gamma \Rightarrow \Gamma' \vdash \alpha \wedge \Box_x\alpha)$, where $x \in \{1, 2\}$.

Claim. Each R_x is transitive.

Proof. Assume $\Gamma R_x \Gamma' \wedge \Gamma' R_x \Gamma''$ and $\Box_x\alpha \in \Gamma$. This implies that both $\Box_x\alpha$ and α are in $FL(\varphi)$. By the definition of $\Gamma R_x \Gamma'$ we have $\Gamma' \vdash \alpha \wedge \Box_x\alpha$, which implies $\Gamma' \vdash \Box_x\alpha$. As $\Box_x\alpha \in FL(\varphi)$ and Γ' is maximally consistent set, we get $\Box_x\alpha \in \Gamma'$. Now we use again the definition of $\Gamma' R_x \Gamma''$ and we get that $\Gamma'' \vdash \alpha \wedge \Box_x\alpha$. Hence $\Gamma R_x \Gamma''$

So far we have defined a finite set W with two transitive relations R_1, R_2 on it. Let $R_{1\vee 2}$ denote the transitive closure of the union of relations R_1 and R_2 i.e., $R_{1\vee 2} = (R_1 \cup R_2)^+$. At this point we are able to prove the truth lemma with respect to the model $\mathcal{M} = (W, R_1, R_2, R_{1\vee 2}, V)$, where $\Gamma \Vdash p$ iff $p \in \Gamma$. The proof goes by analogy to the proof for the common knowledge operator given in Fagin et al. (1995).

Lemma 1 (Truth). *For every formula* $\alpha \in FL(\varphi)$ *and every point* $\Gamma \in W$ *of the model* $\mathcal{M}$*, the following equivalence holds:* $\Gamma \Vdash \alpha$ *iff* $\alpha \in \Gamma$.

The proof is by induction on the length of formula. The base case follows immediately from the definition of valuation. Assume for all $\alpha \in FL(\varphi)$ with length less then k that: $\Gamma \Vdash \alpha$ iff $\alpha \in \Gamma$.

Let us prove the claim for $\alpha \in FL(\varphi)$ with length equal to k. If α is a conjunction or negation of two formulas then the result easily follows from the definition of the satisfaction relation and the properties of maximal consistent sets, so we can skip the proofs. Assume $\alpha = \Box_1\beta$ and assume $\Gamma \Vdash \alpha$. Take a set $B = \{\gamma : \Box_1\gamma \in \Gamma\} \cup \{\Box_1\gamma : \Box_1\gamma \in \Gamma\} \cup \{\neg\beta\}$. The sub-claim is that B is inconsistent. Assume not, then there exists $\Gamma' \in W$ such that $\Gamma' \supseteq B$. This by definition of the relation R_a means that $\Gamma R_1 \Gamma'$. This is because for every α if $\Box_1\alpha \in \Gamma$ then $\Gamma' \vdash \alpha$ and

$\Gamma' \vdash \Box_1\alpha$ and hence $\Gamma' \vdash \alpha \wedge \Box_1\alpha$. Now as $\neg\beta \in \Gamma'$, by inductive assumption we get $\Gamma' \Vdash \neg\beta$. Hence we get a contradiction with our assumption that $\Gamma \Vdash \Box_1\beta$. So B is inconsistent. This means that there exists $\gamma_{i_1}, \gamma_{i_2}, \dots \gamma_{i_n}, \Box_1\gamma_{j_1}, \dots \Box_1\gamma_{j_m} \in B$ such that $\vdash \gamma_{i_1} \wedge \gamma_{i_2} \wedge \dots \wedge \gamma_{i_n} \wedge \Box_1\gamma_{j_1} \wedge \dots \wedge \Box_1\gamma_{j_m} \to \beta$. Now we take the bigger conjunct, in particular we add $\Box\gamma_i$ for every γ_i occurring in the conjunction, so we get: $\vdash (\gamma_{i_1} \wedge \Box_1\gamma_{i_1}) \wedge (\gamma_{i_2} \wedge \Box_1\gamma_{i_2}) \wedge \dots \wedge (\gamma_{i_n} \wedge \Box_1\gamma_{i_n}) \wedge \Box_1\gamma_{j_1} \wedge \dots \wedge \Box_1\gamma_{j_m} \to \beta$. Applying the necessitation rule for $\Box_1$ and using axiom 4 we get $\vdash \Box_1\gamma_{i_1} \wedge \dots \wedge \Box_1\gamma_{i_n} \wedge \Box_1\gamma_{j_1} \wedge \dots \wedge \Box_1\gamma_{j_m} \to \Box_1\beta$ so $\Gamma \vdash \Box_1\beta$, hence as $\Box_1\beta \in FL(\varphi)$ we conclude that $\Box_1\beta \in \Gamma$.

We just showed the left-to-right direction of our claim for $\alpha = \Box_1\beta$. For the right-to-left implication assume $\Box_1\beta \in \Gamma$. By the definition of R_1 for every Γ' with $\Gamma R_1 \Gamma'$ we have $\Gamma' \vdash \beta \wedge \Box_1\beta$. From this it follows that $\Gamma' \vdash \beta$. As $\beta \in FL(\varphi)$ it follows that $\beta \in \Gamma'$ so by the inductive assumption $\Gamma' \Vdash \beta$.

The most important case is when α is of the form $C_B\beta$. Assume $\Gamma \Vdash \alpha$. Let $D = \{\Gamma \in W : \Gamma \Vdash C_B\beta\}$ and let $\delta = \bigvee_{\Gamma \in D} \hat{\Gamma}$, where $\hat{\Gamma}$ is the conjunction of all formulas inside Γ. Observe that as W is finite $\hat{\Gamma}$ is a formula in our language. We want to show that $\vdash \delta \to \Box_1(\delta \wedge \beta) \wedge \Box_2(\delta \wedge \beta)$. We do it piece by piece.

First we show $\vdash \delta \to \Box_1\beta$. This follows by an analogous argument to the previous claim. So let us take $B = \{\gamma : \Box_1\gamma \in \Gamma\} \cup \{\Box_1\gamma : \Box_1\gamma \in \Gamma\} \cup \{\neg\beta\}$. This set is inconsistent, otherwise there would exist $\Gamma' \in W$ with ΓR_1 *Gamma*$'$ and $\Gamma' \not\Vdash \beta$, which contradicts $\Gamma \Vdash C_B\beta$. From the inconsistency of B by the same argument as in the first claim it follows that $\vdash \hat{\Gamma} \to \Box_1\beta$. As Γ was chosen arbitrarily we have $\vdash \delta \to \Box_1\beta$. Analogously we obtain $\vdash \delta \to \Box_2\beta$.

Now let us show that $\delta \to \Box_1\delta$. For this we take an arbitrary $\Gamma \in D$ and arbitrary $\Gamma' \notin D$ and show $\vdash \hat{\Gamma} \to \Box_1\neg\hat{\Gamma}'$. As $\Gamma \in D$, we have that $\Gamma \Vdash C_B\beta$, while for Γ' we have $\Gamma' \not\Vdash C_B\beta$. This implies that not $\Gamma R_1 \Gamma'$, so by the definition of R_1, there is a formula ψ, such that $\Box_1\psi \in \Gamma$, while $\Gamma' \nvdash \Box_1\psi \wedge \psi$. From $\Gamma' \nvdash \Box_1\psi \wedge \psi$ we conclude that $\Box_1\psi \notin \Gamma'$ or $\psi \notin \Gamma'$. Now as both ψ and $\Box_1\psi$ are in $FL(\varphi)$ we have $\neg\Box_1\psi \in \Gamma'$ or $\neg\psi \in \Gamma'$. This means that $\hat{\Gamma}'$ has the form either $\neg\Box_1\psi \wedge \psi \wedge \bigwedge \gamma_i$ or $\neg\Box_1\psi \wedge \neg\psi \wedge \bigwedge \gamma_i$ or $\Box_1\psi \wedge \neg\psi \wedge \bigwedge \gamma_i$. Then $\neg\hat{\Gamma}'$ is of the form $\Box_1\psi \vee \neg\psi \vee \bigvee \neg\gamma_i$ or $\Box_1\psi \vee \psi \vee \bigvee \neg\gamma_i$ or $\neg\Box_1\psi \vee \psi \vee \bigvee \neg\gamma_i$. In each case $\vdash \Box_1\psi \wedge \psi \to \neg\hat{\Gamma}'$. By applying the necessitation rule we get: $\vdash \Box_1\Box_1\psi \wedge \Box_1\psi \to \Box_1\neg\hat{\Gamma}'$ and by axiom 4 for $\Box_1$ we conclude $\vdash \Box_1\psi \to \Box_1\neg\hat{\Gamma}'$. Now as $\Box_1\psi \in \Gamma$, we have $\vdash \hat{\Gamma} \to \Box_1\neg\hat{\Gamma}'$ and as Γ and Γ' were taken arbitrarily we get $\vdash \bigvee_{\Gamma \in D} \hat{\Gamma} \to \bigwedge_{\Gamma' \notin D} \Box_1\neg\hat{\Gamma}'$. It is not difficult to prove that $\vdash \bigwedge_{\Gamma' \notin D} \Box_1\neg\hat{\Gamma} \leftrightarrow \Box_1 \bigvee_{\Gamma \in D} \hat{\Gamma}$, so we obtain the desired result $\vdash \delta \to \Box_1\delta$. Analogously we can prove $\vdash \delta \to \Box_2\delta$.

Now combining $\vdash \delta \to \Box_1\beta$ and $\vdash \delta \to \Box_1\delta$ yields $\vdash \delta \to \Box_1(\delta \wedge \beta)$ and analogously $\vdash \delta \to \Box_2(\delta \wedge \beta)$. So we have $\vdash \delta \to \Box_1(\delta \wedge \beta) \wedge \Box_2(\delta \wedge \beta)$. Now we apply the induction rule to obtain $\vdash \delta \to C_B\beta$. In particular we have $\vdash \hat{\Gamma} \to C_B\beta$. The last validity implies that $C_B\beta \in \Gamma$. So we have proved the left-to-right direction of the truth lemma for the case $\alpha = C_B\beta$.

For the other direction assume $C_B\beta \in \Gamma$. Let us show by induction on k that if Γ' is reachable from Γ in k steps then both $C_B\beta$ and β are in Γ'.

Case for $k = 1$: Without loss of generality we can assume that $\Gamma R_1 \Gamma'$. By the axiom (*Equi*) we have $\vdash C_B\beta \to \Box_1\beta \wedge \Box_1 C_B\beta$. Now by construction both $\Box_1\beta, \Box_1 C_B\beta \in FL(\varphi)$. This implies that $\Box_1 C_B\beta \in \Gamma$ and $\Box_1\beta \in \Gamma$. By the definition of R_1 we get $\Gamma' \vdash \Box_1\beta \wedge \beta$ and $\Gamma' \vdash \Box_1 C_B\beta \wedge C_B\beta$. This implies that $\Gamma' \vdash \beta$ and $\Gamma' \vdash C_B\beta$ and as β and $C_B\beta$ are in $FL(\varphi)$ we derive $\beta \in \Gamma'$ and $C_B\beta \in \Gamma'$.

Assume the induction hypothesis holds for $k \leq n$ and let us verify the case $k = n$. So we have $\Gamma R_x \Gamma_1 R_x \dots R_x \Gamma_{n-1} R_x \Gamma'$, where $x \in \{1, 2\}$. By the induction hypothesis both $C_B\beta$ and β are in Γ_{n-1}, so by the same argument as in the case of $k = 1$ we obtain $\beta \in \Gamma'$, hence $\Gamma \Vdash C_B\beta$. This finishes the truth lemma.

Now if we take $\Gamma_{\neg\varphi}$ to be a maximally consistent set containing $\neg\varphi$, by the truth lemma we it follows that $\mathcal{M}, \Gamma_{\neg\varphi} \nVdash \varphi$. This finishes the completeness proof.

We have seen that every non-theorem of $\mathbf{K4_2^C}$ is falsified on a finite, bi-transitive frame. The following theorem shows that every non-theorem of $\mathbf{K4_2^C}$ can be falsified on a frame (W^t, R_1^t, R_2^t, V^t), where for each $k \in \{1, 2\}$ the pair (W^t, R_k^t) is a transitive tree. Let us first recall the definition of tree.

Definition 3. A frame (W, R) is called a *tree* if:

(1) it is rooted i.e., there is a unique point (the root) $r \in W$ such that for every $v \in W$ holds $v \neq r \Rightarrow rR^+v$,
(2) every element distinct from r has a unique immediate predecessor; that is, for every $v \neq r$ there is a unique v' such that $v'Rv$ and for every v'' we have that $v''Rv \Rightarrow v''Rv'$,
(3) R is acyclic; that is, for every $v \in W$ we have $\neg vR^+v$.

If in addition R is transitive i.e., $R = R^+$, then (W, R) is called a transitive tree.

Theorem 1. *The modal logic* $\mathbf{K4_2^C}$ *has the tree model property.*

Proof. Suppose $\nvdash \varphi$. From Theorem 2 we know that φ can be falsified in a finite, transitive, bi-relational Kripke model. Moreover, we can assume that this model is rooted. Let $\mathcal{M} = (W, R_1, R_2, V)$ be the model and w be the root where φ is falsified. Let us unravel the frame (W, R_1, R_2) around w. As a result we get a frame (W^t, R_1', R_2') where both (W^t, R_1') and (W^t, R_2') are trees. This is a standard technique in modal logic (Blackburn et al. 2006). The underlying set W^t consists of all finite strings of the form $\langle w, w_1, \dots, w_n\rangle$, where each $w_i \in W$ and $w(R_1 \cup R_2)w_1 \wedge w_i(R_1 \cup R_2)w_{i+1}$ for every $i \leq n-1$. The relation R_k' ($k \in \{1, 2\}$) is defined in the following way: $\langle w, w_1, \dots, w_n\rangle R_k' \langle w, w_1', \dots, w_m'\rangle$ iff $m = n+1$, $w_i = w_i'$ for every $i \leq n$ and $w_n R_k w_m$. To spell this out, one sequence is in the R_k' relation with another if the second sequence takes the first sequence and adds as a tail an element which is an R_k-successor of the tail of the first sequence. The relation R_k^t is defined as a transitive closures of R_k' i.e., $R_k^t = (R_k')^+$ for each $k \in \{1, 2\}$. We define the model $\mathcal{M}^t = (W^t, R_1^t, R_2^t, V^t)$, where the valuation V^t is defined by reflecting the valuation V, so $\langle w_1, \dots, w_n\rangle \Vdash p$ iff $w_n \Vdash p$. It is easy to see that the function $f : W^t \to W$ which sends each element $\langle w_1, \dots, w_n\rangle$ of W^t to its tail w_n, is a bounded morphism from the model $\mathcal{M}^t = (W^t, R_1^t, R_2^t, V^t)$ to the model $\mathcal{M} = (W, R_1, R_2, V)$. At this point we can say that if φ does not contain the common belief operator C_B then $\mathcal{M}^t, w \nVdash \varphi$. This is because the bounded morphism preserves the satisfaction of formulas. But we can not yet say that the defined bounded morphism f preserves formulas containing C_B. In fact it does. We can easily show that the function f defined above is a bounded morphism between the extended models $\mathcal{M}^t = (W^t, R_1^t, R_2^t, (R_1^t \cup R_2^t)^+, V^t)$ and $\mathcal{M} = (W, R_1, R_2, (R_1 \cup R_2)^+, V)$.

Note 1. Observe that the relation $(R_1^t \cup R_2^t)^+$ does not contain cycles and in particular it is irreflexive. This is because if $\langle w, w_1, \ldots, w_n \rangle (R_1^t \cup R_2^t)^+ \langle w, v_1, \ldots, v_m \rangle$ then m is strictly greater than n.

The main reason for introducing $\mathbf{K4_2^C}$ was to mimic the infinitary operator C_B^ω by finitary C_B. Though we cannot claim that on a logical level C_B and C_B^ω are equivalent, we can establish a semantical equivalence, in particular on Kripke structures.

Theorem 2. *For any transitive bi-relational Kripke model $\mathcal{M} = (W, R_1, R_2, V)$ and point w: $\mathcal{M}, w \Vdash C_B\varphi$ iff $\mathcal{M}, w \Vdash C_B^\omega \varphi$.*

Proof. The proof follows easily from Definitions 1 and 2 inasmuch as both operators exactly depend on $(R_1 \cup R_2)$ – paths of finite length starting at w.

2.5 Common Belief as Equilibrium

We mentioned that common belief can also be understood as an equilibrium concept.[2] On Kripke structures the equilibrium concept coincides with common belief by infinite iteration, while in general the equilibrium concept has a much closer connection to the logic $\mathbf{K4_2^C}$. It can be formalized in the modal μ-calculus in the following way:

$$C_\nu\varphi = \nu.p(\Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1 p \wedge \Box_2 p).$$

The greatest fixpoint ν is defined as the fixpoint of a descending approximation sequence defined over the ordinals. Denote by $|\varphi|$ the truth set of φ in the appropriate model $\mathcal{M}$ where evaluation occurs:

$$|C_\nu^0\varphi| = |\Box_1\varphi \wedge \Box_2\varphi|;$$
$$|C_\nu^{k+1}\varphi| = |\Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1 C_\nu^k\varphi \wedge \Box_2 C_\nu^k\varphi|;$$

[2]For the remainder of this section and later on for Theorem 9 we assume some familiarity with the modal μ-calculus. Lack of space hinders a fuller treatment, however for more details on the modal μ-calculus we refer to Blackburn et al. (2006, Part 3, Chapter 4); see also the discussion in van Benthem and Sarenac (2004).

$$|C_\nu^\lambda \varphi| = |\bigcap_{k<\lambda} C_\nu^k \varphi|, \text{ for } \lambda \text{ a limit ordinal.}$$

We obtain $|C_\nu \varphi| = |C_\nu^\gamma \varphi|$, where γ is a least ordinal for which the approximation procedure halts: i.e. $|C_\nu^\gamma \varphi| = |C_\nu^{\gamma+1} \varphi|$. Halting is guaranteed because the occurrence of the propositional variable p in operator $F(p)$, where $F(p) = \Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1 p \wedge \Box_2 p$, is positive. Hence by the Knaster-Tarski theorem the sequence will always reach a greatest fixpoint. Then the semantics of the operator C_ν is defined in the following way:

$$\mathcal{M}, w \Vdash C_\nu\varphi \text{ iff } w \in |C_\nu^\gamma \varphi|$$

In general this procedure may take more than ω steps, but in the case of Kripke structures the situation is simpler. The following property relates the different operators on Kripke models.

Theorem 3. *For every bi-relational Kripke model $\mathcal{M} = (W, R_1, R_2, V)$ and a point $w \in W$ the following condition holds: $\mathcal{M}, w \Vdash C_B^\omega \varphi$ iff $\mathcal{M}, w \Vdash C_\nu \varphi$.*

Proof. Observe that we can rewrite $C_B^\omega \varphi = \Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1\Box_1\varphi \wedge \Box_1\Box_2\varphi \wedge \Box_2\Box_1\varphi \wedge \Box_2\Box_2\varphi \wedge \Box_1\Box_1\Box_1\varphi \wedge \Box_1\Box_1\Box_2\varphi \ldots$ in the following way: $\Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1(\Box_1\varphi \wedge \Box_2\varphi) \wedge \Box_2(\Box_1\varphi \wedge \Box_2\varphi) \wedge \ldots$. Hence $|C_B^\omega \varphi| = |C_\nu^\omega \varphi|$. It is known that on Kripke structures the stabilization process does not need more than ω steps (van Benthem and Sarenac 2004) i.e. $|C_\nu \varphi| = |C_\nu^\omega \varphi|$. Hence $w \Vdash C_\nu \varphi$ iff $w \Vdash C_B^\omega \varphi$.

It follows that on transitive bi-relational Kripke structures the three operators C_B, C_B^ω and C_ν coincide.

2.6 A Note on the Semantics of Lismont and Mongin

In their paper (Lismont and Mongin 1994), Lismont and Mongin develop a neighborhood semantics for logics extended with a common belief operator. As a basis for the semantics they consider the class of augmented neighbourhood structures, i.e. the neighborhood function $N_i : W \to PP(W)$ for each agent $i \in \{1, 2\}$ has the following properties: for an arbitrary world $w \in W$, $N_i(w)$ contains the set W, it is closed under supersets and arbitrary intersections (the original work is presented for the finite set of agents we just simplify it here for the case of two agents). It is well known that there is a satisfaction preserving correspondence between augmented neighbourhood structures and Kripke structures and therefore one can reduce the completeness problem of a logic in neighborhood semantics to Kripke completeness (Chellas 1980; Hansen et al. 2009), although the main point is the definition of the semantics for the common belief operator in these terms. In the paper it is given by the following clause:

$$N_{C_\nu} = N_E \circ (N_{C_\nu} \cap B)$$

where N_E stands for the semantics of the collective belief operator. In the case of two agents, $N_E(w) = N_1(w) \cap N_2(w)$ for every $w \in W$. The composition $\circ$ of neighborhood functions is defined in the following way $U \in (N \circ M)(w)$ iff $\{v \mid U \in M(v)\} \in N(w)$ for each $U \subseteq W$ and $w \in W$. Additionally B is the neighborhood function defined by $U \in B(w)$ iff $w \in U$.

On the one hand we can see that N_{C_v} is defined as a fixpoint, although it is not claimed to be the greatest fixpoint. On the other hand the definition of composition of neighborhood functions suggests the operator is treated as the infinite intersection of iterated modalities, exactly as in definition of the operator C_B^{ω}. Therefore the definition of N_{C_v} embraces both the fixpoint definition and the iteration of individual modalities at once and indeed on the class of augmented neighborhood structures these two definitions collapse to the two definitions of common belief operator on Kripke frames which we know to coincide. In general, however, the situation may be different. For example this is the case on the class of topological structures from van Benthem and Sarenac (2004).

In the next section we turn to the topological semantics for our common belief logic. It is natural to ask whether and how this is related to the neighbourhood semantics of Lismont and Mongin (1994). At present we do not have a precise answer to this question. One might look at topologies as special cases of neighborhood structures, where indeed neighborhoods are simply open neighborhoods of points in a topological sense. But this does not provide us with our derived set topological semantics, i.e. given a neighborhood model (W, N, V) the truth set $\{w \mid \{v \in W \mid v \Vdash p\} \in N(w)\}$ of the modality $\Box p$ taken in the neighborhood semantics is not the same as the set of all colimits of the set $\{w \mid w \Vdash p\}$ in the topology obtained from the neighborhood function N. In fact the problem is that the class of neigbourhood structures that correspond to topological structures preserving the satisfaction of modal formulas has not yet been studied. Observe that here we deal with the derived set topological semantics, and we are supposing that neighborhood structures should preserve the satisfaction of formulas with respect to this d-semantics, and not with respect to the standard topological semantics.

3 Topological Semantics

The idea of a derived set topological semantics originates with the McKinsey-Tarski paper (McKinsey and Tarski 1944). This idea was taken further in Esakia (2001). The following works contain some important results in this direction: Bezhanishvil et al. (2005), Shehtman (1990), Lucero-Bryan (2011), and Gabelaia (2004). The derived set topological semantics for $\mathbf{K4_2^C}$ is provided by the class of all bi-topological spaces. In the same way, as it is done in van Benthem and Sarenac (2004) for the common knowledge operator, we interpret the common belief operator on the intersection topology. On the other hand, different from C_K, for which the semantics is given using the interior of the intersection of the two topologies, we provide the semantics of $C_B\varphi$ as a set of all colimits of $|\varphi|$ in the intersection topology. As a main result we prove the soundness and completeness of the logic $\mathbf{K4_2^C}$ with respect to the class of all T_D*-intersection closed*, bi-topological spaces where each topology satisfies the T_D separation axiom. We start with the basic definitions.

Definition 4. A pair (X, Ω) is called a topological space if X is a set and Ω is a collection of subsets of X with the following properties:

(1) $X, \varnothing \in \Omega$,
(2) $A, B \in \Omega$ implies $A \cap B \in \Omega$,
(3) $A_i \in \Omega$ implies $\bigcup A_i \in \Omega$.

Elements of Ω are called opens or open sets of the topological space.

Definition 5. A topological space (X, Ω) is called an *Alexandroff space* if an arbitrary intersection of opens is open, that is $A_i \in \Omega$ implies $\bigcap A_i \in \Omega$. (X, Ω) is called a T_D*-space* if every point $x \in X$ can be represented as an intersection of some open set A and some closed set B.

We now define the colimit operator (or the set of all colimit points Engelking 1977) of a set in a topological space. This is needed to give the semantics of modal formulas in an arbitrary topological space.

Definition 6. Given a topological space (X, Ω) and a set $A \subseteq X$ we will say that $x \in X$ is a colimit point of A if there exists an open neighborhood U_x of x such that $U_x - \{x\} \subseteq A$. The set of all colimit points of A will be denoted by $\tau(A)$ and will be called the colimit set of A.

In words, a point x belongs to the colimit points of a set A iff some open set B around x is contained in $A \cup \{x\}$. The colimit set provides a semantics for the box modality, consequently the semantics for diamond is provided by the dual of the colimit set, which is called the *derived* set. The derived set of A is denoted by $der(A)$. So we have $\tau(A) = X - der(X - A)$. Again a point x belongs to the set of limit points of a set A iff every open set B around x intersects with $A - \{x\}$. Below we list some examples and properties of the colimit and derivative operators.

Example 1. Let R be a set of all reals and $A \subseteq R$ be as follows: $A = \{\frac{1}{m} \mid m \geq 1\}$. Then $der(A) = \{0\}$.

Example 2. Let X be an arbitrary set and let $\Omega = \{U \mid U \subseteq X\}$, i.e. Ω is a discrete topology on X. Then for an arbitrary set $A \subseteq X$ we have the set of all colimit points $\tau(A)$ of a set A is equal to X.

Example 3. Let X be an arbitrary set and let $\Omega = \{\emptyset, X\}$, i.e. Ω is a trivial topology on X. Then for an arbitrary set $A \subseteq X$ the set of all colimit points $\tau(A)$ of A is calculated as follows: If $X - A$ is a singleton or if $A = X$ then $\tau(A) = A$ otherwise $\tau(A) = \emptyset$.

Fact 4 (Engelking 1977; Esakia 2004). *For a given topological space* (X, Ω) *the following properties hold:*

(1) $Int(A) = \tau(A) \cap A \subseteq \tau\tau(A)$*, where Int denotes the interior operator,*
(2) $\tau(X) = X$ *and* $\tau(A \cap B) = \tau(A) \cap \tau(B)$*,*
(3) If Ω *is a* T_d*-space then* $\tau(A) \subseteq \tau\tau(A)$*,*
(4) If $\Omega_1 \subseteq \Omega_2$ *then* $\tau_1(A) \subseteq \tau_2(A)$ *where* τ_i*,* $i \in \{1,2\}$ *is a colimit operator of the corresponding topology* Ω_i*.*

The following links T_D-spaces and irreflexive transitive relational structures. This result is a special case of a more general correspondence between weakly-transitive and irreflexive relational structures and all *Alexandroff* spaces (Esakia 2001).

Fact 5 (Esakia 2004). *There is a one-to-one correspondence between Alexandroff,* T_D*-spaces and transitive, irreflexive relational structures.*

Let us briefly describe the correspondence. We first introduce the downset operator. Let (X, R) be a Kripke frame. The downset operator R^{-1} is defined in the following way: for any $A \subseteq X$ we set $R^{-1}(A) := \{x | (\exists y)(y \in A \wedge xRy)\}$. Now if we are given an irreflexive, transitive order (X, R) it is possible to prove that the downset operator R^{-1} satisfies all the properties of the topological derivative operator for T_D-spaces. Hence we get a T_D-space (X, Ω_R), where Ω_R is the topology obtained from the derivative operator R^{-1}. Conversely with every *Alexandroff* T_D-space (X, Ω), one can associate an irreflexive and transitive relational structure (X, R_Ω), where $xR_\Omega y$ iff $x \in der(\{y\})$. Moreover we have that (X, Ω_{R_Ω}) is homeomorphic to (X, Ω) and (X, R_{Ω_R}) is order isomorphic to (X, R).

Fact 6 (Esakia 2004). *The set A is open in* (X, Ω_R) *iff* $x \in A$ *implies that the implication* $(xRy \Rightarrow y \in A)$ *holds for every* $y \in X$*.*

This correspondence can be directly generalized to Kripke frames with more than one transitive and irreflexive relation. Of course then we will have one Alexandroff T_D-space for each irreflexive and transitive order. Below we prove the proposition which builds a bridge between Kripke and topological semantics for $\mathbf{K4_2^C}$.

Proposition 3. *If* R_1 *and* R_2 *are two irreflexive and transitive orders on X and* $(R_1 \cup R_2)^+$ *is also irreflexive and transitive, then* $\Omega_{(R_1 \cup R_2)^+} \cong \Omega_{R_1} \cap \Omega_{R_2}$*.*

Before starting the proof, observe that $(R_1 \cup R_2)^+$ may not be irreflexive even if both R_1 and R_2 are. For example: let $X = \{x, y\}$ and $R_1 = \{(x, y)\}$ and $R_2 = \{(y, x)\}$ then $(R_1 \cup R_2)^+ = \{(x, y), (y, x), (x, x), (y, y)\}$. On the topological side this example shows that T_D-spaces do not form a lattice. That is why in Proposition 3 we require $(R_1 \cup R_2)^+$ to be irreflexive and transitive.

Proof. Assume that $A \in \Omega_{(R_1 \cup R_2)^+}$. By Fact 6 this means that if $x \in A$ then for every y such that $x(R_1 \cup R_2)^+y$ it holds that $y \in A$. Since $R_i \subseteq (R_1 \cup R_2)^+$ for each $i \in \{1, 2\}$, it holds that $xR_1y \Rightarrow y \in A$ and $xR_2y \Rightarrow y \in A$ for every $y \in X$. Hence $A \in \Omega_1 \cap \Omega_2$ according to Fact 6.

Conversely assume $A \in \Omega_1 \cap \Omega_2$. This means that $x \in A \Rightarrow (x(R_1 \cup R_2)y \Rightarrow y \in A)$. Now take an arbitrary y such that $x(R_1 \cup R_2)^+y$. By definition this means that there is a $(R_1 \cup R_2)$-path $\langle x_1, x_2, \ldots x_n \rangle$ starting at x going to y. But this means that each member of this path is in A because A is open in the intersection of the two topologies. Hence $y \in A$ and hence $A \in \Omega_{(R_1 \cup R_2)^+}$

Next we give a definition of the satisfaction relation of modal formulas in the derived set topological semantics. Observe that this definition is given in a standard modal language i.e., without the common belief operator. Recall that a topological model is a tuple $\mathcal{M} = (W, \Omega, V)$ where $V : Prop \to P(W)$ is a valuation function.

Definition 7. The satisfaction of a modal formula in a topological model $\mathcal{M} = (W, \Omega, V)$ at a point $w \in W$ is defined in the following way:

- $\mathcal{M}, w \Vdash p$ iff $w \in V(p)$,
- Boolean cases are standard,
- $\mathcal{M}, w \Vdash \Box\varphi$ iff $w \in \tau(V(\varphi))$, where τ is a colimit operator of Ω.

Fact 7 (Esakia 2004). *The correspondence mentioned in Fact 5 preserves the truth of modal formulas, i.e.* $(W, R, V), x \Vdash \alpha$ *iff* $(W, \Omega_R, V), x \Vdash \alpha$.

Note that in Fact 7, the symbol $\Vdash$ on the left hand side denotes the satisfaction relation on Kripke models, while on the right hand side it denotes the satisfaction relation on topological frames in the derived set semantics. Now we extend the satisfaction relation to the language with the common belief operator.

Definition 8. The satisfaction of a modal formula on a bi-topological model $\mathcal{M} = (W, \Omega_1, \Omega_2, V)$ at a point $w \in W$ is defined in the following way:

$\mathcal{M}, w \Vdash p$ iff $w \in V(p)$,
$\mathcal{M}, w \Vdash \alpha \wedge \beta$ iff $\mathcal{M}, w \Vdash \alpha$ and $\mathcal{M}, w \Vdash \beta$,
$\mathcal{M}, w \Vdash \neg\alpha$ iff $\mathcal{M}, w \nVdash \alpha$,
$\mathcal{M}, w \Vdash \Box_i\varphi$ iff $w \in \tau_i(V(\varphi))$, where τ_i is a colimit operator of Ω_i, $i \in \{1, 2\}$,
$\mathcal{M}, w \Vdash C_B\varphi$ iff $w \in \tau_{1\wedge 2}(V(\varphi))$, where $\tau_{1\wedge 2}$ is a colimit operator in $\Omega_1 \cap \Omega_2$.

As an immediate corollary of Proposition 3 and a many-modal version of Fact 7, we get the following proposition.

Proposition 4. *If* R_1 *and* R_2 *are two irreflexive and transitive orders and* $(R_1 \cup R_2)^+$ *is also topological then for every formula* α *in* $\mathbf{K4_2^C}$ *the following holds:*

$$(W, R_1, R_2, V), x \Vdash \alpha \text{ iff } (W, \Omega_{R_1}, \Omega_{R_2}, V), x \Vdash \alpha.$$

Now it is clear that we can reduce the topological completeness problem to Kripke completeness if for every non-theorem $\mathbf{K4_2^C} \nvdash \varphi$ we can find a bi-relational topological counter-model (W, R_1, R_2, V) with $(R_1 \cup R_2)^+$ being also a topological relation.

Definition 9. The triple (X, Ω_1, Ω_2) is a T_D-intersection closed bi-topological space if each of the topologies Ω_1, Ω_2 and $\Omega_1 \cap \Omega_2$, satisfies the T_D-separation axiom.

Theorem 8. $\mathbf{K4_2^C}$ *is sound and complete with respect to the class of all T_D-intersection closed, bi-topological, Alexandroff spaces.*

Proof. (Soundness) Take an arbitrary T_D-intersection closed, bi-topological model $\mathcal{M} = (X, \Omega_1, \Omega_2, V)$. From (2) and (3) of Fact 4 it follows that $K4$-axioms are valid for each box. Let us show that at each point $x \in X$, the equilibrium axiom is satisfied. Assume that $\mathcal{M}, x \Vdash C_B p$. Hence by Definition 8 we have $x \in \tau_{1\wedge 2}|p|$. By (4) of Fact 4 we get $x \in \tau_1|p|$ and $x \in \tau_2|p|$. By (3) we have $\tau_{1\wedge 2}|p| \subseteq \tau_{1\wedge 2}\tau_{1\wedge 2}|p| \subseteq \tau_1\tau_{1\wedge 2}|p|$. Analogously $\tau_{1\wedge 2}|p| \subseteq \tau_2\tau_{1\wedge 2}|p|$. Hence we have $x \Vdash \Box_1 p \wedge \Box_2 p \wedge \Box_1 C_B p \wedge \Box_2 C_B p$.

For the other direction assume that $x \in \tau_1\tau_{1\wedge 2}|p| \cap \tau_1|p| \cap \tau_2\tau_{1\wedge 2}|p| \cap \tau_2|p|$. By (2) of Fact 4 we get $x \in \tau_1(\tau_{1\wedge 2}|p| \cap |p|) \cap \tau_2(\tau_{1\wedge 2}|p| \cap |p|)$. By (1) of Fact 4 we conclude $x \in \tau_1(Int_{1\wedge 2}|p|) \cap \tau_2(Int_{1\wedge 2}|p|)$, where $Int_{1\wedge 2}$ denotes the interior operator in the intersection topology. By the definition of colimit there exists $U_x^1 \in \Omega_1$ such that $x \in U_x^1$ and $U_x^1 - \{x\} \subseteq Int_{1\wedge 2}|p|$ and there exists $U_x^2 \in \Omega_2$ such that $x \in U_x^2$ and $U_x^2 - \{x\} \subseteq Int_{1\wedge 2}|p|$. Hence $(U_x^1 \cup U_x^2) - \{x\} \subseteq Int_{1\wedge 2}|p|$. Let us show that $Int_{1\wedge 2}|p| \cup \{x\}$ is open in $\Omega_1 \cap \Omega_2$. Since $U_x^1 \in \Omega_1$ and $Int_{1\wedge 2}|p| \in \Omega_1$ we have $U_x^1 \cup Int_{1\wedge 2}|p| = Int_{1\wedge 2}|p| \cup \{x\} \in \Omega_1$. Analogously we show that $Int_{1\wedge 2}|p| \cup \{x\} \in \Omega_2$. Hence $x \in \tau_{1\wedge 2}|p|$.

Let us show that the induction rule is valid in the class of all T_D-intersection closed bi-topological spaces. The proof goes by contraposition. Assume not $\vdash p \to C_B q$. This means that for some T_D-intersection closed, bi-topological model $\mathcal{M} = (X, \Omega_1, \Omega_2, V)$ and a point $x \in X$ it holds that: $x \Vdash p$ while $x \nVdash C_B q$. We want to show that not $\vdash p \to \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$. It suffices to find a T_D-intersection closed bi-topological model which falsifies the formula. For such a model one could take $\mathcal{M}' = (X, \Omega_1 \cap \Omega_2, \Omega_1 \cap \Omega_2, V)$. Indeed as $(X, \Omega_1, \Omega_2, V)$ is T_D-intersection closed, the topology $\Omega_1 \cap \Omega_2$ satisfies the T_D-separation axiom. Besides since in $\mathcal{M}'$ both topologies are the same, their intersection is also $\Omega_1 \cap \Omega_2$ and hence again is a T_D-space. Now it is immediate that $\mathcal{M}', x \nVdash p \to \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$. This is because by construction of $\mathcal{M}'$ we have $\mathcal{M}', x \nVdash \Box_i q$ iff $\mathcal{M}, x \nVdash C_B q$ for every $x \in X$ and $i \in \{1, 2\}$.

(Completeness) Assume $\mathbf{K4_2^C} \nvdash \varphi$. According to Theorem 1 there exist a tree model $M^t = (W^t, R_1^t, R_2^t, V)$ which falsifies φ. We know that $(R_1 \cup R_2)^+$ is an irreflexive and transitive order (see Note 1). By applying Proposition 4 it follows that the formula φ is falsified in the corresponding bi-topological model $(W^t, \Omega_{R_1^t}, \Omega_{R_2^t}, V)$, which is T_D-intersection closed because of Fact 5, Proposition 3 and Note 1.

We can now show how the semantical definition of common belief $C_B\varphi$ as a colimit of the intersection topology meshes with the general equilibrium concept: on topological models the two operators C_B and C_ν coincide.

Theorem 9. *For every bi-topological model $\mathcal{M} = (X, \Omega_1, \Omega_2, V)$ and an arbitrary formula φ the following equality holds: $\nu.p(\tau_1(|\varphi|) \cap \tau_2(|\varphi|) \cap \tau_1(p) \cap \tau_2(p)) = \tau_{1\wedge 2}(|\varphi|)$.*

Proof. That $\tau_{1\wedge 2}(|\varphi|)$ is a fixpoint of the operator $F(p) = \tau_1(|\varphi|) \cap \tau_2(|\varphi|) \cap \tau_1(p) \cap \tau_2(p)$ follows from the soundness proof of the equilibrium axiom, see Theorem 8.

Now let us show that $\tau_{1\wedge 2}(|\varphi|)$ is the greatest fixpoint of $F(p)$. Take an arbitrary fixpoint B of the operator $F(p)$. That B is a fixpoint immediately implies that $B \subseteq \tau_1(|\varphi|) \cap \tau_2(|\varphi|) \cap \tau_1(B) \cap \tau_2(B)$. By (1) of Fact 4 we have $B \subseteq Int_i(B) = \tau_i(B) \cap B$ for each $i \in \{1, 2\}$. Hence $B = Int_{1\wedge 2}(B)$ where $Int_{1\wedge 2}$ is the interior operator in the intersection topology of the two topologies. Now let us show that for every $x \in B$ the set $\{x\} \cup (B \cap |\varphi|)$ is open in the intersection of the two topologies. Take an arbitrary point $y \in \{x\} \cup (B \cap |\varphi|)$. Since $y \in B \subseteq \tau_1(|\varphi|)$ we know that there exists an open neighborhood $U^1_y \in \Omega_1$ of y such that $U^1_y - \{y\} \subseteq |\varphi|$. This means that $B \cap U^1_y \in \Omega_1$ and $B \cap U^1_y \subseteq \{x\} \cup (B \cap |\varphi|)$. This means that for every point $y \in \{x\} \cup (B \cap |\varphi|)$ there is an open neighborhood $B \cap U^1_y \in \Omega_1$ of y such that $B \cap U^1_y \subseteq \{x\} \cup (B \cap |\varphi|)$ hence $\{x\} \cup (B \cap |\varphi|) \in \Omega_1$. In exactly the same way we show that $\{x\} \cup (B \cap |\varphi|) \in \Omega_2$. Hence $\{x\} \cup (B \cap |\varphi|) \in \Omega_1 \cap \Omega_2$. This means that $x \in \tau_{1\wedge 2}(|\varphi|)$ since there exists an open neighborhood $U_{1\wedge 2} = \{x\} \cup (B \cap |\varphi|) \in \Omega_1 \cap \Omega_2$ with $U_{1\wedge 2} - \{x\} \in |\varphi|$.

4 From Belief to Knowledge

Let us now look briefly at the connection between the logics of common knowledge $\mathbf{S4^C_2}$ and common belief $\mathbf{K4^C_2}$. This connection generalizes the existing splitting translation between **S4**-logics and **K4**-logics.[3] As a result we obtain a validity preserving translation from $\mathbf{S4^C_2}$ formulas to $\mathbf{K4^C_2}$ formulas in which common knowledge is expressed in terms of common belief.

Definition 10. The normal modal logic $\mathbf{S4^C_2}$ is defined in a modal language with infinite set of propositional letters $p, q, r \ldots$ and connectives $\vee, \wedge, \neg, \Box_1, \Box_2, C_K$, where the formulas are constructed in a standard way.

- The axioms are all classical tautologies, each box satisfies all **S4** axioms and in addition we have the equilibrium axiom for the common knowledge operator:

$$(equi) : C_K p \leftrightarrow p \wedge \Box_1 C_K p \wedge \Box_2 C_K p$$

- The rules of inference are: Modus-ponens, Substitution, Necessitation for $\Box_1$ and $\Box_2$ and the induction rule:

$$(ind) : \frac{\vdash \varphi \rightarrow \Box_1(\varphi \wedge \psi) \wedge \Box_2(\varphi \wedge \psi)}{\vdash \varphi \rightarrow C_K \psi}$$

  for arbitrary formulas φ and ψ of the language.

The Kripke semantics for the modal logic $\mathbf{S4^C_2}$ is provided by reflexive and transitive, bi-relational Kripke frames. To interpret the common knowledge operator C_K, the reflexive, transitive closure of a union relation is used.

[3]For a discussion of the splitting translation and its application in non-monotonic modal logics, see the authors' (Pearce and Uridia 2011a).

Definition 11. The reflexive, transitive closure $R^\star$ of a relation $R \subseteq W \times W$ is defined in the following way: $R^\star = R^+ \cup \{(w, w) | w \in W\}$.

The satisfaction of formulas is defined as follows.

Definition 12. For a given bi-relational Kripke model $\mathcal{M} = (W, R_1, R_2, V)$ the satisfaction of a formula at a point $w \in W$ is defined inductively as follows:

$w \Vdash p$ iff $w \in V(p)$,
$w \Vdash \alpha \wedge \beta$ iff $w \Vdash \alpha$ and $w \Vdash \beta$,
$w \Vdash \neg\alpha$ iff $w \nVdash \alpha$,
$w \Vdash \Box_i\varphi$ iff $(\forall v)(wR_iv \Rightarrow v \Vdash \varphi)$,
$w \Vdash C_K\varphi$ iff $(\forall v)(w(R_1 \cup R_2)^\star v \Rightarrow v \Vdash \varphi)$.

Fact 10 (Fagin et al. 1995). *The modal logic* $\mathbf{S4_2^C}$ *is sound and complete with respect to the class of all finite, reflexive, bi-transitive Kripke frames.*

Definition 13. Consider the following function from the set of formulas in $\mathbf{S4_2^C}$ to the set of formulas in $\mathbf{K4_2^C}$.

$Sp(p) = p$ for every propositional letter p,
$Sp(\neg\alpha \vee \beta) = \neg Sp(\alpha) \vee Sp(\beta)$,
$Sp(\Box_i\alpha) = \Box_i Sp(\alpha) \wedge Sp(\alpha)$,
$Sp(C_K\alpha) = C_B Sp(\alpha) \wedge Sp(\alpha)$.

Theorem 11. $\vdash_{\mathbf{S4_2^C}} \varphi$ *iff* $\vdash_{\mathbf{K4_2^C}} Sp(\varphi)$.

Proof. We prove the theorem by a semantical argument using the Kripke completeness results, see Proposition 2 and Fact 10. Let us first show by induction on the length of a formula that for every bi-relational Kripke model $\mathcal{M} = (W, R_1, R_2, V)$ and every $w \in W$ the following holds:

$$(a) \quad \mathcal{M}^\star = (W, R_1^\star, R_2^\star, V), w \Vdash \varphi \text{ iff } \quad \mathcal{M}^+ = (W, R_1^+, R_2^+, V), w \Vdash Sp(\varphi).$$

The only nonstandard case is when $\varphi = C_K\psi$. Assume $\mathcal{M}^\star, w \Vdash C_K\psi$. By the definition of $(R_1 \cup R_2)^\star$ this means that $\mathcal{M}^\star, w \Vdash \psi$ and for every w' such that $w(R_1 \cup R_2)^\star w'$, we have $\mathcal{M}^\star, w' \Vdash \psi$. Now by the induction hypothesis we have that $\mathcal{M}^+, w \Vdash \psi$ and $\mathcal{M}^+, w' \Vdash \psi$. Since w' was an arbitrary $(R_1 \cup R_2)^\star$ successor of w we have $\mathcal{M}^+, w \Vdash C_B\psi$. This is because $(R_1 \cup R_2)^\star \supseteq (R_1 \cup R_2)^+$. Hence we obtain $\mathcal{M}^+, w \Vdash C_B\psi \wedge \psi$. The converse direction follows by the same argument.

Now assume $\vdash_{\mathbf{S4_2^C}} \varphi$. By Fact 10 this means that φ is valid in every reflexive and transitive, bi-relational model. Take an arbitrary transitive, bi-relational model $\mathcal{M}$. Then by assumption we have $\mathcal{M}^\star \Vdash \varphi$. Hence by (a) we have that $\mathcal{M} \Vdash Sp(\varphi)$. As $\mathcal{M}$ was an arbitrary transitive, bi-relational model, from Proposition 2 we infer that $\vdash_{\mathbf{K4_2^C}} Sp(\varphi)$. Conversely, suppose $\vdash_{\mathbf{K4_2^C}} Sp(\varphi)$. Then by Proposition 2, $Sp(\varphi)$ is valid in the class of all transitive, bi-relational models. Take an arbitrary reflexive and transitive, bi-relational model $\mathcal{N}$. Then $\mathcal{N} \Vdash Sp(\varphi)$ because $\mathcal{N} = \mathcal{N}^+$. So by (a) we have that $\mathcal{N}^\star \Vdash \varphi$. Now as $\mathcal{N}$ was reflexive and transitive, $\mathcal{N}^\star = \mathcal{N}$, hence $\mathcal{N} \Vdash \varphi$. And since $\mathcal{N}$ was an arbitrary reflexive and transitive, bi-relational model, by Fact 10 we have $\vdash_{\mathbf{S4_2^C}} \varphi$.

5 Conclusions

Our main aim in this chapter has been to extend the work of van Benthem and Sarenac (2004) on the topological semantics for common knowledge by interpreting a common belief operator on the intersection of two topologies in a bi-topological model. In particular we considered a logic $\mathbf{K4_2^C}$ of common belief for normal agents, first under a Kripke, relational semantics, showing it to have the finite model property and the tree model property. We then showed that $\mathbf{K4_2^C}$ is the modal logic of all T_D-intersection closed, bi-topological spaces with a derived set interpretation of modalities and we saw how the common knowledge logic $\mathbf{S4_2^C}$ can be embedded in $\mathbf{K4_2^C}$ via the splitting translation that maps $C_K p$ into $p \wedge C_B p$.

A worthwhile exercise for the future would be to undertake a more detailed comparison of our topological approach with the neighborhood systems of Lismont and Mongin (1994) that we became aware of after finishing the first version of this chapter. Another direction for the future would be to look for concrete topological structures which would fully capture the behavior of the logic $\mathbf{K4_2^C}$ or some of its extensions.

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