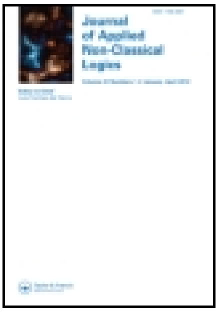

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David Pearce [a] & Levan Uridia [b]

[a] Universidad Politécnica de Madrid , Spain

[b] International Black Sea University and Georgian Institute of Cybernetics , Tbilisi , Georgia

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Algebraic semantics for modal and superintuitionistic non-monotonic logics

David Pearce[a*] and Levan Uridia[b]

[a] *Universidad Politécnica de Madrid, Spain;* [b] *International Black Sea University and Georgian Institute of Cybernetics, Tbilisi, Georgia*

The paper provides a preliminary study of algebraic semantics for modal and superintuitionistic non-monotonic logics. The main question answered is: how can non-monotonic inference be understood algebraically?

Keywords: non-monotonic logic; equilibrium logic; Boolean algebras with operators; Heyting algebras

1. Introduction

In the thirty years since their inception, non-monotonic logics have been widely investigated among logicians and computer scientists. Early motivation for these logics included the aim to reason about actions and defaults, as well as to model the belief of agents with full introspection (McCarthy, 1980; McDermott & Doyle, 1980; Moore, 1985; Reiter, 1980). Subsequently, non-monotonic systems based on classical, modal and on non-classical logics were developed and studied. Logic programming provided another important domain for non-monotonic reasoning (NMR) and led to a rich interaction between NMR and logic-based programming languages. Of particular interest in the area of logic programming are non-monotonic logics based on non-classical logics, including superintuitionistic logics, since they provide a logical foundation for reasoning with stable models or with well-founded models and are therefore applicable to LP paradigms such as *answer set programming* (ASP; Cabalar, Odintsov, Pearce, & Valverde, 2006; Pearce, 2006). The most obvious difference between non-monotonic and monotonic logics concerns the consequence relation. Since non-monotonic consequence, denoted by $\mid\!\sim$, does not satisfy the monotonicity axiom, $\Gamma \mid\!\sim \varphi$ need not imply $\Gamma' \mid\!\sim \varphi$ for a larger set of formulas $\Gamma' \supseteq \Gamma$. Another difference, making algebraic analysis more complicated for non-monotonic logics, is that the substitution rule does not hold.

Algebraic semantics has played a pivotal role in the study of traditional logical systems, classical and non-classical. The same cannot be said for systems of non-monotonic reasoning. Although NMR is well established in artificial intelligence and some areas of computer science, algebraic methods have hardly been exploited in this area. While there have been some works relating algebraic semantics to non-monotonic modal logics (Ghosh, 2004; Truszczynski, 2006a,b), the main question has remained unanswered: how can non-monotonic inference be understood algebraically? This paper takes some first steps in developing an algebraic semantics for the well-known family of non-monotonic logics based on normal modal logics and on superintuitionistic logics. In particular, we describe algebraic structures that correspond to so-called *expansions* of

*Corresponding author. Email: david.pearce@upm.es

sets of formulas. From these we obtain an algebraic counterpart of non-monotonic consequence in these logics.

It is well known that every normal modal logic L has an algebraic counterpart based on Boolean algebras with operators. The axioms of the logic are translated into the equations of the corresponding algebraic theory. All in all we get a variety V_L of Boolean algebras with operators determined by these equations. In the literature this variety is often referred to as the variety of L-algebras. The variety of L-algebras contains a special algebra, the free algebra on countable generators, also called the Lindenbaum-Tarski algebra of the logic L. The Lindenbaum-Tarski algebra of the logic is not easy to investigate because of its complex structure. One of the main directions in algebraic modal logic is to reduce the study of Lindenbaum-Tarski algebra to some other, simpler algebras of the variety. Among good references for algebraic modal logic we can mention Chagrov and Zakharyaschev (1997), and Blackburn, de Rijke, and Venema (2001). Unfortunately, in general there seems to be no direct analogue of the Lindenbaum-Tarski algebra for non-monotonic logics. Nevertheless there are algebraic analogues of expansions on which non-monotonic consequence relations are based. In this paper we formulate the algebraic counterparts of expansions, which we call *algebraic expansions*, and thereby provide a basis for the study of non-monotonic logic by algebraic techniques. A major obstacle we encounter is that non-monotonic logics are not substitution closed. This means that two isomorphic algebraic models may not be the same from a non-monotonic point of view. It also implies that algebraic expansions do not form an equational class.

The paper is organised in the following way: in Section 2 we recall non-monotonic modal and superintuitionistic logics, expansions, non-monotonic consequence relations and mention some important examples. In Section 3 we provide an algebraic study of non-monotonic logics discussed in Section 2. We define the notion of algebraic expansion, study the consequence relation, and prove the two main theorems of the paper: one concerning normal modal non-monotonic logics and one for superintuitionistic logics. In the last section we provide some conclusions, describe some possible advantages and disadvantages of the given semantics, and list some topics for future work.

2. Non-monotonic logics

In this work we consider two different classes of non-monotonic logics. The first class comprises the non-monotonic normal modal logics defined in the sense of McDermott (1982) and the second class consists of non-monotonic superintuitionistic logics defined by Pearce (1999, 2006). We make a distinction between these classes because their underlying languages differ. In the first case we deal with a propositional modal language with one modal operator, while in the second case we have simply a propositional language without modalities. Because of this difference the non-monotonic inference relations in these two classes are defined in distinct ways.

The basic modal language is given by an infinite set $Prop$ of propositional letters and the following connectives, $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\Box$. The set of formulas, $Form$, is constructed in a standard way, in particular, $Prop \subseteq Form$ and if $\alpha, \beta \in Form$, then also $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$, $\Box\alpha \in Form$. We will use the following abbreviations: $\bot \equiv \alpha \wedge \neg\alpha$ and $\alpha \leftrightarrow \beta \equiv (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. If we restrict this language to one without the box operator, we get a propositional language. Depending on the axioms we introduce we obtain different logics.

A modal logic is called *normal* if it contains all classical tautologies and the axiom schema $K : \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$, and is closed under the inference rules of modus ponens, substitution and necessitation. Many normal modal logics have been investigated and applied to model different concepts in philosophy, proof theory, computer science, and so forth. Among the more well-known ones are $S5$, a classical epistemic logic, $KD45$, a classical doxastic logic, and

$S4$, a topological modal logic (for further details, see for example Blackburn, van Benthem, & Wolter, 2006).

The language of superintuitionistic logics is simpler. In particular we have just the propositional language without modalities. The rules of inference are modus ponens and substitution. As a basis we can take the usual axioms for intuitionistic logic, IC (see, for example, van Dalen, 2004):

$$\alpha \to (\beta \to \alpha),$$
$$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)),$$
$$(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma))),$$
$$(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma)),$$
$$(\alpha \to \neg\beta) \to (\beta \to \neg\alpha),$$
$$(\alpha \wedge \beta) \to \alpha,$$
$$(\alpha \wedge \beta) \to \beta,$$
$$\alpha \to (\alpha \vee \beta),$$
$$\beta \to (\alpha \vee \beta),$$
$$\neg(\alpha \to \alpha) \to \beta.$$

Introducing additional axioms gives rise to what are called *superintuitionistic* logics. Of special interest here is the logic HT of *here-and-there* which is obtained by adding to IC the axiom schema $(\neg\alpha \to \beta) \to (((\beta \to \alpha) \to \beta) \to \beta)$ to the intuitionistic logic IC. As another example, the logic KC is obtained by adding to IC the axiom of weak excluded middle $\neg\neg\varphi \vee \neg\varphi$. If we add the axiom of excluded middle, $\neg\varphi \vee \varphi$, we obtain the classical propositional calculus PC.

In the following subsections we will give a general definition of non-monotonic inference for the class of all normal modal logics and for the class of superintuitionistic logics.

2.1. Non-monotonic modal logics

Take an arbitrary monotonic normal modal logic L. The non-monotonic logic $\mathfrak{L}_L$ is based on the same language as L. Formulas are also built in the same way. For axioms and rules of inference the situation is different because the non-monotonic forcing relation is not the same as the standard one. The definition of non-monotonic forcing $\mid\!\sim$ is based on the notion of expansion of a set of formulas. Expansions for non-monotonic modal logics are intuitively analogous to maximal consistent sets for their monotonic modal counterparts. Below we give the fixpoint definition of the expansion of an L-consistent set of formulas I. First let us recall the definition of the L-consequence relation $\vdash_L$.

Definition 1. *Let I be a consistent set of formulas in L. We will say that the formula φ is an L-consequence of I ($I \vdash_L \varphi$) if there exists a finite set of formulas $\psi_1, \ldots, \psi_n$ such that for every k, $1 \leq k \leq n$ holds:*

- *ψ_k is a substitution instance of an axiom of L.*
- *$\varphi_k \in I$.*
- *φ_k is the result of applying modus ponens to formulas ψ_i and ψ_j, for some $i, j \leq k$.*
- *φ_k is the result of applying necessitation to a formula ψ_i, for some $i \leq k$.*

We define $Cn_L(I)$ to be the set of all L-consequences of the set of formulas I, i.e., $Cn_L(I) = \{\varphi : \vdash_L \varphi\}$.

Note. *Note that L-consequence does differ from standard consequence $\vdash$. In fact we get the definition of $\vdash$ if we drop the last case in Definition 1.*

Definition 2. *Let I be a consistent set of formulas in L. A set of formulas E is said to be an L-*expansion *of I if it satisfies the following equation:*

$$E = Cn_L[I \cup \{\neg\Box\varphi : \varphi \notin E\}].$$

If the context makes it clear we drop may drop reference to L and refer simply to expansion.

From the definition it is clear that for a given L a set I may have several L-expansions. Moreover it may happen that the set of all L-expansions of the set I is uncountable. When L is clear from context we use $Ex(I)$ to denote the set of all expansions of I. Below we state the main property of expansions, which mainly reflects the analogy between the maximally consistent sets containing I and expansions of I.

Fact 3. *Let L be a normal modal logic and I a consistent set of formulas in L. Then for any expansion E of I, the following holds: $E \vdash_L \varphi$ iff $\varphi \in E$.*

The usual (sceptical) non-monotonic inference relation, $\mid\!\sim$, makes use of all expansions of a given set of formulas I; we define it for an arbitrary normal modal non-monotonic logic $\mathfrak{L}_L$.

Definition 4. *For a normal modal non-monotonic logic $\mathfrak{L}_L$ and L-consistent set of formulas, I, we define, $I \mid\!\sim \varphi$ iff $\forall E \in Ex(I), E \vdash \varphi$.*

Observe that together with Fact 3. we have an equivalent definition of non-monotonic forcing: $I \mid\!\sim \varphi$ iff $\forall E \in Ex(I), \varphi \in E$.

Example 5. *Let $\mathfrak{L}_L$ be the logic $KD45$. Then $\mid\!\sim$ corresponds to consequence in the well-known system of autoepistemic logic (Schwarz, 1991).*

2.2. Non-monotonic superintuitionistic logics

Unlike in the case of modal logics, there is no standard approach to defining non-monotonic extensions of superintuitionistic logics. However in Pearce (1999) a fixpoint condition analogous to that of Definition 2. was used to define a kind of expansion for different (non-modal) logics; we call this a *completion.*[1] Let H be a superintuitionistic logic with consequence relation Cn_H. Let I be a set of formulas in H.

Definition 6. *A set X of H-formulas is said to be an H-*completion *of I if and only if $X = Cn_H[I \cup \{\neg\varphi : \varphi \notin X\}]$.*

As in the case of modal non-monotonic logics we can define a similar notion of non-monotonic inference.

Definition 7. *For a superintuitionistic logic H and H-consistent set of formulas I we define: $I \mid\!\sim_H \varphi$ iff $\varphi \in X$ for all H-completions X of I.*

Example 8. *Let HT be the superintuitionistic logic of* here-and-there *(see, for example, Pearce, 2006). Then HT-completions correspond to equilibrium models in the sense of Pearce (2006) and hence $\mid\!\sim_{HT}$ captures inference in equilibrium logic. This system provides a logical foundation for answer set programming due to the correspondence with stable models (Gelfond & Lifschitz, 1988; Pearce, 2006).*

Example 9. *Let C be classical propositional logic. Then it is easy to see that completions correspond to ordinary classical models and therefore $\mid\!\sim_C$ is the same as ordinary monotonic inference in classical logic. This is analogous to the case of the modal system $S5$ where no new logic is obtained in the 'non-monotonic' case.*[2]

3. Algebraic models

Before dealing with non-monotonic logics, let us first recall some features of the algebraic semantics for monotonic modal and superintuitionistic logics. To emphasise the importance of the algebraic approach, we will restate some well-known theorems and examples from the literature. In the second part of the section we will develop the algebraic semantics for non-monotonic superintuitionistic and modal logics.

3.1. Monotonic Logics

It is well known that algebraic models for monotonic normal modal logics are based on Boolean algebras with operators. A Boolean algebra with operator has the form (B, τ), where $B = (B, \wedge, -, 0, 1)$ is a Boolean algebra and $\tau : B \to B$ is a unary operator. The normality condition of modal logic is expressed by the following two equations: $\tau(1) = 1$ and $\tau(a \wedge b) = \tau(a) \wedge \tau(b)$. In general, from an arbitrary normal modal logic L we can construct the variety V_L of Boolean algebras with operator which will serve as algebraic models for the logic L. This variety is obtained by translating the axioms of a given modal logic. Below we present the translation $*$ from modal formulas to polynomials. $p^* \equiv p$, $(\alpha \wedge \beta)^* \equiv \alpha^* \wedge \beta^*$, $(\neg\alpha)^* \equiv -\alpha^*$, $(\Box\alpha)^* \equiv \tau(\alpha^*)$. Now for an arbitrary normal modal logic L we have the following fact:

Fact 10. (Burris & Sankappanavar, 2012)**.** $\vdash_L \varphi$ *iff* $\models \varphi^* = 1$.

Here $\vdash_L$ stands for provability in L and $\models$ means that the equation holds in every algebra in the corresponding variety V_L.

As a good example of a Boolean algebra with operator we can take the Lindenbaum-Tarski algebra (or free algebra over countable generators). We briefly recall it here. Given a normal modal logic L, we consider the class of all formulas factorised by logical equivalence. This yields the elements of the free algebra $\mathcal{F}_L$ of equivalence classes of formulas. Here the class $[\varphi]$ of a formula φ is defined in the following way: $[\varphi] = \{\psi : \vdash \psi \leftrightarrow \varphi\}$, the consequence relation $\vdash$ being that of the logic L. Now the Boolean operations on classes are induced in a standard way: $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$, $[\varphi] \vee [\psi] = [\varphi \vee \psi]$, $-[\varphi] = [\neg\varphi]$ and $\tau[\varphi] = [\Box\varphi]$. It is well known that the free algebra $\mathcal{F}_L$ of any normal modal logic L belongs to the variety V_L. Now instead of scanning the whole variety we can restrict to the Lindenbaum-Tarski algebra. So we get the following extension of Fact 10:

$$\vdash_L \phi \text{ iff } \models_{V_L} \phi^* = 1 \text{ iff } \models_{\mathcal{F}_L} \phi^* = 1.$$

Up to this point checking the validities $\vdash_L \phi$, $\models_{V_L} \phi^* = 1$ and $\models_{\mathcal{F}_L} \phi^* = 1$ are equally complex tasks, but in some cases we can find reductions. For example in the case of algebraic models of PC propositional calculus which is given by the variety BA of all Boolean algebras, the problem of checking $\models_{BA} \phi^* = 1$ can be reduced to the problem of checking the equation $\phi^* = 1$ in a single algebra, which is a two-element Boolean algebra. While this is probably the only example of such a trivial reduction, the next example shows we may still identify useful reductions to a smaller class of algebras.

Another important example of Boolean algebras with an operator is provided by the variety MA of all monadic algebras. We will later make use of this variety for defining algebraic expansions for modal formulas. The class of monadic algebras was extensively studied by Halmos and Givant (1998) as structures where universal and existential quantification could be algebraically defined by operators. We give the definition of monadic algebras below. Due to its close connection with quantifiers, the unary operator on monadic algebras is usually denoted by $\forall$, so we will stick to this notation and use $\forall$ instead of τ.

Definition 11. *An algebra* $(B, \wedge, -, \forall, 0, 1)$ *is a monadic algebra if* $(B, \wedge, -, 0, 1)$ *is a Boolean algebra and* $\forall$ *is a unary operator on* B *such that for every* $p, q \in B$ *the following holds:*

$A1 \quad \forall(1) = 1,$
$A2 \quad \forall(p \wedge q) = \forall(p) \wedge \forall(q),$
$A3 \quad \forall(p) \leq \forall\forall(p),$
$A4 \quad \forall(p) \leq p,$
$A5 \quad p \leq \forall\exists(p)$

where $\exists(p) = -\forall(-p)$ *and the ordering* $\leq$ *is defined in a standard way* $a \leq b$ *iff* $a \wedge b = a$.

For readability we will omit the full representation $(B, \wedge, -, \forall, 0, 1)$ and just write $(B, \forall)$ meaning that B is a Boolean algebra and $\forall$ is a unary operator satisfying above-mentioned equalities. It is known that MA is semisimple and locally finite. We will not enter into too many details from the theory of universal algebra, but restrict ourselves to the definitions of the main concepts we need here (for further information see, for example, Burris & Sankappanavar, 2012).

To provide an algebraic semantics for non-monotonic normal modal logics we do not need all monadic algebras but just simple ones. An algebra is called simple if it has only two (unit and trivial) congruence relations. There exists a very elegant characterisation of simple monadic algebras.

Fact 12. (Halmos & Givant, 1998). *A Boolean algebra with operator* $(B, \forall)$ *is a simple monadic algebra iff the operator* $\forall : B \to B$ *satisfies the following two conditions:*

- $\forall(1) = 1$.
- $\forall(a) = 0$ *for every* $a \neq 1$.

The variety MA has all the good properties one could ask for. If we return to Fact 10. and its extension and apply this to monadic algebras we obtain:

$$\vdash_{S5} \phi \ \text{ iff } \ \models_{MA} \ \phi^* = 1 \ \text{ iff } \ \models_{\mathcal{F}_{MA}} \ \phi^* = 1.$$

And here we have two kinds of reduction. First, $\mathcal{F}_{MA}$ on finite generators is finite and therefore for checking the validity of a concrete formula $\phi(p_1, \ldots, p_n)$ it is enough to check the equality $\phi(p_1, \ldots, p_n)^* = 1$ in the free monadic algebra over n generators, hence in a finite algebra. The other reduction which is possible is that the variety MA is generated by the class of all finite simple algebras. Hence $\models_{MA} \ \phi^* = 1$ iff $\models_{MA_{sf}} \ \phi^* = 1$ where MA_{sf} is the class of all finite and simple monadic algebras.

Below we recall some standard notions from the theory of algebras which with slight adjustments are common both for Boolean algebras with operators as well as for Heyting algebras.

Definition 13. *Given two Boolean algebras with operators,* (B, τ) *and* (B', τ'), *a function* $h : B \to B'$ *is called a homomorphism if it preserves all the operations, i.e., for any two elements* $a, b \in B$ *the following holds:* $h(a \vee b) = h(a) \vee h(b)$, $h(-a) = -h(a)$, $h(0) = 0'$, $h(1) = 1'$, $h(\tau(a)) = \tau'(h(a))$. *If* h *in addition is a surjection, we will say that it is an epimorphism.*

We can now introduce τ-filters which are in one-to-one correspondence with epimorphisms.

Definition 14. *Given a Boolean algebra with operator* (B, τ), *a subset* $F \subseteq B$ *is called a* τ*-filter if:*

$$a \in F \text{ and } a \leq b \text{ implies } b \in F,$$
$$a, b \in F \text{ implies } a \wedge b \in F,$$
$$a \in F \text{ implies } \tau(a) \in F.$$

For example τ-filters in Lindenbaum-Tarski algebras are characterised by L-consequence relations as in Definition 1. The following holds:

Fact 15. *A subset $F \subseteq \mathcal{F}_L$ is a τ-filter iff $F = |Cn_L[A]|$ for some set of formulas $A \subseteq Form$. Here $|Cn_L[A]| = \{[\varphi] : \varphi \in Cn_L[A]\}$. In other words it is just the set of elements corresponding to the consequences of A in the Lindenbaum-Tarski algebra.*

It is well known that τ-filters are in one-to-one correspondence with congruences and hence epimorphisms. Briefly, this correspondence is as follows. Given a Boolean algebra with operator (B, τ) and a τ-filter F, the congruence $\sim_F$ is defined by: $a \sim_F b$ iff $a \leftrightarrow b \in F$, where $a \leftrightarrow b$ stands for $(-a \vee b) \wedge (-b \vee a)$. Conversely given a congruence $\sim$ on B we construct the τ-filter $F_\sim$ in the following way:

$$F_\sim = \{a \in B : a \sim 1\}.$$

We refer to Buris and Sankappanavar (2012) for the proofs of these facts. As a corollary we have the following property:

Fact 16. *For every Boolean algebra with operator (B, τ), there is a one-to-one correspondence between the set of all epimorphisms defined on this algebra and the set of all τ-filters of this algebra.*

The analogous fact to Fact 15 we have for Heyting or intermediate algebras.

Fact 17. *A subset $F \subseteq \mathcal{F}_{HA}$ is a filter iff $F = |Cn_{Int}[A]|$ for some set of formulas $A \subseteq Form$. Here $|Cn_{Int}[A]| = \{[\varphi] : \varphi \in Cn_{Int}[A]\}$, where $\mathcal{F}_{HA}$ denotes the Lindenbaum-Tarski algebra for intuitionistic logic and Cn_{Int} stands for intuitionistic consequence operator.*

We recall one more fact from universal algebra. We make use of this fact in the proof of the main theorem of the next section.

Fact 18. (Burris & Sankappanavar, 2012, Lemma 6.14). *Suppose A is an algebra and θ_1, θ_2 are congruences on A with $\theta_1 \subseteq \theta_2$. Then $\theta_1/\theta_2 = \{([a]_{\theta_1}, [b]_{\theta_1}) \in (A/\theta_1)^2 : (a, b) \in \theta_2\}$ is a congruence on A/θ_1.*

3.2. Algebraic models for non-monotonic modal logics

In this section we define the algebraic analogues for expansions of formulas in both the normal modal and the superintuitionistic cases. We call them *algebraic expansions*.

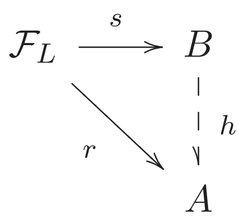
To express algebraic analogues of expansions in normal modal logics we define below special maps which we call *monadic-surjections*.

Definition 19. *We say that a surjective homomorphism $s : A \twoheadrightarrow B$ between two L-algebras is a monadic-surjection for the set of formulas I if B is a simple monadic algebra and $s([I]) = 1$.*

Note. *Monadic surjections may not exist in some varieties. For example the variety of Gödel-Löb algebras (Boolos, 1980) does not contain simple monadic algebras and hence does not allow monadic surjections either.*

The algebraic models for a non-monotonic modal logic $\mathfrak{L}_L$ are provided by pairs $(\mathcal{F}_L, s)$, where $\mathcal{F}_L$ is the Lindenbaum-Tarski algebra of the underlying monotonic modal logic L and s is a monadic-surjection defined on $\mathcal{F}_L$ satisfying some additional properties which we mention in the definition. For simplicity, instead of writing the equivalence class $[\varphi]$ we will simply write φ having in mind that it is the element of Lindenbaum-Tarski algebra. This does not cause confusion as far as the classes are determined by their members.

Definition 20. *We say that a pair* $(\mathcal{F}_L, s)$ *is an* algebraic expansion *for a consistent set of formulas* I *if* $\mathcal{F}_L$ *is the Lindenbaum-Tarski algebra of the logic* L *and* $s : \mathcal{F}_L \to s(\mathcal{F}_L)$ *is a monadic-surjection for* I *such that for any epimorphism* $r : \mathcal{F}_L \twoheadrightarrow A$ *with* $r(I) = 1$ *and* $r^{-1}(1) \supseteq \{\neg\Box\varphi : \varphi \notin s^{-1}(1)\}$*, there is a homomorphism* $h : B \to A$*, such that the following diagram commutes:*



Below we define the validity of a modal formula in an arbitrary algebraic expansion $(\mathcal{F}_L, s)$.

Definition 21. *We say that a formula* φ *is valid in a given algebraic expansion* $(\mathcal{F}_L, s)$ (*notation* $(\mathcal{F}_L, s) \models \varphi$) *if* $s([\varphi]) = 1$. *We say that a formula is valid in a class of algebraic expansions* M *if for every* $(\mathcal{F}_L, s) \in M$, $(\mathcal{F}_L, s) \models \varphi$. *We denote this by* $\models_M \varphi$*, while* $\models \varphi$ *will stand for validity in all algebraic expansions.*

Let M_I denote the class of all algebraic expansions for I. Now we are ready to prove the main theorem of this section which establishes the link between validity in algebraic expansions and non-monotonic validity.

Theorem 22. *For a given non-monotonic logic* $\mathfrak{L}_L$ *and* L*-consistent set of formulas* I*, the following holds:* $I \mid\sim \varphi$ *iff* $\models_{M_I} \varphi$.

Proof. We first show that there is a one-to-one correspondence between the set $Ex(I)$ of all expansions of I and the set M_I. Let $E \in Ex(I)$. By the definition of expansion, $E = Cn_L[I \cup \{\neg\Box\varphi : \varphi \notin E\}]$. By Fact 15, E is a τ-filter on $\mathcal{F}_L$. By Fact 16, the set $\{(b, b')|b \leftrightarrow b' \in E\}$ is a congruence on $\mathcal{F}_L$. Let us denote this congruence by $\sim$. We immediately obtain an epimorphism $s : \mathcal{F}_L \twoheadrightarrow \mathcal{F}_L/\sim$. To avoid confusion let $\Box$ be the unary operator in $\mathcal{F}_L$ and let $\forall$ be the operator of the factor algebra $\mathcal{F}_L/\sim$. First of all as $I \subseteq E$ we conclude that $s(I) = 1$.

Now let us show that $\mathcal{F}_L/\sim$ is a simple monadic algebra. The top element in $\mathcal{F}_L/\sim$ is the equivalence class (under congruence $\sim$) $[\varphi]_\sim$, where $\varphi \in E$. Hence $\forall(1) = [\Box\varphi]_\sim$. But since E is closed under L-consequence, $\varphi \in E$ implies that $\Box\varphi \in E$ and hence $[\varphi]_\sim = [\Box\varphi]_\sim$ so we infer that $\forall(1) = 1$ in the algebra $\mathcal{F}_L/\sim$. Now assume $a \neq 1$. This, by the construction of the congruence, means that for every $\varphi \in a$ (recall that a is an equivalence class of elements of $\mathcal{F}_L$ hence equivalence class of classes of formulas) it holds that $\varphi \leftrightarrow \top \notin E$ hence $\varphi \notin E$ then $\neg\Box\varphi \in E$. Hence $[\neg\Box\varphi]_\sim = 1$. This means that $-\forall[\varphi]_\sim = 1$. From this it follows that $\forall a = 0$. Hence $\mathcal{F}_L/\sim$ is a simple monadic algebra.

Now take an arbitrary epimorphism $r : \mathcal{F}_L \to A$ with $r(I) = 1$ and $r^{-1}(1) \supseteq \{\neg\Box\varphi : \varphi \notin s^{-1}(1)\}$. Since $r^{-1}(1)$ is τ-filter (Fact 16) we get that $r^{-1}(1) \supseteq Cn_L[I \cup \{|\neg\Box\varphi| : |\varphi| \notin s^{-1}(1)\}]$. Now by Fact 18. there is a naturally defined morphism $h : \mathcal{F}_L/\sim \to A$ so that $hs = r$. Hence we have shown that the pair $(\mathcal{F}_L, s)$ is an algebraic expansion. It is immediate from the construction that algebraic I-expansions corresponding to two distinct I-expansions are not the same. Hence the correspondence is injective.

For surjectivity take an arbitrary algebraic expansion $(\mathcal{F}_L, s)$ of the set of formulas I. Let us show that $s^{-1}(1)$ is an expansion of I. This means we need to show the following equality: $s^{-1}(1) = Cn_L\{I \cup \{\neg\Box\varphi : \varphi \notin s^{-1}(1)\}\}$. First let us show the right-to-left inclusion: $s^{-1}(1) \supseteq Cn_L\{I \cup \{\neg\Box\varphi : \varphi \notin s^{-1}(1)\}\}$. By assumption and the definition of algebraic expansion we immediately have that $s^{-1}(1) \supseteq I$. Let us show that $s^{-1}(1) \supseteq \{\neg\Box\varphi : \varphi \notin s^{-1}(1)\}$. For this assume $\varphi \notin s^{-1}(1)$ hence $s(\varphi) \neq 1$. By Fact 12 the following holds: $\forall s(\varphi) = 0$ and hence $-\forall s(\varphi) = 1$. Now as s is a homomorphism, $s(\neg\Box\varphi) = 1$ so $\neg\Box\varphi \in s^{-1}(1)$. Now by Fact 16, $s^{-1}(1)$ is closed under L-consequence. Hence we obtain the desired inclusion.

Now let us show the left-to-right inclusion $s^{-1}(1) \subseteq Cn_L\{I \cup \{\neg\Box\varphi : \varphi \notin s^{-1}(1)\}\}$. Assume $\varphi \in s^{-1}(1)$. As far as $Cn_L\{I \cup \{\neg\Box\varphi : \varphi \notin s^{-1}(1)\}\}$ is a filter. By Fact 16 there is an epimorphism $r : \mathcal{F}_L \twoheadrightarrow A$, where A is a factoralgebra $\mathcal{F}_L/Cn_L\{I \cup \{\neg\Box\varphi : \varphi \notin s^{-1}(1)\}\}$. Besides we have that $r^{-1}(1) = Cn_L\{I \cup \{\neg\Box\varphi : \varphi \notin s^{-1}(1)\}\}$. Now by Definition 20 there is a homomorphism $h : s(\mathcal{F}_L) \to A$ with $r = hs$. As homomorphisms send top elements to top elements, we have that $hs(\varphi) = 1$ and hence $r(\varphi) = 1$. Hence $\varphi \in r^{-1}(1) = Cn_L\{I \cup \{\neg\Box\varphi : \varphi \notin s^{-1}(1)\}\}$.

So we have seen that algebraic expansions are in one-to-one correspondence with normal expansions. Besides with this correspondence for every expansion E the following holds: $\varphi \in E$ iff $s(\varphi) = 1$ in a factor algebra $\mathcal{F}_L/\sim$. From this the theorem immediately follows. □

3.3. Algebraic expansions for superintuitionistic logics

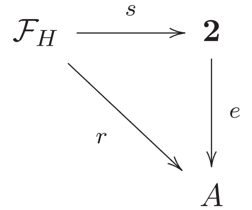
We now turn to the algebraic analogues of expansions for superintuitionistic logics. The idea is based on the well-known Tarski-style view of a logic as a free algebra over countable generators. We prove that algebraic expansions are exact analogues of expansions—in other words we obtain something like a completeness theorem which states that non-monotonic provability defined by algebraic expansions is exactly equivalent to normal non-monotonic provability.

Definition 23. *Given a superintuitionistic logic H, we say that $\mathcal{F}_H = (F, \wedge, \to, 1_F)$ is a free H-algebra over the set of generators S if $S \subseteq \mathcal{F}_H$ and for every function $r : S \to G$, where G is an underlying set of superintuitionistic H-algebra $\mathcal{G} = (G, \wedge, \to, 1_G)$, there is a homomorphism $h : \mathcal{F} \to \mathcal{G}$ such that $r = h_{|S}$.*

There is another representation of free algebras which is based on the factorising term algebra (the algebra of all intuitionistic formulas analogous to that described at the beginning of the section for modal algebras) by the equations brought from superintuitionistic axioms. This representation reveals better the idea of identifying the free algebra over the countable set of generators with the logic itself. Throughout, when we say formulas we mean the corresponding classes in the free algebra.

Definition 24. *We say that a pair $(\mathcal{F}_H, s)$ is algebraic I-expansion for the H-consistent set of formulas I if the following two conditions hold:*

(1) *$\mathcal{F}_H$ is the free H-algebra over a countable set of generators, and*
(2) *$s : \mathcal{F}_H \to \mathbf{2}$ is a homomorphism to the two-element Boolean algebra such that $s(I) = 1$ and for every epimorphism $r : \mathcal{F}_H \twoheadrightarrow A$ with $r^{-1}(1) \supset I \cup \{\neg\varphi : \varphi \notin s^{-1}(1)\}$ the following diagram commutes*



where e is naturally defined embedding of $\mathbf{2}$ into A.

Remark. *Observe that there is only one homomorphism from $\mathcal{F}_H$ to $\mathbf{2}$ which is not an epimorphism. This is the constant function which sends every element of $\mathcal{F}_H$ to the top element of $\mathbf{2}$. We will denote the top and bottom element of $\mathbf{2}$ by 1 and 0 respectively. As we will see later such algebraic expansions correspond to inconsistent expansions.*

Let M_I denote the class of all algebraic expansions for I.

Theorem 25. *Given a superintuitionistic logic H, there is a one-to-one correspondence between I-expansions and algebraic I-expansions, i.e.,* $Ex(I) \simeq M_I$.

Proof. Let $K \in Ex(I)$. By the definition of expansion $K = Cn_H[I \cup \{\neg\varphi : \varphi \notin K\}]$. By Fact 17. K is a filter on $\mathcal{F}_H$. By Fact 16 the set $\{(b, b')|b \leftrightarrow b' \in K\}$ is a congruence on $\mathcal{F}_H$. Let us denote this congruence by $\sim$. We immediately get an epimorphism $s : \mathcal{F}_H \twoheadrightarrow \mathcal{F}_H/\sim$. We have two cases. The first case is when $Cn_H[I \cup \{\neg\varphi : \varphi \notin K\}]$ is not consistent, i.e., K is equal to the whole algebra $\mathcal{F}_H$. In this case $\sim = \{(b, b')|b \leftrightarrow b' \in \mathcal{F}_H\} = \mathcal{F}_H \times \mathcal{F}_H$ and $\mathcal{F}_H/\sim$ is the one-element trivial algebra **1**, which exists in every variety of superintuitionistic algebras. Then we take the desired algebraic I-expansion to be $is : \mathcal{F}_H \to \mathbf{2}$, where i is the embedding of **1** into **2**. The other case is when K is a proper filter. In this case we want to show that $\mathcal{F}_H/\sim$ is isomorphic to **2**. Take any element $\varphi \in K$ then $\varphi \leftrightarrow \top = \varphi \in K$ hence $\varphi \sim \top$. Now for any $\varphi \notin K$ we have that $\varphi \leftrightarrow \bot = \neg\varphi \in K$ since K is expansion, so $\varphi \sim \bot$ for every $\varphi \notin K$. As a result we conclude that $\mathcal{F}_H/\sim \cong \mathbf{2}$ where $\cong$ stands for isomorphism of the corresponding algebras. It is immediate from the construction that algebraic I-expansions corresponding to two distinct I-expansions are not the same. Hence the correspondence is injective.

For surjectivity take an arbitrary algebraic I-expansion $(\mathcal{F}_H, s)$. Our claim is that $s^{-1}(1)$ is an I-expansion. Let us show that $s^{-1}(1) = Cn_H[I \cup \{\neg\varphi : \varphi \notin s^{-1}(1)\}]$. By the definition of s we have that $s^{-1}(1) \supseteq I$. Now if $s(\varphi) \neq 1$ then $s(\varphi) = 0$ and $-s(\varphi) = 1$ hence $s(\neg\varphi) = 1$. So we get that $s^{-1}(1) \supseteq \{\neg\varphi : \varphi \notin s^{-1}(1)\}$. Now as $s^{-1}(1)$ is a filter we see that $s^{-1}(1) \supseteq Cn_H[I \cup \{\neg\varphi : \varphi \notin s^{-1}(1)]$. □

Definition 26. *We say that a formula* φ *is valid in a given algebraic I-expansion* $(\mathcal{F}_H, s)$ *(notation* $(\mathcal{F}_H, s) \models \varphi$*) if* $s(|\varphi|) = 1$.

Theorem 27. *For a given superintuitionistic non-monotonic logic* $\mathfrak{L}_L$ *and H-consistent set of formulas I, the following holds:* $I \mid\!\sim \varphi$ *iff* $\models_{M_I} \varphi$.

Proof. This follows directly from the previous theorem. ■

4. Discussion and future work

At first sight algebraic semantics for non-monotonic logics may appear to be a less promising field of study than it is in the case of monotonic logics. The main complication that arises is the need to use free algebras in the definition. As we saw, in the monotonic case the logic is identified with the corresponding free algebra over a countable generator (Lindenbaum-Tarski algebra) and the algebraic study is directed to reducing the analysis of the free algebra to some smaller algebras, such as simple or finite algebras, etc. In the non-monotonic case this seems to be a complicated task due to the differences of the substitution rule. Although accomplishing this task may give us much simpler and more algebraic flexible semantics. Let us mention two possible directions for how this may be realised. These directions have not yet been explored, so we list them as topics for future work. One possibility is to examine Stone duality. Stone duality is very well studied for superintuitionistic algebras and Boolean algebras with operators (Davey & Priestley, 2001; Esakia, 1985). In both cases the dual objects are Stone spaces with a relation, which give rise to Kripke frames in simplified cases. Now if we look at Definition 19. and take the dual objects of monadic surjections, we get the following triple:

$$\mathcal{F}_L^* \xleftarrow{s^*} B^* \quad .$$

It is known that the dual of a simple monadic algebra is a cluster, i.e., a rooted $S5$ Kripke frame. Duals of free algebras are quite well studied in several cases, for example for the logics $S4$, $S5$, HA, etc. (Bezhanishvili, 2006; D'Antona & Marra, 2006; Di Nola, Grigolia, & Panti, 1998). Now, since s was a surjection in the definition, the dual $s*$ is an injection—besides, it is a p-morphism and therefore the cluster B^* is mapped to the maximum of $\mathcal{F}_L^*$. Already this picture brings us very close to the definition of minimal models that can be used to capture non-monotonic expansions (Marek & Truszczynski, 1993). This suggests that it may be worthwhile to examine the link between minimal models and dual algebraic expansions.

Another interesting direction arises when the variety V_L is locally finite. In this case the finitely generated free algebra is finite and hence every surjective image is also finite. So if we manage to reduce the study of non-monotonic inference for concrete formulas to the free algebra containing only as many variables as are contained in the formula, then for locally finite varieties we obtain the result that in Definition 20 everything is finite and hence we should have decidability.

These two directions for future work may help to establish the usefulness and practical applicability of algebraic semantics for non-monotonic logics.

5. Conclusion

In this paper we have established a link between algebraic semantics and fixpoint semantics for some well-known non-monotonic logics, making it possible to use algebraic methods in the study of non-monotonic systems. Since algebraic expansions have a rather complex structure, it may not always be easy to apply them to solve concrete problems. In addition, the fact that free algebras appear in the definition means that some standard algebraic techniques may not be applicable. For this reason we adopt caution and suggest that while algebraic semantics will not replace other approaches, it may nevertheless complement existing methods and may perhaps provide some new insights. As topics worthy of study in the future, we mentioned the well-known duality theory for Boolean algebras with operators: dual structures of algebraic expansions may be closely related to the important concept of minimal models. Furthermore, the Stone duals of free algebras are well-understood for some concrete modal and superintuitionistic logics, and efficient algorithms for building dual structures are known. This suggests that an algebraic approach together with the well-established apparatus of duality theory may provide some new techniques and results for tackling some logical and mathematical questions in non-monotonic reasoning.

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Notes

1. This was called *negation stable extension* in Pearce (1999).
2. As discussed in Pearce (2006), it seems to follow from results of Mauricio Osorio that varying the logic H for other superintuitionistic logics weaker than HT does not lead to non-monotonic systems different from equilibrium logic. An analogous situation arises in modal logic, where many different monotonic systems share the same non-monotonic extensions (Marek & Truszczynski, 1993).

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