The bitwisted Cartesian model for the free loop fibration

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A R T I C L E   I N F O

A B S T R A C T

Using the notion of truncating twisting function from a simplicial set to a cubical set a special, bitwisted, Cartesian product of these sets is defined. For the universal truncating twisting function, the (co)chain complex of the corresponding bitwisted Cartesian product agrees with the standard Cartier (Hochschild) chain complex of the simplicial (co)chains. The modelling polytopes F_n are constructed. An explicit diagonal on F_n is defined and a multiplicative model for the free loop fibration $\Omega Y \to \Lambda Y \to Y$ is obtained. As an application we establish an algebra isomorphism $H^*(\Lambda Y; \mathbb{Z}) \approx S(U) \otimes \Lambda(s^{-1}U)$ for the polynomial cohomology algebra $H^*(Y; \mathbb{Z}) = S(U)$.

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1. Introduction

Let ΛY denote the free loop space of a topological space Y, i.e. the space of all continuous maps from the circle S^1 into Y, and let $\Omega Y \to \Lambda Y \xrightarrow{\xi} Y$ be the free loop fibration. Since ξ can be viewed as obtained from the path fibration $\Omega Y \to PY \to Y$ by means of the conjugation action $ad : \Omega Y \times \Omega Y \to \Omega Y$ [12], one could apply [9] to construct for ξ the twisted tensor product $C_*(Y) \otimes_{\tau_*} \Omega C^*(Y)$. However, the induced action $ad_* : \Omega C_*(Y) \otimes \Omega C_*(Y) \to \Omega C_*(Y)$ is hardly to write down by explicit formulas; an alternative way for modelling ΛY is the Cartier chain complex $\Lambda C_*(Y)$ of the singular simplicial chain coalgebra $C_*(Y)$ thought of as a specific, *bitwisted*, tensor product $\Lambda C_*(Y) = C_*(Y)_{\tau_*} \otimes_{\tau_*} \Omega C_*(Y)$ [5,12]. Accordingly, for a twisting truncating function $\tau : X \to Q$ from a 1-reduced simplicial set to a monoidal cubical set Q, we modify the twisted Cartesian product $X \times_\tau Q$ from [9] to obtain a *bitwisted Cartesian product* $X_\tau \times_\tau Q$ such that $\Lambda C_*(X) = C_*^\odot(X_\tau \times_\tau Q)$ whenever $Q = \boldsymbol{\Omega} X$, a monoidal cubical set, and $\tau = \tau_U : X \to \boldsymbol{\Omega} X$, the universal truncating twisting function, constructed in [9]. Dually, for the Hochschild complex $\Lambda C^*(X)$ of the simplicial cochain algebra $C^*(X)$ we get the inclusion of cochain complexes $\Lambda C^*(X) \subset C_\odot^*(X_\tau \times_\tau \boldsymbol{\Omega} X)$ (here we have equality when the graded sets have finite type). The required model for ΛY is obtained by taking $X = \mathrm{Sing}^1 Y$, the Eilenberg 1-subcomplex of the singular simplicial set $\mathrm{Sing}\, Y$.

We construct polytopes F_n referred to as *freehedra* and introduce the notion of an F_n-*set*. The motivation is that the bitwisted Cartesian product $X_\tau \times_\tau \boldsymbol{\Omega} X$ above admits such a combinatorial structure in a canonical way. As the standard simplices Δ^n, $n \geqslant 0$, are the modelling polytopes for a simplicial set, the Cartesian products $F_i \times I^j$, $i, j \geqslant 0$, with I^j the

standard j-cube serve as modelling polytopes for an F_n-set. A universal example for an F_n-set is the *singular* complex $\mathrm{Sing}^F Y$, obtained as the set of continuous maps $I^k \times F_m \times I^\ell \to Y$ for $k, \ell, m \geqslant 0$. The normalized (co)chain complex $C_*^\odot(Y)$ ($C_\odot^*(Y)$) of $\mathrm{Sing}^F Y$ also gives the singular (co)homology of Y. In particular, we construct a map of F_n-sets

$$\Upsilon : \mathrm{Sing}^1 Y \, {}_\tau \times_\tau \boldsymbol{\Omega} \, \mathrm{Sing}^1 Y \to \mathrm{Sing}^F \Lambda Y$$

that extends the cubical map $\omega : \boldsymbol{\Omega} \, \mathrm{Sing}^1 Y \to \mathrm{Sing}^I \Omega Y$ realizing Adams' cobar equivalence $\omega_* : \Omega C_*(Y) \to C_*^\square(\Omega Y)$ [1]: The image $\Upsilon(\sigma_m, \sigma_n')$ for $m, n > 1$ consists of singular $(F_m \times I^n)$-polytopes that are purely determined by a choice of measuring homotopy for the *commutativity* of the standard loop product of ΩY by the inclusion $\Omega Y \subset \Lambda Y$. Such a relationship is possible since F_m admits a representation as an explicit subdivision of the cube $I^m = I^{m-1} \times I$.

We construct an explicit diagonal Δ_F for F_n. In a standard way this diagonal together with the Serre diagonal of the cubes yields the diagonal of an arbitrary F_n-set. Consequently, the F_n-set structure of the bitwisted Cartesian product $\mathrm{Sing}^1 Y \, {}_\tau \times_\tau \boldsymbol{\Omega} \, \mathrm{Sing}^1 Y$, as a by-product, determines a comultiplication on the Cartier complex $\Lambda C_*(Y)$. Dually, we obtain a multiplication on the Hochschild complex $\Lambda C^*(Y)$. More precisely, a combinatorial analysis of Δ_F shows that the multiplication on the Hochschild complex $\Lambda C^*(Y)$ involves a canonical homotopy G-algebra (hga) structure on the simplicial cochain algebra $C^*(Y)$ [3,6,7] (compare [9]). Thus, applying the cohomology functor for Υ we obtain a natural algebra isomorphism

$$\Upsilon^* : HH_*\big(C^*(Y; \Bbbk)\big) \xleftarrow{\approx} H^*(\Lambda Y; \Bbbk)$$

in which on the left-hand side the product is given by formula (6.5) and $\Bbbk$ is a commutative ring with identity. Note also that the multiplication on the Hochschild complex $\Lambda C^*(Y)$ is not associative, but it can be extended to an A_∞-algebra structure [20].

As an application of the above algebra isomorphism we establish the fact that the standard shuffle product on the Hochschild complex ΛH with $H = H^*(Y; \Bbbk)$ polynomial is *geometric*. More precisely, given a cocycle $z \in C^n(Y; \Bbbk)$, the product $z \smile_1 z \in C^{2n-1}(Y; \Bbbk)$ is a cocycle for n even or for n odd too, when z^2 is of the second order. In any case, the class $[z \smile_1 z] \in H^{2n-1}(Y; \Bbbk)$ is defined and is denoted by $Sq_1[z]$. Clearly, if $H = H^*(Y; \Bbbk)$ is a polynomial algebra and $\Bbbk$ has no 2-torsion, then H is evenly graded and Sq_1 is identically zero on H. However, H may have odd dimensional generators for $\Bbbk = \mathbb{Z}/2\mathbb{Z}$. We have the following theorem that generalizes a well-known result when $\Bbbk$ is a field.

Theorem 1. *Let $H = H^*(Y; \Bbbk) = S(U)$ be a polynomial algebra such that $Sq_1 = 0$ on H. Let $\Lambda(s^{-1}U)$ be the exterior algebra over the desuspension of the polynomial generators U. Then there are algebra isomorphisms*

$$H(\Lambda Y; \Bbbk) \approx S(U) \otimes \Lambda\big(s^{-1}U\big) \approx H(Y; \Bbbk) \otimes H(\Omega Y; \Bbbk).$$

Note that in an ungraded setting of $H = H(Y; \Bbbk)$ the middle term above is interpreted as the module of differential forms denoted by $\Omega_{H|\Bbbk}$ [10]. Using a *strong homotopy commutative* structure of $C^*(Y; \Bbbk)$ the multiplication on $\Lambda C^*(Y; \Bbbk)$ is constructed in [14] for $\Bbbk$ to be a field (see also [13] for references). Though there is a close relationship between the strong homotopy commutative and hga structures, the explicit formula for the product on the Hochschild chain complex $\Lambda C^*(Y)$ in terms of the hga operations is heavily used here (see Lemma 1). Moreover, the proof of the above theorem suggests further calculations of the Hochschild homology for a more wide class of spaces having, for example, the cohomology isomorphic to smooth algebras.

Restricted to finitely generated polynomial algebras, i.e., U is finite dimensional, Theorem 1 agrees with the solution of the Steenrod problem given in [2]. At the end of the paper we include an example showing that there is a tensor product algebra C but with perturbed differential such that the E_∞-term of the spectral sequence of C is isomorphic as algebras with the E_∞-term of the Serre spectral sequence of the free loop fibration with the base $Y = \mathbb{C}P^\infty \times \mathbb{C}P^\infty$, nevertheless $H^*(C)$ differs from $H^*(\Lambda Y)$ as algebras.

Finally, note that a large part of the paper consists of an editing excerpt from the earlier preprint [17] where in particular the polytopes F_n were constructed. In the meantime 3-dimensional polytope F_3 appeared in [4] (see also [11]).

2. Some preliminaries and conventions

We adopt the notions and notations from [9].

2.1. Cobar and bar constructions

Let $\Bbbk$ be a commutative ring with identity. For a $\Bbbk$-module M, let $T(M)$ be the tensor algebra of M, i.e. $T(M) = \bigoplus_{i=0}^\infty M^{\otimes i}$. An element $a_1 \otimes \cdots \otimes a_n \in M^{\otimes n}$ is denoted by $[a_1|\cdots|a_n]$. We denote by $s^{-1}M$ the desuspension of M, i.e. $(s^{-1}M)_i = M_{i+1}$.

Let (C, d_C, Δ) be a 1-reduced dgc, i.e. $C_0 = \Bbbk$, $C_1 = 0$. Denote $\bar{C} = s^{-1}(C_{>0})$. Let $\Delta = Id \otimes 1 + 1 \otimes Id + \Delta'$. The (reduced) cobar construction ΩC on C is the tensor algebra $T(\bar{C})$, with differential $d = d_1 + d_2$ defined for $\bar{c} \in \bar{C}_{>0}$ by

$$d_1[\bar{c}] = -\big[\overline{d_C(c)}\big] \quad \text{and} \quad d_2[\bar{c}] = \sum (-1)^{|c'|}[\bar{c}'|\bar{c}''], \quad \text{for } \Delta'(c) = \sum c' \otimes c'',$$

extended as a derivation. The acyclic cobar construction $\Omega(C; C)$ is the twisted tensor product $C \otimes \Omega C$ in which the tensor differential is twisted by the universal twisting cochain $\tau_* : C \to \Omega C$ being an inclusion of degree -1.

Let (A, d_A, μ) be a 1-reduced dga. The (reduced) bar construction BA on A is the tensor coalgebra $T(\bar{A})$, $\bar{A} = s^{-1}(A_{>0})$, with differential $d = d_1 + d_2$ given for $[\bar{a}_1 | \cdots | \bar{a}_n] \in T^n(\bar{A})$ by

$$d_1[\bar{a}_1 | \cdots | \bar{a}_n] = -\sum_{i=1}^{n} (-1)^{\epsilon_{i-1}} [\bar{a}_1 | \cdots | \overline{d_A(a_i)} | \cdots | \bar{a}_n],$$

and

$$d_2[\bar{a}_1 | \cdots | \bar{a}_n] = -\sum_{i=1}^{n-1} (-1)^{\epsilon_i} [\bar{a}_1 | \cdots | \overline{a_i a_{i+1}} | \cdots | \bar{a}_n],$$

where $\epsilon_i = \epsilon_i^a = |a_1| + \cdots + |a_i| + i$. The acyclic bar construction $B(A; A)$ is the twisted tensor product $A \otimes BA$ in which the tensor differential is twisted by the universal twisting cochain $\tau^* : BA \to A$ being a projection of degree 1.

2.2. Cartier and Hochschild chain complexes

Let (C, d_C, Δ) be a 1-reduced dgc and let $\Delta = Id \otimes 1 + \Delta_1 = 1 \otimes Id + \Delta_2$. The (normalized) *Cartier complex* ΛC of C [5] is $C \otimes \Omega C$ with differential d defined by $d = d_C \otimes 1 + 1 \otimes d_{\Omega C} + \theta_1 + \theta_2$, where

$$\theta_1 \big(v \otimes [\bar{c}_1 | \cdots | \bar{c}_n] \big) = -\sum (-1)^{|v'_1|} v'_1 \otimes [\bar{v}''_1 | \bar{c}_1 | \cdots | \bar{c}_n],$$

$$\theta_2 \big(v \otimes [\bar{c}_1 | \cdots | \bar{c}_n] \big) = \sum (-1)^{(|v'_2|+1)(|v''_2|+\epsilon_n^c)} v''_2 \otimes [\bar{c}_1 | \cdots | \bar{c}_n | \bar{v}'_2],$$

$$\Delta_i(v) = \sum v'_i \otimes v''_i, \quad i = 1, 2.$$

The homology of ΛC is called the Cartier homology of a dgc C and is denoted by $HH_*(C)$.

Note that the components θ_1 and θ_2 above can be thought of as obtained by applying the universal twisting cochain $\tau_* : C \to \Omega C$ on the tensor product $C \otimes \Omega C$ twice:

$$C \otimes \Omega C \xrightarrow{\Delta \otimes 1} C \otimes C \otimes \Omega C \xrightarrow{1 \otimes \tau_* \otimes 1 + (1 \otimes 1 \otimes \tau_*)T} C \otimes \Omega C \otimes \Omega C \xrightarrow{1 \otimes \mu} C \otimes \Omega C,$$

where $T : C \otimes (C \otimes \Omega C) \to (C \otimes \Omega C) \otimes C$; consequently, ΛC is a *bitwisted tensor product* $C {}_{\tau_*}\!\otimes_{\tau_*} \Omega C$.

The (normalized) *Hochschild complex* ΛA of a 1-reduced associative dg algebra A [10] is $A \otimes BA$ with differential d defined by $d = d_A \otimes 1 + 1 \otimes d_{BA} + \theta^1 + \theta^2$, where

$$\theta^1 \big(u \otimes [\bar{a}_1 | \cdots | \bar{a}_n] \big) = -(-1)^{|u|} u a_1 \otimes [\bar{a}_2 | \cdots | \bar{a}_n],$$

$$\theta^2 \big(u \otimes [\bar{a}_1 | \cdots | \bar{a}_n] \big) = (-1)^{(|a_n|+1)(|u|+\epsilon_{n-1}^a)} a_n u \otimes [\bar{a}_1 | \cdots | \bar{a}_{n-1}].$$

The homology of ΛA is called the Hochschild homology of a dga A and is denoted by $HH_*(A)$.

Dually, the components θ^1 and θ^2 can be thought of as obtained by applying the universal twisting cochain $\tau^* : BA \to A$ on the tensor product $A \otimes BA$ twice; consequently, ΛA is a *bitwisted tensor product* $A {}_{\tau^*}\!\otimes_{\tau^*} BA$.

3. The polytopes F_n

It is well known that the standard cube I^n can be viewed as obtained from the standard simplex Δ^n with vertices $(v_0, v_1, \ldots, v_n)$ by a truncating procedure starting either at the minimal vertex v_0 or at the maximal vertex v_n; One gets a polytope, called a freehedron and denoted by F_n, when the truncations start at both vertices v_0 and v_n simultaneously; since these two truncations do not meet, we can begin the truncation procedure at vertex v_0 to obtain first the standard cube I^n, and then continue the same procedure at vertex v_n to recover F_n. Thus, we obtain the canonical cellular projection, a "healing" map, $\varphi : F_n \to \Delta^n$ such that it factors through the projections $\phi : F_n \to I^n$ and $\psi : I^n \to \Delta^n$ (see Figs. 1–3).

It is convenient, to regard F_n as a subdivision of I^n and to give the following inductive construction of F_n whose faces are labelled three types of face operators d_i^0, d_i^1 and d_i^2 with $d_1^1 = d_1^2$. Let F_0 and F_1 be a point and an interval respectively. If F_{n-1} has been constructed, let e_i^ϵ denote the face $(x_1, \ldots, x_i, \epsilon, x_{i+1}, \ldots, x_{n-1}) \subset I^n$ where $\epsilon = 0, 1$ and $1 \leqslant i \leqslant n$. Then F_n is the subdivision of $F_{n-1} \times I$ given below and its various $(n-1)$-faces are labelled as indicated:

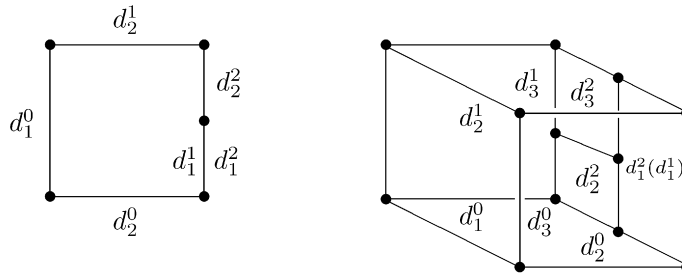| Face of F_n	Face operator	
e_i^0	d_i^0,	$1 \leqslant i \leqslant n$
e_i^1	d_i^1,	$2 \leqslant i \leqslant n$
$d_i^2 \times I$	d_i^2,	$1 \leqslant i \leqslant n-2$
$d_{n-1}^2 \times [0, \frac{1}{2}]$	d_{n-1}^2	
$d_n^2 \times [\frac{1}{2}, 1]$	d_n^2	

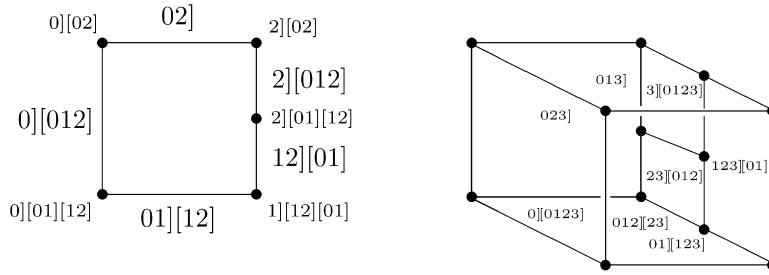**Fig. 1.** F_n as a subdivision of $F_{n-1} \times I$ for $n = 2, 3$.

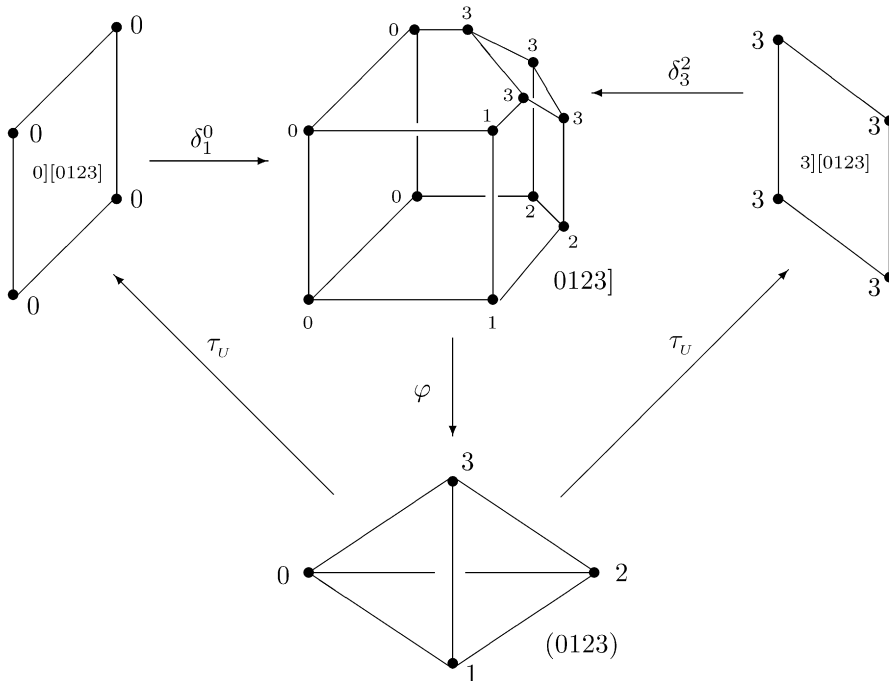**Fig. 2.** The combinatorial description of freehedra $F_2 = 012]$ and $F_3 = 0123]$.

Fig. 3. The two-fold interpretation of the universal truncating twisting function τ_U.

Thus, F_2 is a pentagon, F_3 has eight 2-faces (4 pentagon and 4 quadrilateral), 18 edges and 12 vertices (compare [4]). In particular, the sequence of codimension 1 faces of F_n, $n \geqslant 1$, is an arithmetic progression with difference 3.

3.1. The singular F_n-set

Before we give the notion of an *abstract F_n-set* below, let consider its universal example, the *singular F_n-set* $\mathrm{Sing}^F Y$ of a topological space Y, i.e. the set of all continuous maps $\{I^k \times F_m \times I^\ell \to Y\}_{m,k,\ell \geqslant 0}$. The face and degeneracy operators are defined as follows.

Obviously, the face $d_i^1(F_m)$ is homeomorphic to F_{m-1} for each i, while according to the orientation of the cube I^m the faces $d_i^0(F_m)$ and $d_i^2(F_m)$ are homeomorphic to $F_{i-1} \times I^{m-i}$ and $I^{i-1} \times F_{m-i}$ respectively; let these homeomorphisms be realized by the following inclusions

$$\bar{\delta}_i^0 : F_{i-1} \times I^{m-i} \hookrightarrow F_m,$$

$$\bar{\delta}_i^1 : F_{m-1} \hookrightarrow F_m,$$

$$\bar{\delta}_i^2 : I^{i-1} \times F_{m-i} \hookrightarrow F_m$$

with $\bar{\delta}_1^2 = \bar{\delta}_1^1$. Let $\iota_i^\epsilon : I^k \hookrightarrow I^{k+1}$ be the inclusion defined by $\iota_i^\epsilon(I^k) = e_i^\epsilon$. Given $k, \ell, m \geqslant 0$ with $m + \ell = r_1$, $k + m + \ell = r_2$, define the inclusions

$$\delta_i^\epsilon : I^{k_{\epsilon,i}} \times F_{m_{\epsilon,i}} \times I^{\ell_{\epsilon,i}} \hookrightarrow I^k \times F_m \times I^\ell$$

by

$$\delta_i^\epsilon = \begin{cases} 1 \times \bar{\delta}_i^0 \times 1, & \epsilon = 0, & (k, m, \ell)_{0,i} = (k, m - i, \ell + i - 1), & 1 \leqslant i \leqslant m, \\ 1 \times \bar{\delta}_i^1 \times 1, & \epsilon = 1, & (k, m, \ell)_{1,i} = (k, m - 1, \ell), & 1 \leqslant i \leqslant m, \\ 1 \times 1 \times \iota_{i-m}^\epsilon, & \epsilon = 0, 1, & (k, m, \ell)_{\epsilon,i} = (k, m, \ell - 1), & m < i \leqslant r_1, \\ \iota_{i-r_1}^\epsilon \times 1 \times 1, & \epsilon = 0, 1, & (k, m, \ell)_{\epsilon,i} = (k - 1, m, \ell), & r_1 < i \leqslant r_2, \\ 1 \times \bar{\delta}_i^2 \times 1, & \epsilon = 2, & (k, m, \ell)_{2,i} = (k + i - 1, m - i, \ell), & 1 \leqslant i \leqslant m. \end{cases}$$

Then define the face operators

$$d_i^\epsilon : (\mathrm{Sing}^F Y)^{m,n} \to (\mathrm{Sing}^F Y)^{m_{\epsilon,i}, n_{\epsilon,i}}$$

for $f \in (\mathrm{Sing}^F Y)^{m,n}$, $f : I^k \times F_m \times I^\ell \to Y$, $n = k + \ell$, by $d_i^\epsilon(f) = f \circ \delta_i^\epsilon$, $\epsilon = 0, 1, 2$.

Given $1 \leqslant i \leqslant n + 1$, define the degeneracy operators

$$\eta_i : (\mathrm{Sing}^F Y)^{m,n} \to (\mathrm{Sing}^F Y)^{m,n+1}$$

by

$$\eta_i(f) = \begin{cases} f \circ (1 \times 1 \times \bar{\eta}_i), & 1 \leqslant i \leqslant \ell + 1, \\ f \circ (\bar{\eta}_{i-\ell} \times 1 \times 1), & \ell + 1 < i \leqslant n + 1, \ n = k + \ell, \end{cases}$$

with $\bar{\eta}_i : I^{r+1} \to I^r$, $\bar{\eta}_i(x_1, \ldots, x_{r+1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{r+1})$.

Thus, we obtain the singular F_n-set

$$\left\{ \mathrm{Sing}^F Y = \bigcup_{k,m,\ell \geqslant 0} \left[(\mathrm{Sing}^F Y)^{m,k+\ell} = \{ I^k \times F_m \times I^\ell \to Y \} \right], \ d_i^0, d_i^1, d_i^2, \eta_i \right\}$$

the face and degeneracy operators of which satisfy the following equalities:

$$d_i^0 d_j^0 = d_{j-1}^0 d_i^0, \quad i < j,$$

$$d_i^1 d_j^1 = d_{j-1}^1 d_i^1, \quad i < j \text{ and } (i, j) \neq (1, 2) \text{ for } m > 0,$$

$$d_i^1 d_j^0 = \begin{cases} d_{j-1}^0 d_i^1, & i < j, \\ d_j^0 d_{i+1}^1, & i \geqslant j, \end{cases}$$

$$d_i^2 d_j^0 = d_{j-i}^0 d_i^2,$$

$$d_i^2 d_j^1 = \begin{cases} d_{j-i}^1 d_i^2, & i < j - 1, \\ d_{m+n+j-i-2}^1 d_{i+1}^2, & i \geqslant j - 1 > 0, \end{cases}$$

$$d_i^2 d_j^2 = d_{m+n-i}^0 d_{i+j}^2,$$

$$d_i^\epsilon \eta_j = \begin{cases} \eta_{j-1} d_i^\epsilon, & i < m + j, \\ 1, & i = m + j, \\ \eta_j d_{i-1}^\epsilon, & i > m + j, \ \epsilon = 0, 1, \end{cases}$$

$$d_i^2 \eta_j = \eta_j d_i^2,$$

$$\eta_i \eta_j = \eta_{j+1} \eta_i, \quad i \leqslant j. \tag{3.1}$$

Note that the compositions $d_1^1 d_2^1$ and $d_1^1 d_1^1$ that are eliminated from the second equality above are involved in the fifth and sixth ones as $d_1^2 d_2^1 = d_{m+n-1}^2 d_2^2$ and $d_1^2 d_1^2 = d_{m+n-1}^2 d_2^2$ by taking into account that $d_1^1 = d_1^2$.

3.2. Abstract freehedral sets

For a general theory of polyhedral sets, see for example [15,8]. Motivated by the underlying combinatorial structure of the bitwisted Cartesian product ΛX below we give the following

Definition 1. An F_n-*set* $\mathcal{CH}$ is a bigraded set $\mathcal{CH} = \{\mathcal{CH}^{m,n}, \ m, n \geqslant 0\}$ with total grading $m+n$ and three types of faces operators d_i^0, d_i^1, d_i^2 with $d_1^1 = d_1^2$ for $m > 0$,

$$d_i^0 : \mathcal{CH}^{m,n} \to \mathcal{CH}^{i-1,m+n-i}, \quad 1 \leqslant i \leqslant m,$$
$$d_i^0 : \mathcal{CH}^{m,n} \to \mathcal{CH}^{m,n-1}, \qquad m < i \leqslant m+n,$$
$$d_i^1 : \mathcal{CH}^{m,n} \to \mathcal{CH}^{m-1,n}, \qquad 1 \leqslant i \leqslant m,$$
$$d_i^1 : \mathcal{CH}^{m,n} \to \mathcal{CH}^{m,n-1}, \qquad m < i \leqslant m+n,$$
$$d_i^2 : \mathcal{CH}^{m,n} \to \mathcal{CH}^{m-i,n+i-1}, \quad 1 \leqslant i \leqslant m,$$

and degeneracy operators

$$\eta_i : \mathcal{CH}^{m,n} \to \mathcal{CH}^{m,n+1}, \quad 1 \leqslant i \leqslant n+1$$

satisfying structural identities (3.1).

A *morphism* of F_n-sets is a family of maps $f = \{f_{m,n}\}$, $f_{m,n} : \mathcal{CH}^{m,n} \to \mathcal{CH}'^{m,n}$, commuting with all face and degeneracy operators.

Note that unlike d_i^0 and d_i^1 the face operator d_i^2 is defined only for $1 \leqslant i \leqslant m$. Consequently, for $m = 0$ we have only of two types of face operators d_i^ϵ, $\epsilon = 0, 1$, acting on $\mathcal{CH}^{0,n}$ that satisfy the standard *cubical* relations; so that, in every F_n-set the subset $\{\mathcal{CH}^{0,r}\}_{r \geqslant 0}$ together with the operators d^0, d^1 and η_i forms a cubical set. Furthermore, the operators d_1^0 and d_m^2 have the image in this subset.

The F_n-set structural relations also can be conveniently verified by means of the following combinatorics of the polytopes F_m (compare with Proposition 3.2 in [9]). The top dimensional cell of F_m is identified with the set $0, 1, \ldots, m]$, while any proper q-face u of F_m is expressed as (see Fig. 2)

$$u = i_{s_t}, \ldots, i_{s_{t+1}}][i_{s_{t+1}}, \ldots, i_{s_{t+2}}] \ldots [i_{s_{k-1}}, \ldots, i_{s_k}, m][0, i_1, \ldots, i_{s_1}][i_{s_1}, \ldots, i_{s_2}] \cdot [i_{s_{t-1}}, \ldots, i_{s_t}],$$
$$0 < i_1 < \cdots < i_{s_t} < \cdots < i_{s_k} < m, \quad q = s_k - k + 1,$$

where for $t = 0$ the face u is assumed to have the form

$$u = i_1, \ldots, i_{s_1}][i_{s_1}, \ldots, i_{s_2}] \ldots [i_{s_{k-1}}, \ldots, i_{s_k}, m]$$

with $0 \leqslant i_1 < \cdots < i_{s_k} < m$.

Proposition 1. *Let the face operators d^0, d^1, d^2 act on $a_0, a_1, \ldots, a_m][b_0, \ldots, b_{n+1}]$ (thought of as the top cell of $F_m \times I^n$) by*

$$a_0, a_1, \ldots, a_m][b_0, \ldots, b_{n+1}] \xrightarrow{d_i^0} \begin{cases} a_0, a_1, \ldots, a_{i-1}][a_{i-1}, \ldots, a_m][b_0, \ldots, b_{n+1}], & 1 \leqslant i \leqslant m, \\ a_0, a_1, \ldots, a_m][b_0, \ldots, b_j][b_j, \ldots, b_{n+1}], & i = m+j, \end{cases}$$

$$a_0, a_1, \ldots, a_m][b_0, \ldots, b_{n+1}] \xrightarrow{d_i^1} \begin{cases} a_1, \ldots, a_m][b_0, \ldots, b_{n+1}][a_0, a_1], & i = 1, \\ a_0, a_1, \ldots, \hat{a}_{i-1}, \ldots, a_m][b_0, \ldots, b_{n+1}], & 2 \leqslant i \leqslant m, \\ a_0, a_1, \ldots, a_m][b_0, b_1, \ldots, \hat{b}_j, \ldots, b_{n+1}], & i = m+j, \end{cases}$$

$$a_0, a_1, \ldots, a_m][b_0, \ldots, b_{n+1}] \xrightarrow{d_i^2} a_i, \ldots, a_m][b_0, \ldots, b_{n+1}][a_0, a_1, \ldots, a_i], \quad 1 \leqslant i \leqslant m.$$

Then the relations among d^ϵ's for $\epsilon = 0, 1, 2$ agree with the F_n-set identities.

Proof. It is straightforward. □

A degeneracy operator η_i is thought of as adding a formal element $*$ to the set $a_0, a_1, \ldots, a_m][b_0, b_1, \ldots, b_{n+1}]$ at the $(m+i+2)$st place:

$$a_0, a_1, \ldots, a_m][b_0, b_1, \ldots, b_{n+1}] \xrightarrow{\eta_i} a_0, a_1, \ldots, a_m][b_0, b_1, \ldots, b_i, *, b_{i+1}, \ldots, b_{n+1}]$$

with the convention $[b_0, b_1, \ldots, b_i, *][*, b_{i+1}, \ldots, b_{n+1}] = [b_0, b_1, \ldots, b_{n+1}]$ that guarantees the equality $d_{m+i}^0 \eta_i = Id = d_{m+i}^1 \eta_i$.

4. A diagonal on the freehedra F_n

For the freehedron F_n define its integral chain complex $(C_*(F_n), d)$ with differential

$$d = \sum_{i=1}^{n} (-1)^i (d_i^0 - d_i^1) + \sum_{i=2}^{n} (-1)^{(i-1)n} d_i^2. \tag{4.1}$$

Now define the map

$$\Delta_F : C_*(F_n) \to C_*(F_n) \otimes C_*(F_n),$$

for $u_n \in C_n(F_n)$, $n \geqslant 0$, by

$$\Delta_F(u_n) = \sum_{(K,L)} \operatorname{sgn}(K,L) d_{j_p}^0 \ldots d_{j_1}^0 (u_n) \otimes d_{i_q}^1 \ldots d_{i_1}^1 (u_n)$$

$$+ \sum_{\substack{(K',L') \\ r+1 \leqslant i_q \leqslant p+1 \\ 0 \leqslant r < n}} (-1)^{j_{(r)} + r + (p+1)(i_q+1)} \cdot \operatorname{sgn}(K',L') d_{j_p}^0 \ldots d_{j_{r+1}}^0 d_{j_r}^2 \ldots d_{j_1}^2 (u_n) \otimes d_{i_q}^2 d_{i_{q-1}}^1 \ldots d_{i_1}^1 (u_n),$$

$$(K,L) = (i_q < \cdots < i_1; 1 = j_p < \cdots < j_1) \quad \text{and}$$

$$(K',L') = (1 < i_{q-1} < \cdots < i_1; j_1 + 1 < \cdots < j_{(r)} + 1 < j_p + j_{(r)} < \cdots < j_{r+1} + j_{(r)}) \tag{4.2}$$

with $j_{(k)} = j_1 + \cdots + j_k$ are unshuffles of the set $\{1, \ldots, n\}$.

Proposition 2. *The map defined by formula* (4.2) *is a chain map.*

Proof. The proof is straightforward. $\square$

Note that the components of the first summand of (4.2) together with ones of the second for $(r, i_q) = (0, 1)$ agree with the components of the diagonal of the standard cube I^n [19,9].

Using Proposition 1 formula (4.2) can be rewritten in the following combinatorial form (compare (6.3)):

$$\Delta_F(0, 1, \ldots, n]) = \Sigma (-1)^{\epsilon_1} 0, 1, \ldots, j_1][j_1, \ldots, j_2][j_2, \ldots, j_3] \ldots [j_p, \ldots, n] \otimes j_1, j_2, \ldots, j_p, n]$$

$$+ \Sigma (-1)^{\epsilon_2} j_r, \ldots, j_{r+1}][j_{r+1}, \ldots, j_{r+2}] \ldots [j_p, \ldots, n][0, 1, \ldots, j_1][j_1, \ldots, j_2] \ldots [j_{r-1}, \ldots, j_r]$$

$$\otimes j_t, \ldots, j_p, n][0, j_1, \ldots, j_r, \ldots, j_t], \tag{4.3}$$

where the last tensor factor for a fixed r varies from $j_{r+1}, \ldots, j_p, n][0, j_1, \ldots, j_{r+1}]$ to $n][0, j_1, \ldots, j_p, n]$.

For example, formulas (4.2) and (4.3) for F_2 and F_3 read:

$$\Delta_F(u_2) = \big(d_1^0 d_2^0 \otimes Id + Id \otimes d_1^1 d_2^1 - d_1^0 \otimes d_2^1 + d_2^0 \otimes d_1^1 + (d_2^0 + d_1^2) \otimes d_2^2\big)(u_2 \otimes u_2);$$

$$\Delta_F(u_3) = \big(d_1^0 d_2^0 d_3^0 \otimes Id + Id \otimes d_1^1 d_2^1 d_3^1 + d_1^0 \otimes d_2^1 d_3^1 - d_2^0 \otimes d_1^1 d_3^1 + d_3^0 \otimes d_1^1 d_2^1 + d_1^0 d_2^0 \otimes d_3^1 - d_1^0 d_3^0 \otimes d_2^1 + d_2^0 d_3^0 \otimes d_1^1$$

$$\qquad - (d_2^0 + d_1^2) \otimes d_2^2 d_3^1 + (d_3^0 + d_2^2) \otimes d_2^2 d_2^1 - (d_2^0 d_3^0 + d_2^0 d_1^2) \otimes (d_2^2 - d_3^2) + d_2^0 d_2^2 \otimes d_3^2\big)(u_3 \otimes u_3)$$

and

$$\Delta_F(012]) = 0][01][12] \otimes 012] + 012] \otimes 2][02] - 0][012] \otimes 02] + 01][12] \otimes 12][01] + (01][12] + 12][01]) \otimes 2][012];$$

$$\Delta_F(0123]) = 0][01][12][23] \otimes 0123] + 0123] \otimes 3][03] + 0][0123] \otimes 03] - 01][123] \otimes 13][01]$$

$$\qquad + 012][23] \otimes 23][02] + 01][12][23] \otimes 123][01] + 0][01][123] \otimes 013] - 0][012][23] \otimes 023]$$

$$\qquad - (01][123] + 123][01]) \otimes 3][013] + (012][23] + 23][012]) \otimes 3][023]$$

$$\qquad - (01][12][23] + 12][23][01]) \otimes (23][012] - 3][0123]) + 23][01][12] \otimes 3][0123].$$

The diagonal Δ_F is compatible with the AW diagonal of the standard simplex Δ^n under the cellular map $\varphi : F_n \to \Delta^n$. To see this it is also convenient to represent φ combinatorially as:

$$i_{s_t}, \ldots, i_{s_{t+1}}][i_{s_{t+1}}, \ldots, i_{s_{t+2}}] \ldots [i_{s_{k-1}}, \ldots, i_{s_k}, n][0, i_1, \ldots, i_{s_1}][i_{s_1}, \ldots, i_{s_2}] \ldots [i_{s_{t-1}}, \ldots, i_{s_t}] \xrightarrow{\varphi} (i_{s_t}, \ldots, i_{s_{t+1}}).$$

In particular, the faces $0][0, 1, \ldots, n]$ and $n][0, 1, \ldots, n]$ of F_n, i.e. d_1^0 and d_n^2, go to the minimal and maximal vertices $0 \in \Delta^n$ and $n \in \Delta^n$ respectively (see Fig. 3).

Note that the diagonal Δ_F on $C_*(F_n)$ is not coassociative, and hence, the product on $C^*(F_n)$ is not associative, however since the acyclicity of F_n there exists an A_∞-algebra structure on $C^*(F_n)$ (see Section 6.2 below).

4.1. The diagonal on an F_n-set

Given an F_n-set $\mathcal{CH}$, define the chain complex $(C_*(\mathcal{CH}), d)$ of $\mathcal{CH}$ with coefficients in $\Bbbk$ and with differential $d_r : C_r(\mathcal{CH}) \to C_{r-1}(\mathcal{CH})$ given by

$$d_r = \bigoplus_{\substack{r=m+n \\ m,n \geqslant 0}} d_{m,n}, \quad d_{m,n} = \sum_{i=1}^{m+n} (-1)^i (d_i^0 - d_i^1) + \sum_{i=2}^{m} (-1)^{(i-1)(m+n)} d_i^2.$$

The *normalized chain complex* of $\mathcal{CH}$ is $(C_*^{\odot}(\mathcal{CH}), d) = (C_*(\mathcal{CH}), d)/D$, where D is the subcomplex of $C_*(\mathcal{CH})$ formed by degeneracies. Then apply (4.2) to make $C_*^{\odot}(\mathcal{CH})$ as a dg *coalgebra*. In particular, given the singular F_n-set $\operatorname{Sing}^F Y$, we get the dg coalgebra $C_*^{\odot}(\operatorname{Sing}^F Y)$ denoted by $(C_*^{\odot}(Y), d)$. Then the cellular composition

$$I^k \times F_m \times I^\ell \xrightarrow{1 \times \phi \times 1} I^k \times I^m \times I^\ell = I^{k+m+\ell} \xrightarrow{\psi} \Delta^{k+m+\ell}$$

induces a chain map $C_*(Y) \to C_*^{\odot}(Y)$ to obtain the following

Proposition 3. *There are the natural isomorphisms of the homologies*

$$H_*(Y) \approx H_*^{\odot}(Y) = H_*\big(C_*^{\odot}(Y), d\big)$$

and the cohomology algebras

$$H^*(Y) \approx H_{\odot}^*(Y) = H^*\big(C_{\odot}^*(Y), d\big).$$

5. Truncating twisting functions and bitwisted Cartesian products

5.1. The Cartier–Hochschild set ΛX

Given a 1-reduced simplicial set (X, ∂_i, s_i), i.e. $X = \{X_0 = X_1 = \{*\}, X_2, X_3, \ldots\}$, recall the definition of a *truncating twisting function* [9]:

Definition 2. Let X be a 1-reduced simplicial set and Q be a monoidal cubical set. A sequence of functions $\tau = \{\tau_n : X_n \to Q_{n-1}\}_{n \geqslant 1}$ of degree -1 is a *truncating twisting function* if it satisfies:

$$
\begin{aligned}
\tau(x) &= e, & x &\in X_1, \\
d_i^0 \tau(x) &= \tau \partial_{i+1} \ldots \partial_n(x) \cdot \tau \partial_0 \ldots \partial_{i-1}(x), & x &\in X_n, \quad 1 \leqslant i < n, \\
d_i^1 \tau(x) &= \tau \partial_i(x), & x &\in X_n, \quad 1 \leqslant i < n, \\
\eta_n \tau(x) &= \tau s_n(x), & x &\in X_n, \quad n \geqslant 1.
\end{aligned}
$$

A useful characterization of a truncating twisting function is that the monoidal map $f : \Omega X \to Q$ defined by $f(\bar{x}_1 \cdots \bar{x}_k) = \tau(x_1) \cdots \tau(x_k)$ is a map of cubical sets, where ΩX is the monoidal cubical set constructed in [9] such that ΩX is related with X by the universal truncating twisting function $\tau_U : X \to \Omega X$, $x \to \bar{x}$.

Definition 3. Let $X = \{X_m, \partial_i, s_i\}$ be a 1-reduced simplicial set, Q be a monoidal cubical set, and $L = \{L_n, d_i^\epsilon, \eta_i\}$ be a Q-bimodule via $Q \times L \to L$ and $L \times Q \to L$. Let $\tau = \{\tau_k : X_k \to Q_{k-1}\}_{k \geqslant 1}$ be a truncating twisting function. The *bitwisted Cartesian product* $X_\tau \times_\tau L$ is the bigraded set

$$X_\tau \times_\tau L = X \times L/_\sim,$$

where $(s_m(x), y) \sim (x, \eta_1(y))$, $(x, y) \in X_m \times L_n$, and endowed with the face d_i^0, d_i^1, d_i^2 and degeneracy η_i operators defined by

$$
\begin{aligned}
d_i^0(x, y) &= \begin{cases} (\partial_1 \ldots \partial_m(x), \tau(x) \cdot y), & i = 1, \\ (\partial_i \ldots \partial_m(x), \tau \partial_0 \ldots \partial_{i-2}(x) \cdot y), & 1 < i \leqslant m, \\ (x, d_{i-m}^0(y)), & m < i \leqslant m+n, \end{cases} \\
d_i^1(x, y) &= \begin{cases} (\partial_{i-1}(x), y), & 1 \leqslant i \leqslant m, \\ (x, d_{i-m}^1(y)), & m < i \leqslant m+n, \end{cases} \\
d_i^2(x, y) &= \big(\partial_0 \ldots \partial_{i-1}(x), y \cdot \tau \partial_{i+1} \ldots \partial_m(x)\big), \quad 1 \leqslant i \leqslant m, \\
\eta_i(x, y) &= \big(x, \eta_i(y)\big), \quad 1 \leqslant i \leqslant n+1.
\end{aligned}
\tag{5.1}
$$

Using Proposition 1 it is easy to verify that $(X_\tau \times_\tau L, d_i^0, d_i^1, d_i^2, \eta_i)$ forms an F_n-set. In particular,

$$d_1^2(x, y) = \big(\partial_0(x), y \cdot \tau \partial_2 \ldots \partial_m(x)\big) = \big(\partial_0(x), y \cdot e\big) = \big(\partial_0(x), y\big) = d_1^1(x, y).$$

Take in the above definition $Q = L = \boldsymbol{\Omega} X$, $\tau = \tau_U$, and the bimodule actions inducing by the monoidal product on the cubical set $\boldsymbol{\Omega} X$ to give the following

Definition 4. Given a 1-reduced simplicial set X, *the Cartier–Hochschild set $\boldsymbol{\Lambda} X$ is the bitwisted Cartesian product $X_{\tau_U} \times_{\tau_U} \boldsymbol{\Omega} X$* endowed with the F_n-set structure via (5.1).

Remark 1. 1. Note that in the Cartier–Hochschild set $\boldsymbol{\Lambda} X$ one implies the identity $d_1^0(x, e) = d_m^2(x, e) = \tau_U(x) = \bar{x}$ for any simplex $x \in X_m$.
2. The operators d^0, d^1 just subject to the defining identities of a cubical set.

6. The bitwisted Cartesian model for the free loop fibration

Let $\Omega Y \to \Lambda Y \xrightarrow{\xi} Y$ be the free loop space fibration on a topological space Y. Let $\mathrm{Sing}^1 Y \subset \mathrm{Sing} Y$ be the Eilenberg 1-subcomplex generated by the singular simplices that send the 1-skeleton of the standard n-simplex Δ^n to the base point y of Y. Denote by $C_*(Y)$ the quotient coalgebra $C_*(\mathrm{Sing}^1 Y)/C_{>0}(\mathrm{Sing} y)$, the chain complex of Y. Let $\mathrm{Sing}^I \Omega Y$ be the singular cubical set of ΩY and $C_*^\square(\Omega Y)$ be the (normalized) chain complex of $\mathrm{Sing}^I \Omega Y$. Then Adams' map $\omega_* : \Omega C_*(Y) = C_*(\boldsymbol{\Omega} \mathrm{Sing}^1 Y) \to C_*^\square(\Omega Y)$ is realized by a monoidal cubical map $\omega : \boldsymbol{\Omega} \mathrm{Sing}^1 Y \to \mathrm{Sing}^I \Omega Y$ [9]. Obviously, we have the short sequence of singular F_n-sets

$$\mathrm{Sing}^F \Omega Y \to \mathrm{Sing}^F \Lambda Y \xrightarrow{\xi_\#} \mathrm{Sing}^F Y.$$

On the other hand, there is the short sequence of sets

$$\boldsymbol{\Omega} \mathrm{Sing}^1 Y \to \boldsymbol{\Lambda} \mathrm{Sing}^1 Y \xrightarrow{p} \mathrm{Sing}^1 Y$$

where the maps are the natural inclusion and projection respectively. Let

$$\iota : \mathrm{Sing}^I \Omega Y \to \mathrm{Sing}^F \Omega Y$$

be the inclusion determined via the identification $(\mathrm{Sing}^I \Omega Y)_* = (\mathrm{Sing}^F \Omega Y)^{0,*}$.
We have the following

Theorem 2. *Let $\Omega Y \to \Lambda Y \xrightarrow{\xi} Y$ be the free loop fibration.*

(i) *There are natural maps of sets $\tilde{\varphi}$, Υ, $\tilde{\omega}$ such that*

$$
\begin{array}{ccccc}
\mathrm{Sing}^F \Omega Y & \longrightarrow & \mathrm{Sing}^F \Lambda Y & \xrightarrow{\xi_\#} & \mathrm{Sing}^F Y \\
\tilde{\omega} \uparrow & & \Upsilon \uparrow & & \tilde{\varphi} \uparrow \\
\boldsymbol{\Omega} \mathrm{Sing}^1 Y & \longrightarrow & \boldsymbol{\Lambda} \mathrm{Sing}^1 Y & \xrightarrow{p} & \mathrm{Sing}^1 Y,
\end{array}
\tag{6.1}
$$

$\tilde{\varphi} : \mathrm{Sing}^1 Y \to \mathrm{Sing}^F Y$ is a map induced by the composition $I^k \times F_m \times I^\ell \xrightarrow{1 \times \phi \times 1} I^{k+m+\ell} \xrightarrow{\psi} \Delta^{k+m+\ell}$, while Υ is a map of F_n-sets, and $\tilde{\omega} = \iota \circ \omega$; moreover, these maps are homotopy equivalences whenever Y is simply connected.
(ii) *The chain complex $C_*^\odot(\boldsymbol{\Lambda} \mathrm{Sing}^1 Y)$ of the bitwisted Cartesian product*

$$\boldsymbol{\Lambda} \mathrm{Sing}^1 Y = \mathrm{Sing}^1 Y_{\tau_U} \times_{\tau_U} \boldsymbol{\Omega} \mathrm{Sing}^1 Y$$

coincides with the Cartier complex $\Lambda C_*(Y)$ of the chain coalgebra $C_*(Y)$.

Proof. To define the map $\Upsilon : \boldsymbol{\Lambda} \mathrm{Sing}^1 Y \to \mathrm{Sing}^F \Lambda Y$, it is convenient to apply a homotopically equivalent description of the free loop fibration ξ (cf. [12]). Namely, ξ is thought of as the associated fibre bundle with the universal bundle $G \to EG \to BG$ via the conjugation action $G \times G \to G$, $(a, b) \to a^{-1}ba$, where G has the homotopy type of ΩY and let $\pi : EG \times G \to (EG \times G)/\sim = \Lambda BG$ be the quotient map.

Fix a section $s : Y \to \Lambda BG$. Choose its factorization $s : Y \xrightarrow{s'} EG \times G \xrightarrow{\pi} \Lambda BG$ (s' does not need to be continuous). Fix a homotopy $\chi_k : I^k \times I \to I^k$ contracting I^k at the minimal vertex. Let $\rho : F_m \to I^m = I^{m-1} \times I$ be a cellular projection obtained by dilatation of the face $d_m^2(F_m)$ to $d_1^1(I^m)$. Given $(\sigma_m, \sigma_n') \in (\boldsymbol{\Lambda} \mathrm{Sing}^1 Y)^{m,n}$, let $\Upsilon(\sigma_m, \sigma_n') \in (\mathrm{Sing}^F Y)^{m,n}$ be a map $f : F_m \times I^n \to \Lambda BG$ defined as follows: Consider the compositions

$$\zeta : F_m \times I^n \xrightarrow{\rho \times 1} I^{m-1} \times I \times I^n \xrightarrow{p_1} I^{m-1} \times I^n \xrightarrow{\tilde{\omega}(\bar{\sigma}_m \cdot \sigma'_n)} G,$$

$$h : F_m \times I^n \xrightarrow{\rho \times 1} I^{m-1} \times I \times I^n \xrightarrow{p_2} I^{m-1} \times I \xrightarrow{\chi_{m-1}} I^{m-1} \xrightarrow{\tilde{\omega}(\bar{\sigma}_m)} G,$$

where p_1 and p_2 are canonical projections. Given $(u, v) \in F_m \times I^n$, let

$$f(u, v) = \pi\big(x_u \cdot h(u, v), y_u \cdot \zeta(u, v)\big),$$

where $(x_u, y_u) = (\varphi \circ \sigma_m \circ s')(u) \in EG \times G$. Since $\varphi \circ \sigma_m \circ s : F_m \to \Lambda BG$ is continuous, it is easy to verify that f will be continuous as well. The relation $(xa, ab) \sim (x, ba)$ in $(EG \times G)/\sim$ guarantees the compatibility $d_i^2(\sigma_m, \sigma'_n)$ with $d_i^2(f)$ under Υ. Thus, for $m = 0, 1$ the map $\Upsilon(\sigma_m, e) : F_m \to \Lambda BG$ is constant to the base point $\lambda_0 = \pi(x_0, e)$, where e denotes the unit of the monoid $\mathrm{Sing}^I \Omega Y$ too, and x_0 is the base point of EG. On the other hand, we have $\Upsilon(\sigma_0, \sigma'_n) : I^n \xrightarrow{\tilde{\omega}(\sigma'_n)} G \hookrightarrow \Lambda BG$ for $n \geqslant 0$.

The proof of Υ being a homotopy equivalence (after the geometric realization) immediately follows, for example, from the comparison of the standard spectral sequences for ξ and p using the fact that $\tilde{\omega}_*$ and $\tilde{\varphi}_*$ are homology isomorphisms.

(ii) The proof is straightforward as the proof of the identification isomorphism $C_*^\square(\mathbf{P}\mathrm{Sing}^1 Y) = \Omega(C_*(Y); C_*(Y))$ in [9]. Only we remark that the differential of $\Lambda C_*(Y)$ differs from the one of the above acyclic cobar construction by the component θ_2 that agrees with the component d^2 of the differential d of the chain complex $C_*^\odot(\Lambda \mathrm{Sing}^1 Y)$ under the required identification here. $\square$

Thus, by passing to chain complexes in diagram (6.1) we obtain the following comultiplicative model of the free loop fibration ξ formed by dgc's.

Theorem 3. *For the free loop space fibration* $\Omega Y \to \Lambda Y \xrightarrow{\xi} Y$ *there is a comultiplicative model formed by dgc's which is natural in* Y:

$$
\begin{array}{ccc}
C_*^\odot(\Omega Y) \longrightarrow C_*^\odot(\Lambda Y) \xrightarrow{\xi_*} C_*^\odot(Y) \\
\tilde{\omega}_* \uparrow \qquad \Upsilon_* \uparrow \qquad \tilde{\varphi}_* \uparrow \\
\Omega C_*(Y) \longrightarrow \Lambda C_*(Y) \xrightarrow{p_*} C_*(Y).
\end{array}
\tag{6.2}
$$

Obviously, one obtains the dual statement for cochain complexes involving the Hochschild dg algebra $\Lambda C^*(Y)$.

6.1. The canonical homotopy G-algebra structure on $C^*(X)$

Recall that there exists a canonical hga structure $\{E_{k,1} : C^*(X)^{\otimes k} \otimes C^*(X) \to C^*(X)\}_{k \geqslant 0}$ on the simplicial cochain algebra $C^*(X)$ [3,7,9] which, in particular, defines an associative multiplication μ_E on the bar construction $BC^*(X)$. It is convenient to view these operations as the dual of the cooperations $E^{k,1}$ on the simplicial chain coalgebra $C_*(X)$. In turn, these cooperations can be obtained by a combinatorial analysis of the diagonal of I^n:

$$\Delta[0, 1, \ldots, n+1] = \Sigma(-1)^\epsilon [0, 1, \ldots, j_1][j_1, \ldots, j_2][j_2, \ldots, j_3] \ldots [j_p, \ldots, n+1] \otimes [0, j_1, j_2, \ldots, j_p, n+1], \tag{6.3}$$

where the summands $[01 \ldots n+1] \otimes [0, n+1]$ and $[01][12][23] \ldots [n, n+1] \otimes [01 \ldots n+1]$ form the primitive part of the diagonal.

Regard the blocks of natural numbers above as faces of the standard $(n+1)$-simplex and discard the expression of the form $[j, j+1]$ to obtain Baues' formula for a 1-reduced simplicial set X and a generator $\sigma \in C_{n+1}(X)$:

$$E^{k,1}(\sigma) = \Sigma(-1)^\epsilon \big(\sigma(0, 1, \ldots, j_{s_1}) \otimes \sigma(j_{s_2}, \ldots, j_{s_3}) \otimes \cdots \otimes \sigma(j_{s_k}, \ldots, n+1)\big) \otimes \sigma(0, j_1, j_2, \ldots, j_p, n+1), \tag{6.4}$$

where $\sigma(i_1, \ldots, i_r)$ denotes the suitable face of σ (i.e. $[i_1, \ldots, i_r] = \tau_U \sigma(i_1, \ldots, i_r)$) and $k \leqslant p$.

Now let λ_E denote an inducing multiplication by (4.2) on the complex $\Lambda C^*(X)$. Apply formulas (4.3), (6.3), (6.4) to write down λ_E in terms of operations $E_{k,1}$. Namely, given two elements $u \otimes [\bar{a}_1| \cdots |\bar{a}_m]$ and $v \otimes [\bar{b}_1| \cdots |\bar{b}_n]$ in $\Lambda C^*(X)$, we get

$$\lambda_E\big((u \otimes [\bar{a}_1| \cdots |\bar{a}_m]) \otimes (v \otimes [\bar{b}_1| \cdots |\bar{b}_n])\big)$$

$$= \sum_{p=0}^{m} (-1)^{\varepsilon_1} u \cdot E_{p,1}(a_1, \ldots, a_p; v) \otimes \mu_E\big([\bar{a}_{p+1}| \cdots |\bar{a}_m] \otimes [\bar{b}_1| \cdots |\bar{b}_n]\big)$$

$$+ \sum_{0 \leqslant i \leqslant j \leqslant k \leqslant m} (-1)^{\varepsilon_2} E_{m+i+1-k,1}(a_{k+1}, \ldots, a_m, u, a_1, \ldots, a_i; b_n) \cdot E_{j-i,1}(a_{i+1}, \ldots, a_j; v)$$

$$\otimes \mu_E\big([\bar{a}_{j+1}| \cdots |\bar{a}_k] \otimes [\bar{b}_1| \cdots |\bar{b}_{n-1}]\big),$$

$$\varepsilon_1 = \epsilon_p^a + \big(\epsilon_p^a + \epsilon_m^a\big)|v|,$$

$$\varepsilon_2 = \epsilon_m^a + \big(|u| + \epsilon_k^a\big)\big(\epsilon_k^a + \epsilon_m^a\big) + \big(|v| + \epsilon_{n-1}^b\big)\big(|b_n| + 1\big) + \big(\epsilon_j^a + \epsilon_k^a\big)\big(|v| + 1\big),$$

$$\text{where } \epsilon_r^x = |x_1| + \cdots + |x_r| + r. \tag{6.5}$$

Remark 2. The first summand component of (6.5) agrees with (13) in [9] up to signs: The sign component ϵ_p^a correctly shown above is omitted in [9].

Thus, formula (6.5) gives the product on $\Lambda C^*(Y)$ by setting $X = \text{Sing}^1 Y$.

For example, for $m = n = 1$ we have (up to signs):

$$\lambda_E\big((u \otimes [\bar{a}]) \otimes (v \otimes [\bar{b}])\big) = u \cdot v \otimes \big([\bar{a}|\bar{b}] + [\bar{b}|\bar{a}] + E_{1,1}(a; b)\big) + u \cdot E_{1,1}(a; v) \otimes [\bar{b}] + E_{1,1}(u; b) \cdot v \otimes [\bar{a}]$$

$$+ \big(E_{1,1}(u; b) \cdot E_{1,1}(a; v) + E_{2,1}(u, a; b) \cdot v + E_{2,1}(a, u; b) \cdot v\big) \otimes [\ ]. \tag{6.6}$$

Note that [9] $E_{1,1}$ is in fact Steenrod's original definition of the cochain $\smile_1$ operation. It satisfies the following Hirsch formula

$$c \smile_1 a \cdot b = (c \smile_1 a) \cdot b + (-1)^{|a|(|c|+1)} a \cdot (c \smile_1 b)$$

saying that $\smile_1$ is the left derivation with respect to the $\cdot$ (cup) product on $C^*(Y)$. On the other hand, the map $- \smile_1 c$: $C^*(Y) \to C^*(Y)$ is a derivation only *up to homotopy* with the operation $E_{2,1}$ serving as a suitable homotopy:

$$dE_{2,1}(a, b; c) = E_{2,1}(da, b; c) - (-1)^{|a|} E_{2,1}(a, db; c) + (-1)^{|a|+|b|} E_{2,1}(a, b; dc)$$

$$- (-1)^{|a|} a \cdot b \smile_1 c + (-1)^{|a|+|b|+|b||c|} (a \smile_1 c) \cdot b + (-1)^{|a|} a \cdot (b \smile_1 c),$$

the *Hirsch formula up to homotopy*. In the next subsection we point out the other role of the operation $E_{2,1}$.

6.2. Interaction between the Stasheff and Gerstenhaber higher order operations on $\Lambda C^*(Y)$

Since the diagonal Δ_F is not coassociative, the multiplication λ_E on the Hochschild chain complex $\Lambda C^*(Y)$ is not associative; but Theorem 2(ii) and the acyclicity of $F_m \times I^n$ guarantees the existence of an A_∞-algebra structure on $\Lambda C^*(Y)$. Since formula (4.2), it is expected to construct this structure precisely. Indeed, denoting $A = \Lambda C^*(Y)$ and $x, y, z \in A$ with $x = u \otimes [\ ]$, $y = v \otimes [\ ]$, $z = 1 \otimes [\bar{b}]$, $du = dv = db = 0$, apply (6.6) to obtain the following λ_E products in A (up to sign):

$$(xy)z = u \cdot v \otimes [\bar{b}] + (u \cdot v \smile_1 v) \otimes [\ ] \quad \text{and}$$

$$x(yz) = u \cdot v \otimes [\bar{b}] + \big((u \smile_1 b) \cdot v + u \cdot (v \smile_1 b)\big) \otimes [\ ].$$

From the Hirsch formula up to homotopy immediately follows the equality

$$d\big(E_{2,1}(u, v; b) \otimes [\ ]\big) = (xy)z - x(yz).$$

Consequently, an operation $\varphi^3 : A^{\otimes 3} \to A$ can be chosen with

$$\varphi^3\big(u \otimes [\ ], v \otimes [\ ], 1 \otimes [\bar{b}]\big) = E_{2,1}(u, v; b) \otimes [\ ].$$

Note that the operation $E_{2,1}$ is unavoidable on the *simplicial* cochain algebra $C^*(Y)$ (i.e. there exists no strict Hirsch formula for the both sides simultaneously), and, consequently, its Hochschild chain algebra becomes a natural occurring and simplest example in which there is a non-trivial A_∞-algebra structure on the chain level.

7. Proof of Theorem 1

Recall the definition of an hga $(A, d, \cdot, \{E_{p,q}\})_{p \geqslant 0, q = 0,1}$, in general [6,7,9]. Given $k \geqslant 1$, we have the following defining identities for it:

$$dE_{k,1}(a_1, \ldots, a_k; b) = \sum_{i=1}^{k} (-1)^{\epsilon_{i-1}^a} E_{k,1}(a_1, \ldots, da_i, \ldots, a_k; b) + (-1)^{\epsilon_k^a} E_{k,1}(a_1, \ldots, a_k; db)$$

$$+ \sum_{i=1}^{k-1} (-1)^{\epsilon_i^a} E_{k-1,1}(a_1, \ldots, a_i a_{i+1}, \ldots, a_k; b) + (-1)^{\epsilon_k^a + |a_k||b|} E_{k-1,1}(a_1, \ldots, a_{k-1}; b) \cdot a_k$$

$$+ (-1)^{|a_1|} a_1 \cdot E_{k-1,1}(a_2, \ldots, a_k; b), \tag{7.1}$$

$$E_{k,1}(a_1, \ldots, a_k; b \cdot c) = \sum_{i=0}^{k} (-1)^{|b|(\epsilon_i^a + \epsilon_k^a)} E_{i,1}(a_1, \ldots, a_i; b) \cdot E_{k-i,1}(a_{i+1}, \ldots, a_k; c) \tag{7.2}$$

and

$$\sum_{\substack{k_1+\cdots+k_p=k \\ 1\leqslant p\leqslant k+\ell}} (-1)^\epsilon E_{p,1}\big(E_{k_1,\ell_1}(a_1,\ldots,a_{k_1};b_1'),\ldots,E_{k_p,\ell_p}(a_{k-k_p+1},\ldots,a_k;b_p');c\big) = E_{k,1}\big(a_1,\ldots,a_k;E_{\ell,1}(b_1,\ldots,b_\ell;c)\big),$$

$$b_i' \in \{1,b_1,\ldots,b_\ell\}, \ \ \epsilon = \sum_{i=1}^p \big(|b_i'|+1\big)\big(\varepsilon_{k_i}^a + \varepsilon_k^a\big), \ \ b_i' \neq 1. \tag{7.3}$$

A *morphism* $f:A \to A'$ between two hga's is a dga map f commuting with all $E_{p,q}$. Obviously, formula (6.5) has a sense for the Hochschild chain complex of an arbitrary hga and then f induces a dga map $\Lambda f : \Lambda A \to \Lambda A'$, and, consequently, an algebra map $HH(f):HH(A) \to HH(A')$. However, in the lemma below f is not necessarily an hga map, nevertheless it induces an algebra map on the Hochschild homologies.

We need the following lemma in which for simplicity the subscripts are removed for the operations $E_{k,1}$.

Lemma 1. *Let A, A' be two $\Bbbk$-free hga's and let $f:A \to A'$ be a dga map such that there is a sequence of maps $s = \{s_{k,1} : A^{\otimes k+1} \to A'\}_{k\geqslant 1}$ with*

$$f E_{k,1} - E_{k,1}' f^{\otimes k+1} = -(s_{k,1} + s_{k-1,1})D - d's_{k,1}, \quad D:A^{\otimes k+1} \to A^{\otimes k+1} \oplus A^{\otimes k},$$

where $D(a_1,\ldots,a_k;b) = \sum_{i=1}^k (-1)^{\epsilon_{i-1}^a}(a_1,\ldots,da_i,\ldots,a_k;b) + (-1)^{\epsilon_k^a}(a_1,\ldots,a_k;db) + \sum_{i=1}^{k-1}(-1)^{\epsilon_i^a}(a_1,\ldots,a_i a_{i+1},\ldots,a_k;b)$, and

$$s(a_1,\ldots,a_k;b\cdot c) = \sum_{i=0}^{n-1}(-1)^{(|b|+1)\epsilon_i^a + |b|(\epsilon_k^a+1)}E'(fa_1,\ldots,fa_i;fb)\cdot s(a_{i+1},\ldots,a_k;c)$$

$$+ \sum_{j=1}^n (-1)^{|b|(\epsilon_j^a+\epsilon_k^a)}s(a_1,\ldots,a_j;b)\cdot fE(a_{j+1},\ldots,a_k;c), \tag{7.4}$$

and let $\mathbf{s}:BA\otimes BA \to BA'$ be the extension of a map $\bar{s}:BA\otimes BA \to A'$,

$$\bar{s}\big([\bar{a}_1|\cdots|\bar{a}_n]\otimes[\bar{b}_1|\cdots|\bar{b}_m]\big) = \begin{cases} s(a_1\otimes\cdots\otimes a_n\otimes b_1), & m=1, \\ 0, & otherwise, \end{cases}$$

as a $(\mu_{E'}\circ(Bf\otimes Bf), Bf\circ\mu_E)$-coderivation. Then a map

$$\chi : \Lambda A\otimes\Lambda A \to \Lambda A'$$

defined for $(u\otimes[\bar{a}_1|\cdots|\bar{a}_n])\otimes(v\otimes[\bar{b}_1|\cdots|\bar{b}_m]) \in \Lambda A\otimes\Lambda A$ by

$$\chi\big((u\otimes[\bar{a}_1|\cdots|\bar{a}_m])\otimes(v\otimes[\bar{b}_1|\cdots|\bar{b}_n])\big)$$

$$= \sum_{p=0}^m (-1)^{\nu_1+|u|+\epsilon_p^a+|v|}u\cdot fE(a_1,\ldots,a_p;v)\otimes\mathbf{s}\big([\bar{a}_{p+1}|\cdots|\bar{a}_m]\otimes[\bar{b}_1|\cdots|\bar{b}_n]\big)$$

$$+ \sum_{p=0}^m (-1)^{\nu_1+|u|}u\cdot s(a_1,\ldots,a_p;v)\otimes\big(\mu_{E'}\circ(Bf\otimes Bf)\big)\big([\bar{a}_{p+1}|\cdots|\bar{a}_m]\otimes[\bar{b}_1|\cdots|\bar{b}_n]\big)$$

$$+ \sum_{0\leqslant i\leqslant j\leqslant k\leqslant m}(-1)^{\nu_2+|u|+|b_n|+|v|+\epsilon_j^a+\epsilon_k^a+\epsilon_m^a}fE(a_{k+1},\ldots,a_m,u,a_1,\ldots,a_i;b_n)$$

$$\times fE(a_{i+1},\ldots,a_j;v)\otimes\mathbf{s}\big([\bar{a}_{j+1}|\cdots|\bar{a}_k]\otimes[\bar{b}_1|\cdots|\bar{b}_{n-1}]\big)$$

$$+ \sum_{0\leqslant i\leqslant j\leqslant k\leqslant m}(-1)^{\nu_2+|u|+|b_n|+\epsilon_i^a+\epsilon_k^a+\epsilon_m^a}fE(a_{k+1},\ldots,a_m,u,a_1,\ldots,a_i;b_n)$$

$$\times s(a_{i+1},\ldots,a_j;v)\otimes\big(\mu_{E'}\circ(Bf\otimes Bf)\big)\big([\bar{a}_{j+1}|\cdots|\bar{a}_k]\otimes[\bar{b}_1|\cdots|\bar{b}_{n-1}]\big)$$

$$+ \sum_{0\leqslant i\leqslant j\leqslant k\leqslant m}(-1)^{\nu_2}s(a_{k+1},\ldots,a_m,u,a_1,\ldots,a_i;b_n)\cdot E'(fa_{i+1},\ldots,fa_j;fv)$$

$$\otimes\big(\mu_{E'}\circ(Bf\otimes Bf)\big)\big([\bar{a}_{j+1}|\cdots|\bar{a}_k]\otimes[\bar{b}_1|\cdots|\bar{b}_{n-1}]\big)$$

is a chain homotopy between $\Lambda f\circ\lambda_E$ and $\lambda_{E'}\circ(\Lambda f\otimes\Lambda f)$.

Proof. The proof is straightforward using equality (7.4). $\quad\square$

Remark 3. This lemma emphasizes a role of the explicit formula for the product λ_E in the following way: Since it has a general form $\sum(A_1 \cdot A_2 \otimes A_3)$, the chain homotopy χ admits to be of the form $\chi = A_1 \cdot A_2 \otimes s_3 + A_1 \cdot s_2 \otimes A_3' + s_1 \cdot A_2' \otimes A_3'$, i.e. a standard derivation extension of maps s_i which are thought to be related with the factors by $A_i - A_i' = ds_i + s_i d$.

Using Lemma 1 we have the following comparison proposition.

Proposition 4. *Let* $f : A \to A'$ *be as in Lemma* 1. *Then* f *induces an algebra map*

$$HH(f) : HH(A) \to HH(A')$$

and when $H(f) : H(A) \to H(A')$ *is an isomorphism, so is* $HH(f)$.

Now let fix on H the trivial hga structure, i.e. $\{E_{p,q}\} = \{E_{0,1}, E_{1,0}\}$, while on $C^*(Y; \Bbbk)$ the canonical hga structure $\{E_{p,q}\}$ mentioned in the previous section. Then we construct an auxiliary hga (RH, d) with dg algebra maps

$$H \xleftarrow{\ \rho\ } RH \xrightarrow{\ f\ } C^*(Y; \Bbbk)$$

such that both maps satisfy the hypotheses of Lemma 1 and are cohomology isomorphisms. Then we can apply Proposition 4 to obtain algebra isomorphisms

$$HH_*(H) \xleftarrow{HH(\rho)} HH_*(RH) \xrightarrow{HH(f)} HH_*\big(C^*(Y; \Bbbk)\big).$$

Since the product λ_E on $\Lambda(H, 0, \cdot, \{E_{0,1}, E_{1,0}\})$ coincides with the standard shuffle product, one gets algebra isomorphisms [10]

$$HH_*\big(H, 0, \cdot, \{E_{0,1}, E_{1,0}\}\big) \approx S(U) \otimes \Lambda\big(s^{-1}U\big) \approx H(Y; \Bbbk) \otimes H(\Omega Y; \Bbbk);$$

on the other hand, from the previous section we have an algebra isomorphism

$$HH_*\big(C^*(Y; \Bbbk), d, \cdot, \{E_{p,q}\}\big) \approx H^*(\Lambda Y; \Bbbk).$$

Consequently, Theorem 1 follows.

Thus it remains to define the hga RH and maps ρ, f mentioned above: Indeed, consider a bigraded multiplicative resolution $\rho : (RH, d) \to H$ of H [16,18] such that $R^*H^* = T(V^{*,*})$ with $V^{*,*} = V^{0,*} \oplus \mathcal{E}^{<0,*} \oplus \mathcal{T}^{-2r,*}$, $r \geqslant 1$, $V^{0,*} \approx U^*$, $\mathcal{E}^{-n,*} = \{\mathcal{E}_{k,1}^{-n,*}\}_{1 \leqslant k \leqslant n}$ with $\mathcal{E}_{k,1}^{-n,*}$ spanned on the set of expressions $E_{k,1}(a_1, \ldots, a_k; b)$, $a_r \in R^{-i_r}H^*$, $b \in V^{-j,*}$, $n = \sum_{r=1}^{k} i_r + j$, unless $E_{1,1}(a; a)$, $a \in V^{0,*}$, and subjected to relations (7.3), while $\mathcal{T}^{-2(n-1),*}$, $n \geqslant 2$, is spanned on the set of expressions $a_1 \underset{2}{\curlyvee} a_2 \underset{2}{\curlyvee} \cdots \underset{2}{\curlyvee} a_n$ with $a_i \in V^{0,*}$, $a_i \underset{2}{\curlyvee} a_j = a_j \underset{2}{\curlyvee} a_i$, and $a_i \neq a_j$ for $i \neq j$; the differential d is defined: On $V^{0,*}$ by $dV^{0,*} = 0$; on $\mathcal{E}$ by formula (7.1), and on $\mathcal{T}$ by

$$d(a_1 \underset{2}{\curlyvee} \cdots \underset{2}{\curlyvee} a_n) = \sum_{(\mathbf{i};\mathbf{j})} (a_{i_1} \underset{2}{\curlyvee} \cdots \underset{2}{\curlyvee} a_{i_k}) \underset{1}{\curlyvee} (a_{j_1} \underset{2}{\curlyvee} \cdots \underset{2}{\curlyvee} a_{j_\ell}) \tag{7.5}$$

where the summation is over unshuffles $(\mathbf{i};\mathbf{j}) = (i_1 < \cdots < i_k; j_1 < \cdots < j_\ell)$ of $\underline{n}$ and $\underset{1}{\curlyvee}$ denotes $E_{1,1}$. In particular, $dE_{1,1}(a; b) = dE_{1,1}(b; a) = ab - ba$ and $d(a \underset{2}{\curlyvee} b) = E_{1,1}(a; b) + E_{1,1}(b; a)$ for $a, b \in V^{0,*}$. It is straightforward to check that $H(R^{<0}H^*, d) = 0$ (or see the argument for an analogous resolution, denoted by $R_\delta H$, in [18]). Set $E_{1,1}(a; a) = 0$ for $a \in V^{0,*}$ and extend the operations $E_{k,1}(a_1, \ldots, a_k; b)$, $k \geqslant 1$, by formula (7.2) on RH for any $b \in RH$. Thus, ρ becomes an hga map too. Consequently, $HH(\rho)$ is an isomorphism by Proposition 4.

Since $Sq_1[z] = 0$, denote by $\gamma(z; z) \in C^{2n-2}(Y; \Bbbk)$ a cochain such that

$$d\gamma(z; z) = z \smile_1 z. \tag{7.6}$$

Next define a dga map $f : (RH, d) \to C^*(Y; \Bbbk)$ as follows. First define it on V and then extend multiplicatively. On $V^{0,*}$: by choosing cocycles $f^0 : U^* \to C^*(Y; \Bbbk)$; on $\mathcal{T}$: by $f(a_1 \underset{2}{\curlyvee} a_2) = f^0 a_1 \smile_2 f^0 a_2$, where $\smile_2$ denotes Steenrod's cochain operation, and extend inductively for $a_1 \underset{2}{\curlyvee} \cdots \underset{2}{\curlyvee} a_n$, $n \geqslant 3$; such an extension has no obstructions, since a cocycle in $C^*(Y)$ written by cochain operations in distinct variables is cohomologous to zero; on $\mathcal{E}$: for $E_{k,1}(a_1, \ldots, a_k; b)$, $k \geqslant 1$, with $a_i \in RH$ and $b \in V^{0,*}$ set

$$fE_{k,1}(a_1, \ldots, a_k; b) = \begin{cases} E_{2,1}(fa_1, fa_2; fb) - fa_1 \cdot \gamma(fa_2; fb) - \gamma(fa_1; fb) \cdot fa_2, & k = 2, \\ E_{k,1}(fa_1, \ldots, fa_k; fb), & k \neq 2, \end{cases}$$

for $b = E_{\ell,1}(b_1, \ldots, b_\ell; c)$ and $b' = E_{\ell-1,1}(b_1, \ldots, b_{\ell-1}; c)$, $b'' = E_{\ell-1,1}(b_2, \ldots, b_\ell; c)$ set

$$fE_{k,1}(a_1, \ldots, a_k; b) = \begin{cases} E_{k,1}(fa_1, \ldots, fa_k; fb) + (-1)^k E_{k-1,1}(fa_1, \ldots, fa_{k-1}; fb_1) \cdot \gamma(fa_k; fc) \\ \quad + (-1)^k E_{k-1,1}(fa_1, \ldots, fa_{k-1}; fc) \cdot \gamma(fa_k; fb_1) - \gamma(fa_1; fb_1) \cdot E_{k-1,1}(fa_2, \ldots, fa_k; fc) \\ \quad - \gamma(fa_1; fc) \cdot E_{k-1,1}(fa_2, \ldots, fa_k; fb_1), \quad \ell = 1, \\ E_{k,1}(fa_1, \ldots, fa_k; fb) + (-1)^k E_{k-1,1}(fa_1, \ldots, fa_{k-1}; fb') \cdot \gamma(fa_k; fb_\ell) \\ \quad - \gamma(fa_1; fb_1) \cdot fE_{k-1,1}(a_2, \ldots, a_k; b''), \quad \ell \geqslant 2, \end{cases}$$

where we assume $\gamma(a; b) = 0$ unless $a = b$ with $a \in V^{0,*}$ in which case γ is defined by (7.6); and, finally, for $b \in \mathcal{T}$, set

$$f E_{k,1}(a_1, \ldots, a_k; b) = E_{k,1}(f a_1, \ldots, f a_k; f b).$$

Define maps $s_{k,1} : RH^{\otimes k+1} \to C^*(Y; \Bbbk)$, $k \geqslant 1$, of degree -1 first for $a_1 \otimes \cdots \otimes a_k \otimes b \in RH^{\otimes k} \otimes V$ by

$$s_{k,1}(a_1 \otimes \cdots \otimes a_k \otimes b) = \begin{cases} \gamma(b; b), & k = 1, \ a_1 = b \in V^{0,*}, \\ 0, & \text{otherwise} \end{cases}$$

and then extend them on whole $RH^{\otimes k+1}$ by formula (7.4). It is immediate to verify that f and $s = \{s_{k,1}\}_{k \geqslant 1}$ satisfy the hypotheses of Proposition 4 and, consequently, $HH(f)$ is an isomorphism.

Example 1. Let $A = \Bbbk[x, y]$, $|x| = |y| = 2$, $B = T(\bar{x}, \bar{y}, z)/\{\bar{x}^2, \bar{y}^2\}$, $|\bar{x}| = |\bar{y}| = |z| = 1$, $dz = \bar{x}\bar{y} + \bar{y}\bar{x}$. Take $(A, 0) \otimes (B, d)$ and set $h(z) = x$ to obtain the dga $C = (A \otimes B, d^\otimes + h)$. It is easy to see that the spectral sequence of C is collapsed, and, consequently, its E_∞-term is isomorphic as algebras with the E_∞-term of the Serre spectral sequence of the free loop fibration with the base $Y = \mathbb{C}P^\infty \times \mathbb{C}P^\infty$. However, $H^*(C)$ is isomorphic only additively with $H^*(\Lambda Y)$.

Acknowledgements

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