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MATHEMATICS

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On Homology Groups Based on Kurosh Type Coverings in Categories of Complete Distributive Lattices

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ABSTRACT. **In the category of complete distributive lattices the isomorphism of functional Kolmogorov type groups of homology with projective type groups of homology is proved.**

Key words: lattice, homology, complex.

Let $S(L)=\{e=(L',f')|f':L\rightarrow L'\}$ (an epimorphic $\vee$-homomorphism) be a complete lattice of all subspaces of a complete distributive lattice L ($S(L)$ is the same as the complete lattice $Q(L)$ of all $\vee$-congruences on L). Subspaces of the type $u=(C(x),i)$, where $C(x)=\{z|z\leq x,\ x,z\in L\}$ and $i(z)=x\wedge z$ are open; subspaces of the type $F=(L/C(x),h)$ are closed, where $L/C(x)$ is the factor-lattice of L with respect to $\vee$-congruences: $y\equiv z\Leftrightarrow y\vee x=z\vee x$; $y,z\in L$; h is the natural $\vee$-homomorphism $h:L\rightarrow L/C(x)$ [1,2]). We shall consider only those subspaces which have complements in $S(L)$. The closure $\overline{e}$ and the interior Int$e$ of the subspace e are defined by the equalities $\overline{e}=\wedge\{F\mid F\in S(L),\ e\leq F\}$ and $\mathrm{Int}e=1-\overline{1-e}$, respectively (by 1-$e$ is denoted the complement of the subspace e). The subspaces e and e' are said to be nonintersecting, if $e\wedge e'=0$.

Definition 1. An open subspace $u\in S(L)$ is said to be canonical, if $u=\mathrm{Int}\overline{u}$. A closed subspace $F\in S(L)$ is said to be canonical, if $F=\overline{\mathrm{Int}F}$.

Proposition 1. *The complement in $S(L)$ of the canonical open (closed) subspace is canonically closed (open) subspace.*

Proposition 2. *If F and F' are canonically closed subspaces, then $F\vee F'$ is canonically closed, as well. Consequently, if u and u' are canonically open subspaces, then the same is $u\wedge u'$.*

Proposition 3. *If $F(u)$ is a closed (open) subspace then* $\mathrm{Int}F(\overline{u})$ *is a canonical open (closed) subspace.*

Proposition 4. *If u and u' are canonically open subspaces, then $u\leq u'\Leftrightarrow\overline{u}\leq\overline{u}'$.*

Proposition 5. *If $\alpha=\{F_1,\dots,F_n\}$ is a finite closed covering of L (i.e. $\vee F_i=1$, $F_i\in S(L)$ are closed subspaces) and $u_i=\mathrm{Int}F_i$, $\beta\equiv\{u_i\}$, then $\overline{\beta}=\{\overline{u}_i\}$ is refinement of α ($\overline{\beta}\geq\alpha$), and $\overline{\beta}$ is the covering of L.*

Definition 2. A family of nonintersecting canonical open subspaces $\alpha=\{u_i\}$ for which $\overline{\alpha}=\{\overline{u}_i\}$ is the covering of L, we call Kurosh type covering of the lattice L.

Similar definitions and constructions in the case of topological spaces can be found in [3].

Proposition 6. *If $\alpha=\{u_i\}$ and $\beta=\{u'_j\}$ are Kurosh type coverings of the lattice L, then the same is $\alpha\wedge\beta=\{u_i\wedge u'_j\}$.*

Let $\{\alpha\}$ be a family of all Kurosh type coverings of L and K_α be nerve of the covering

. We say that β is refinement of α, $\alpha\leq\beta$, if $\alpha=\{u_{\alpha_i}\}$, $\beta=\{u_{\beta_j}\}$ if every element u_{β_j} of β is contained in some element u_{α_i} of α. Correspondence $u_{\beta_j}\to u_{\alpha_i}$ defines uniquely projection (simplicial map) $\rho_\alpha^\beta: K_\beta\to K_\alpha$. By Proposition 6, the system $\{\alpha\}$ turns into a directed set. Thus $A=\{K_\alpha,\rho_\alpha^\beta\}$ is the simplicial spectrum with uniquely defined projections; therefore it may be helpful for determination of spectral and projective homological theories.

Let G be topological Abelian group .The spectral homology groups of L which are based on spectrum $A=\{K_\alpha,\rho_\alpha^\beta\}$ and on a simplicial spectrum A' of all finite closed coverings of L over the group of coefficients G, we denote by $H_P^Z(L;G)$ and $H_P^F(L;G)$, respectively.

Theorem 1. *Spectral homology groups* $H_P^Z(L;G)$ *and* $H_P^F(L;G)$ *are isomorphic.*

Proof. Since A is a subspectrum of A', the theorem follows from the fact that the closed Kurosh type covering can be inscribed into every finite closed covering.

Since $A=\{K_\alpha,\rho_\alpha^\beta\}$ is the simplicial single-valued spectrum, the projections ρ_α^β possess the property of transitivity $\rho_\alpha^\beta\circ\rho_\beta^\gamma=\rho_\alpha^\gamma$, when $\alpha\leq\beta\leq\gamma$ not only with respect to the homology classes, but also with respect to the chain. This fact makes it possible to introduce the following projective groups of chains, cycles, bounding cycles:

$$C_P(L;G)=\varprojlim\{C_P(K_\alpha;G),\rho_\alpha^\beta\},$$

$$Z_P(L;G)=\varprojlim\{Z_P(K_\alpha;G),\rho_\alpha^\beta\},$$

$$B_P(L;G)=\varprojlim\{B_P(K_\alpha;G),\rho_\alpha^\beta\}$$

respectively.

If group of coefficients G is compact then $B_P(L;G)=\overline{B_P(L;G)}$.

Definition 3. Denote $H_P(L;G)=Z_P(L;G)/\overline{B_P(L;G)}$ and call as projective homology group of L.

Let $C_P=\{C_{P\beta}\}\in C_P(L;G)$, i.e. $C_{P\beta}=g_\beta^j t_\beta^P$, where g_β^j, $g_\beta^j\in G'$ is the chain coefficient; $C_{P\alpha}=\rho_\alpha^\beta C_{P\beta}$ for $\alpha\leq\beta$ and $t_{\beta j}^P\in K_\beta$. If $\alpha\leq\beta$ and the simplex $t_{\beta j}^P$ is encountered in K_α, then we can easily see that the chain coefficients $C_{P\beta j}$ and $C_{P\alpha i}$ coincide on $t_{\beta j}^P$. In other words, the coefficient depends only on the simplex $t_{\beta j}^P$, and therefore we can write it as the function $g_\beta^j=\varphi_P(t_{\beta j}^P)$ of one oriented simplex $t_{\beta j}^P$ or, likewise,as the function $\varphi_P(\overline{\vartheta}_{\beta j0},...,\overline{\vartheta}_{\beta jP})$ of the ordered set $\overline{\vartheta}_{\beta j0},...,\overline{\vartheta}_{\beta jP}$ of vertices of the simplex $t_{\beta j}^j$; to get rid of that ordering, we define $\varphi_P(\overline{\vartheta}_{\beta j0},...,\overline{\vartheta}_{\beta jP})$ in terms of a skew-symmetric function of all its arguments.

Let K_0 be a nerve of all canonical closed subspaces of L. Then the subcomplex K of the complex K_0 consisting of all simplices, whose vertices have nonintersecting interiors, form a closed subcomplex. If $t^P=(\overline{\vartheta}_0,...,\overline{\vartheta}_P)$ is the simplex from K and $\vartheta_{P+1}=\mathrm{Int}(1-\vee\vartheta_i)$, then $(\overline{\vartheta}_0,...,\overline{\vartheta}_P,\overline{\vartheta}_{P+1})$ is a Kurosh type covering of L, and its nerve has

t^P as its own simplex. Denote this special covering and its nerve by $\alpha(t^P)$ and $K(t^P)$, respectively. To the projective chain c_p, $c_p \in C_p(L;G)$ there corresponds the chain of the complex K given by the relation $'c_P = \sum \varphi_P(t^P)t^P$ or by summation with respect to all sets of subspaces $(\overline{\vartheta}_0,\ldots,\overline{\vartheta}_p)$ with nonintersecting interiors: $'c_P = \frac{1}{p+1}\sum \varphi_P(\overline{\vartheta}_0,\ldots,\overline{\vartheta}_p)\overline{\vartheta}_0,\ldots,\overline{\vartheta}_P$.

The relation $c_p \to 'c_p$ defines the isomorphism $\tau: C_p(L;G) \to 'C_p(K;G) \subset C_p(K;G)$ of groups. If u is an open set in G and $u(t^P)=\{c_p / \varphi_p(t^P) \in u,\ c_p \in C_p(L;G)\}$, then $\{u(t^P)\}$ and $\{\tau(u(t^P))\}$ are subbasis for both groups of chains. Consequently τ is the isomorphism.

If $c_p=\{c_{p\alpha}\}$ is the projective chain, then such is $\partial c_p=\{\partial c_{p\alpha}\}$; for $c_{p\alpha}$ we have $\partial c_{p\alpha} = h_\alpha^j t_{\alpha j}^{P-1}$ $h_\alpha^j = g_\alpha^j[t_{\alpha i}^P, t_{\alpha j}^{P-1}]$. Since the coefficient for $t_{\alpha j}^{p-1}$ in $\partial c_{p\alpha}$ does not depend on α, we can replace K_α by any other nerve containing $t_{\alpha j}^p$. Let the notation be chosen in such a way that $t^{p-1} = (\overline{\vartheta}_0,\ldots,\overline{\vartheta}_{p-1})$, $\alpha = \{\vartheta_0,\ldots,\vartheta_{p-1},\vartheta_{\alpha p},\ldots,\vartheta_{\alpha r}\}$. Then $\alpha(t^P) = \{\overline{\vartheta}_0,\ldots,\overline{\vartheta}_{p-1},\overline{\vartheta}\}$, where $\vartheta = \mathrm{Int}(1 - V\vartheta_i)$, and therefore $\alpha(t^p) \le \alpha$. The projection $\rho: K_\alpha \to K(t^{p-1})$ is defined by the equalities $\rho(\overline{\vartheta}_i) = \overline{\vartheta}_i$, $\rho(\overline{\vartheta}_{\alpha i}) = \overline{\vartheta}$). This and the fact that $\rho\partial C_{p\alpha} = \partial\rho C_{p\alpha}$ imply that the coefficient for t^{p-1} in $\partial C_{p\alpha}$ is $\varphi_{p-1}(\overline{\vartheta}_0,\ldots,\overline{\vartheta}_{p-1}) = \varphi_p(\overline{\vartheta},\overline{\vartheta}_0,\ldots,\overline{\vartheta}_{p-1})$. Therefore ∂C_p can be defined as a chain $\partial C_p = \frac{1}{p}\sum \varphi_p(\overline{\vartheta},\overline{\vartheta}_0,\ldots,\overline{\vartheta}_{p-1})$, $\overline{\vartheta}_0,\ldots,\overline{\vartheta}_{p-1}$ or the boundary operator ∂ over φ_p can be defined by the equality $\partial\varphi_p = \varphi_p(\overline{\mathrm{Int}(1 - V\vartheta_i)}, \overline{\vartheta}_0,\ldots,\overline{\vartheta}_{p-1})$.

Let $\{F\}$ be a set of all closed subspaces of L.

Definition 3. For any integer $p \ge 0$ we call as p-dimensional Kolmogorov type chain of the lattice L over the topological group of coefficients G the function $\varphi_p(F_0,\ldots,F_p)$ with values in G which satisfies the following conditions:

1. φ_p remains unchanged under even permutation of arguments and changes its sign for their odd permutation; $\varphi_p=0$ if two arguments coincide (skew-symmetry).

2. If F_i' and F_i'' have nonintersecting interiors, $\mathrm{Int}F_i' \wedge \mathrm{Int}F_i'' = 0$, then $\varphi_p(F_0,\ldots,F_i' \vee F_i'',\ldots,F_p) = \varphi_p(F_0,\ldots,F_i',\ldots,F_p) - \varphi_p(F_0,\ldots,F_i'',\ldots,F_p)$ (additivity).

3. If $\wedge F_i=0$, then $\varphi_p(F_0,\ldots,F_p)=0$.

We introduce in the group $C_p^K(L;G)$ of all Kolmogorov type chains of L over the group of coefficients G the boundary operator ∂ as follows: $\partial\varphi_p = \varphi_{p-1}(F_0,\ldots,F_{p-1}) = \varphi_p(1,F_0,\ldots,F_{p-1})$, where 1 is the largest element of the lattice L. Obviously, $\partial\partial=0$. Consequently, we have obtained ordinary groups both of cycles $Z_p^K(L;G)$ and of bounding cycles $B_p^K(L;G)$.

We see that in $\varphi_p(F_0,\ldots,F_p)$ all F_i can be replaced by canonical closed subspaces $\overline{IntF_i}$ $\mathrm{Int}F_i$, leaving the values of the function φ_p unchanged. The values of the function φ_p on the all $(F_0,\ldots,F_p)$ are known whenever they are known on $\overline{\vartheta}_0,\ldots,\overline{\vartheta}_p$, where ϑ_i are pairwise nonintersecting canonical open subspaces in L. Let K_0 be a nerve of all canonical closed subspaces and K be a subcomplex of the complex K_0 consisting of all simplices whose vertices have nonintersecting interiors. We can consider φ_p as a function on the set of p-simplexes of the complex K_0. The value of the function φ_p on p-simplices of the complex

It defines the same function $'\varphi_p(\overline{\vartheta}_0,...,\overline{\vartheta}_p)$ as above and, consequently, the projection of the chain c_p, $c_p \in C_p(L;G)$. The correspondence $\varphi_p \to C_p$ defines the isomorphism in the algebraical sense $\tau: C_p^K(L;G) \to C_p(L;G)$. If $\{u\}$ are open sets of the group $C_p(L;G)$, we take as open sets the groups $C_p(L;G)$ of the set $\{\tau^{-1}(u)\}$, turing τ into the isomorphism; the isomorphism τ in its turn provides isomorphisms $Z_p(L;G) = Z_p^K(L;G)$ and $B_p(L;G) = B_p^K(L;G)$.

Definition 4. Denote the factor group $Z_p^K(L;G)/\overline{B_p^K(L;G)}$ by $H_p^K(L;G)$ and call as p-dimensional group the Kolmogorov type homologies over the topological group of coefficients G.

Thus the following theorem is valid.

Theorem 2. *Projective homology groups and functional Kolmogorov type homology groups of the lattice L over the topological group of coefficients G are isomorphic, i.e.*

$$H_p(L;G) = H_p^K(L;G).$$

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მათემატიკა

ზ. თოდუა

კუროშის ტიპის დაფარვებზე დაყრდნობით განმარტებული ჰომოლოგიის ჯგუფების შესახებ სრული დისტრიბუციული მესერების კატეგორიაში

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