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MATHEMATICS

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On some Properties of Presheaves of Modules on a Complete Distributive Lattice

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ABSTRACT. **Theorem of uniqueness for cohomology of presheaves in a class of paracompact, complete distributive lattices is proved.** © *2006 Bull. Georg. Natl. Acad. Sci.*

Key words: lattice, presheaves, cohomology.

Let F be a presheaf of modules on a distributive lattice L; $\alpha=\{x\}_{x\in L}$ be a family of elements from L. A compatible α-family of the presheaf F is defined as a family of elements $\{v_x\in F(x)\}_{x\in\alpha}$ such that $v_x|_{x\wedge x'}=v_{x'}|_{x\wedge x'}$. By $F(\alpha)$ we denote a module of compatible α-families of the presheaf F. If the family $\beta=\{y\}_{y\in L}$ refines the family α, then there exists the homomorphism $F(\alpha)\to F(\beta)$. If α runs through the family of coverings of the fixed element z, $z\in L$, then $\{F(\alpha)\}$ is a direct spectrum, and we assume that $\hat{F}(z)=\varinjlim\{F(\alpha)\}$. If $z'\le z$, and α is some covering of the element z, then $\alpha'=\{x\wedge z'\,|\,x\in\alpha\}$ is the covering of the element z' which refines the covering α. Hence there is the homomorphism $F(\alpha)\to F(\alpha')$ which defines the homomorphism $\hat{F}(z)\to\hat{F}(z')$. $\hat{F}$ is the presheaf on L. There exists a natural transformation $k:F\to\hat{F}$ which assigns the element of the module $\hat{F}(z)$ represented by the compatible β-family $\{v\}$ to the element $v\in F(z)$, where the family β consists of the unique element z; the presheaf $\hat{F}$ is called a completion of the presheaf F.

Proposition 1. *The presheaf F on L is a sheaf if and only if $k:F\simeq\hat{F}$.*

Let F be a presheaf of modules on L, and let $\alpha=\{x_\alpha\}$ be a covering of L. For every integer $p\ge 0$ we define $C^p(\alpha;F)$ as a module of functions ψ which assign to the ordered set of $p+1$ elements $x_{\alpha_0},x_{\alpha_1},\dots,x_{\alpha_p}$ of the covering α the element $\psi(x_{\alpha_0},\dots,x_{\alpha_p})\in F(x_{\alpha_0}\wedge\dots\wedge x_{\alpha_p})$. The coboundary operator $\delta:C^p(\alpha;F)\to C^{p+1}(\alpha;F)$ is defined by

$$(\delta\psi)(x_{\alpha_0},\dots,x_{\alpha_{p+1}})=\sum_{0\le i\le p+1}(-1)^i\psi(x_{\alpha_0},\dots,\hat{x}_{\alpha_i},\dots,x_{\alpha_p})\,|\,x_{\alpha_0}\wedge\dots\wedge x_{\alpha_{p+1}}.$$

Then $\delta^2=0$, and the module $C^\cdot(\alpha;F)=\{C^p(\alpha;F),\delta\}$ is a cochain complex; its module of cohomologies is denoted by $H^{\cdot}(\alpha;F)$. It follows from the definition that $H^0(\alpha;F)=F(\alpha)$.

Let β be the covering of L which refines α, and let the function $\lambda:\beta\to\alpha$ be such that $x_\beta\leq\lambda(x_\beta)$ for all x_β. We define the cochain mapping $\lambda^{\cdot}:C^{\cdot}(\alpha;F)\to C^{\cdot}(\beta;F)$ by assuming that $(\lambda^{\cdot}\psi)(x_{\beta_0},\ldots,x_{\beta_p})=\psi(\lambda(x_{\beta_0}),\ldots,\lambda(x_{\beta_p}))|_{x_{\beta_0}\wedge\cdots\wedge x_{\beta_p}}$. This implies that the homomorphism $\lambda^{\cdot}:H^{\cdot}(\alpha;F)\to H^{\cdot}(\beta;F)$ is defined correctly, and the element $\lambda^{\cdot}\{\psi\}=\{\lambda^{\cdot}\psi\}$ does not depend on the particular choice of functions λ. A collection of modules and homomorphism $\{H^{\cdot}(\alpha;F),\lambda^{\cdot}\}$, where α runs through the coverings of L, is by itself a direct spectrum, and the Cech cohomologies of the complete distributive lattice L with coefficients in the presheaf F is defined by the equality $\check{H}^{\cdot}(L;F)=\varinjlim\{H^{\cdot}(\alpha;F),\lambda^{\cdot}\}$. For every presheaf F we have $\check{H}^0(L;F)=\hat{F}(L)$. If F is a constant presheaf on L, then these groups of cohomologies coincide with those defined in [2] and [3].

Theorem 1. *If* $0\to F'\to F\to F''\to 0$ *is a short exact sequence of presheaves on L, then the corresponding cohomological sequence*

$$\cdots\to\check{H}^p(L;F')\to\check{H}^p(L;F)\to\check{H}^p(L;F'')\to\check{H}^{p+1}(L;F')\to\cdots$$

is exact.

If $(L/\theta,k_\theta)\in S(L)$ is a subspace in L (see [4], [5]) and F is a presheaf on L, then we can define two presheaves $F_{L/\theta}$ and $F^{L/\theta}$ on L such that the short sequence of presheaves $0\to F^{L/\theta}\to F\to F_{L/\theta}\to 0$ is exact. The corresponding cohomological exact sequence is an exact sequence of the Cech cohomology of the pair $(L,L/\theta)$ with the coefficient in the presheaf F, and we assume that $\check{H}^p(L/\theta;F)=\check{H}^p(L;F_{L/\theta})$ and $\check{H}^p(L,L/\theta;F)=\check{H}^p(L;F^{L/\theta})$.

Definition 1. The presheaf F of modules on L is said to be locally zero, if for every element v of the module $F(x)$, $x\in L$, there exists a covering $\alpha=\{x_\alpha\}$ of the element x $(x=\vee x_\alpha)$, such that $v|_{x_\alpha}=0$ for all $x_\alpha\in\alpha$.

This is so if and only if the completion $\hat{F}$ of F is the zero presheaf.

Definition 2. Let α be a system of subspaces in L ([4], [5]), covering $S(L)$; the covering α is said to be locally finite if there exists open covering α' of L (i.e. the elements of the covering α' are open subspaces of L) such that every element of the covering α' has a non-empty intersection only with a finite number of elements of the covering α.

Definition 3. The lattice L is said to be paracompact if $S(L)$ is Hausdorf (see [6]) and every its open covering is refined by a locally finite open covering.

Theorem 2. *If L is a paracompact complete distributive lattice, and F is a locally zero presheaf on L, then* $\check{H}^{\bullet}(L;F)=0$.

Definition 4. The homomorphism $f:F\to F'$ of presheaves on L is said to be a locally isomorphism, if the presheaves $\ker f$ and $\operatorname{coker} f$ are both locally zero.

Corollary 1. *If L is paracompact and* $f:F\to F'$ *is the local isomorphism of presheaves on L, then* $f_*:\check{H}^{\bullet}(L;F)\approx\check{H}^{\bullet}(L;F')$ *is an isomorphism.*

Corollary 2. *If L is paracompact, then the natural homomorphism* $k:F\to\hat{F}$ *induces the isomorphisms* $k_*:\check{H}^{\bullet}(L;F)\approx\check{H}^{\bullet}(L;\hat{F})$.

Definition 5. The presheaf F on L is said to be fine, if for every locally finite open covering $\alpha=\{x\}=\{(L/\theta_x,k_{\theta_x})\}$ of the lattice L there exists the family $\{e_x\}_{x\in\alpha}$ of endomorphisms of the presheaf F such that: a) if $v\in F(y)$, then $e_x(v)|_{(L/\theta_y,k_{\theta_y})-\overline{(L/\theta_x,k_{\theta_x})}}=0$ and b) if the open subspace $(L/\theta_y,k_{\theta_y})$ in L intersects only a finite number of elements of the family $\{\overline{(L/\theta_x,k_{\theta_x})}\}_{x\in\alpha}$, then for the element $v\in F(y)$ the equality $v=\sum_{x\in\alpha}e_x(v)$ holds.

The lattice v-homomorphism $f:M\to L$ and the presheaf F on L define the presheaf f_*F on M, namely, $(f_*M)(x)=F(f(x))$, $x\in M$.

Proposition 2. *Let F be a fine presheaf of modules on L. Then:*

(a) *for every presheaf of modules* F' *on L the presheaf* $F\otimes F'$ *is fine;*

(b) *if* $f:M\to L$ *is the homomorphism of the lattice, then* f_*F *is the fine presheaf on M;*

(c) $\hat{F}$ *is the fine presheaf on L.*

Definition 6. A shrinking of the open covering $\alpha=\{x\}=\{(L/\theta_x,k_{\theta_x})\}$ is an open covering $\beta=\{y\}=\{(L/\theta_y,k_{\theta_y})\}$ of the lattice L whose elements are in one-to-one correspondence with the elements of the covering α such that if to the element $(L/\theta_x,k_{\theta_x})$ is assigned the element $(L/\theta_y,k_{\theta_y})$, then $\overline{(L/\theta_y,k_{\theta_y})}\le(L/\theta_x,k_{\theta_x})$ (see [4], [5]).

Every shrinking of the locally finite open covering is locally finite.

Theorem 3. *If F is a fine presheaf on the paracompact lattice L, then* $\check{H}^{p}(L;F)=0$ *for all* $p>0$.

Let F^* be a cochain complex of presheaves of modules on L; Z^p and B^{p+1} denote, respectively, the kernel and the image of the homomorphism $\delta:F^p\to F^{p+1}$, and let H^p be the presheaf Z^p/B^p.

Proposition 3. *Let F^* be the cochain complex of presheaves of modules on L. For every integer p there exists a functorial with respect to F^* exact sequence*

$$0 \to \ker\left(\check{H}^0\left(L;B^p\right)\right) \to \check{H}^1\left(L;Z^{p-1}\right) \to \check{H}^0\left(L;Z^p\right) \to H^p\left(\hat{F}^*(L)\right) \to 0.$$

Proposition 4. *If F^* is the cochain complex of presheaves of modules on the paracompact L, then for all integers p there exists the functorial with respect to F^* short exact sequence*

$$0 \to \operatorname{Im}\left(\check{H}\left(L;B^p\right)\right) \to \check{H}^1\left(L;Z^{p-1}\right) \to H^p\left(\hat{F}^*(L)\right) \to \ker\left(\check{H}^0\left(L;H^p\right)\right) \to \check{H}^1\left(L;B^p\right) \to 0.$$

Theorem 4. *Let F^* be a nonnegative cochain complex of fine presheaves of modules on the paracompact L. Assume that for some integers $0 \le m < n$ $H^p(F^*)$ is locally zero for $p < m$ and $m < p < n$. Then there are functoral isomorphisms $\check{H}^{p-m}\left(L;H^m(F^*)\right) = H^p\left(\hat{F}^*(L)\right)$, $p < n$, and the functorial monomorphism $\check{H}^{n-m}\left(L;H^m(F^*)\right) \to H^n\left(\hat{F}^*(L)\right)$.*

Theorem 5. *Let L be a complete distributive paracompact lattice, and let $\tau : F^* \to F'^*$ be a cochain mapping of one nonnegative cochain complex of fine presheaves of modules on L into the other one. Let for any number p<n the mapping $\tau_* : H^p(F^*) \to H^p(F'^*)$ be a local isomorphism for p<n and a local monomorphism for p=n. Then the induced mapping $\hat{\tau}_* : H^p(\hat{F}^*(L)) \to H^p(\hat{F}'^*(L))$ is the isomorphism for p<n and monomorphism for p=n.*

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მათემატიკა

ზ. თოდუა

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რეზიუმე. აღნიშნულ კატეგორიაში მოცემულია ჩეხის ტიპის კოჰომოლოგიის ჯგუფის განმარტება კოეფიციენტებით წინარეკონებში. კოეფიციენტთა წინარეკონების მოკლე ზუსტ მიმდევრობას შეესაბამება კოჰომოლოგიური გრძელი ზუსტი მიმდევრობა. თუ მესერი პარაკომპაქტურია, წინარეკონა კი ლოკალურად ნულოვანი ან თხელი, მაშინ კოჰომოლოგიის ჯგუფები შესაბამისად ნულოვანია ან ნულოვანია დადებით განზომილებაში. პარაკომპაქტურ მესერზე თხელი წინარეკონების კოჯაჭვური კომპლექსისათვის დამტკიცებულია ერთადერთობის თეორემა.