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ON THE HOMOTOPY CLASSIFICATION OF MAPS

SAMSON SANEBLIDZE

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To Nodar Berikashvili

Abstract

We establish certain conditions which imply that a map $f : X \to Y$ of topological spaces is null homotopic when the induced integral cohomology homomorphism is trivial; one of them is: $H^*(X)$ and $\pi_*(Y)$ have no torsion and $H^*(Y)$ is polynomial.

1. Introduction

We give certain classification theorems for maps via induced cohomology homomorphism. Such a classification is based on a new aspects of obstruction theory to the section problem in a fibration beginning in [**4**], [**5**] and developed in some directions in [**24**], [**25**]. Given a fibration $F \to E \xrightarrow{\xi} X$, the obstructions to the section problem of ξ naturally lay in the groups $H^{i+1}(X;\pi_i(F)), i \geqslant 0$. A basic method here is to use the Hurewicz homomorphism $u_i : \pi_i(F) \to H_i(F)$ for passing the above obstructions into the groups $H^{i+1}(X;H_i(F)), i \geqslant 0$. In particular, this suggests the following condition on a fibration: The induced homomorphism

$$(1.1)_m \qquad u^* : H^{i+1}(X;\pi_i(F)) \to H^{i+1}(X;H_i(F)),\ 1 \leqslant i < m,$$

is an inclusion (assuming $u_1 : \pi_1(F) \to H_1(F)$ is an isomorphism). Note also that the idea of using the Hurewicz map in the obstruction theory goes back to the paper [**23**]. (Though its main result was erroneous, it became one crucial point for applications of characteristic classes (see [**7**]).)

For the homotopy classification of maps $X \to Y$, the space F in $(1.1)_m$ is replaced by ΩY and we establish the following statements. Below all topological spaces are assumed to be path connected (hence, Y is also simply connected) and the ground coefficient ring is the integers $\mathbb{Z}$. Given a commutative graded algebra (cga) H^* and an integer $m \geqslant 1$, we say that H^* is *m-relation free* if H^i is torsion free for $i \leqslant m$ and also there is no multiplicative relation in H^i for $i \leqslant m+1$; in particular, $H^{2i-1} = 0$ for $1 \leqslant i \leqslant [\frac{m+2}{2}]$. We also allow $m = \infty$ for H to be polynomial on even degree generators.

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Theorem 1. *Let $f : X \to Y$ be a map such that the pair $(X, \Omega Y)$ satisfies* $(1.1)_m$*, X is an m-dimensional polyhedron and $H^*(Y)$ is m-relation free. Then f is null homotopic if and only if*

$$0 = H^*(f) : H^*(Y) \to H^*(X).$$

Theorem 2. *Let X and Y be spaces such that the Hurewicz map $u_i : \pi_i(\Omega Y) \to H_i(\Omega Y)$ is an inclusion for $1 \leqslant i < m$, and* $\mathrm{Tor}\big(H^{i+1}(X), H_i(\Omega Y)/\pi_i(\Omega Y)\big) = 0$ *when $\pi_i(\Omega Y) \neq 0$, X is an m-dimensional polyhedron and $H^*(Y)$ is m-relation free. Then a map $f : X \to Y$ is null homotopic if and only if*

$$0 = H^*(f) : H^*(Y) \to H^*(X).$$

Theorem 3. *Let X be an m-dimensional polyhedron and G a topological group such that $\pi_i(G)$ is torsion free for $1 \leqslant i < m$, and* $\mathrm{Tor}\big(H^{i+1}(X), \mathrm{Coker}\, u_i\big) = 0$, *$u_i : \pi_i(G) \to H_i(G)$ when $\pi_i(G) \neq 0$. Suppose that the cohomology algebra $H^*(BG)$ of the classifying space BG is m-relation free. Then a map $f : X \to BG$ is null homotopic if and only if*

$$0 = H^*(f) : H^*(BG) \to H^*(X).$$

In fact the two last Theorems follow from the first one, since their hypotheses imply $(1.1)_m$, too. A main example of G in Theorem 3 is the unitary group $U(n)$ with $m = 2n$, since u_{2i} is a trivial inclusion and u_{2i-1} is an inclusion given by multiplication by the integer $(i-1)!$ for $1 \leqslant i \leqslant n$. A $U(n)$-principal fibre bundle over X is classified by a map $X \to BU(n)$. Suppose that all its Chern classes are trivial, then $H^*(f) = 0$ and by Theorem 3, f is null homotopic. Therefore the $U(n)$-principal fibre bundle is trivial. Thus, we have in fact deduced the following statement, the main result of [**22**] (compare also [**29**]).

Corollary 1. *Let ξ be a $U(n)$-principal fibre bundle over X with $\dim X \leqslant 2n$ and the only torsion in $H^{2i}(X)$ is relatively prime to $(i-1)!$. Then ξ is trivial if and only if the Chern classes $c_k(\xi) = 0$ for $1 \leqslant k \leqslant n$.*

While the proof of this statement in [**22**] does not admit an immediate generalization for an infinite dimensional X, Theorem 3 does by taking $m = \infty$. Furthermore, for $G = U$ and $X = BU$ recall that $[BU, BU]$ is an abelian group, so we get that two maps $f, g : BU \to BU$ are homotopic if and only if $H^*(f) = H^*(g) : H^*(BU; \mathbb{Q}) \to H^*(BU; \mathbb{Q})$ (compare [**14**], [**21**]). Note also that when $m = \infty$ in Theorem 3, $H^*(Y)$ must have infinitely many polynomial generators (e.g. $Y = BU, BSp$) as it follows from the solution of the Steenrod problem for finitely generated polynomial rings [**1**] (the underlying spaces do not have torsion free homotopy groups in all degrees).

Finally, note that beside obstruction theory we apply a main ingredient of the proof of Theorem 1 is an explicit form of minimal multiplicative (non-commutative) resolution of an m-relation free cga (of a polynomial algebra when $m = \infty$) in total degrees $\leqslant m$ (compare [**24**], [**26**]). Namely, the generator set of the resolution in the above range only consists of monomials formed by $\smile_1$ products. Remark that the idea of using $\smile_1$ product when dealing with polynomial cohomology, especially in the context of homogeneous spaces, has been realized by several authors [**17**], [**9**], [**20**], [**13**] (see also [**18**] for further references).

In sections 2 and 3 we recall certain basic definitions and constructions, including the functor $D(X;H_*)$ [**2**], [**3**], for the aforementioned obstruction theory, and in section 4 prove Theorems 1-3.

2. Functor D(X;H)

Given a bigraded differential algebra $A=\{A^{i,j}\}$ with $d:A^{i,j}\to A^{i+1,j}$ and total degree $n=i+j$, let $D(A)$ be the set [**3**] defined by $D(A)=M(A)/G(A)$ where

$$\begin{aligned} M(A) &= \{a\in A^1 \,|\, da=-aa,\ a=a^{2,-1}+a^{3,-2}+\cdots\},\\ G(A) &= \{p\in A^0 \,|\, p=1+p^{1,-1}+p^{2,-2}+\cdots\}, \end{aligned}$$

and the action $M(A)\times G(A)\to M(A)$ is given by the formula

$$a*p=p^{-1}ap+p^{-1}dp. \tag{2.1}$$

In other words, two elements $a,b\in M(A)$ are on the same orbit if there is $p\in G(A)$, $p=1+p'$, with

$$b-a=ap'-p'b+dp'. \tag{2.2}$$

Note that an element $a=\{a^{*,*}\}$ from $M(A)$ is of total degree 1 and referred to as *twisting*; we usually suppress the second degree below. There is a distinguished element in the set $D(A)$, the class of $0\in A$, and denoted by the same symbol.

There is simple but useful (cf. [**24**])

Proposition 1. *Let $f,g:A^{*,*}\to B^{*,*}$ be two dga maps that preserve the bigrading. If they are (f,g)-derivation homotopic via $s:A^{i,j}\to B^{i-1,j}$, i.e., $f-g=sd+ds$ and $s(ab)=(-1)^{|a|}fasb+sagb$, then $D(f)=D(g):D(A)\to D(B)$.*

Proof. Given $a\in M(A)$, apply the (f,g)-derivation homotopy s to get $fa-ga=dsa+sda=dsa+s(-aa)=dsa+fasa-saga$. From this we deduce that fa and ga are equivalent by (2.2) for $p'=-sa$. □

Another useful property of D is fixed by the following comparison theorem [**2**], [**3**]:

Theorem 4. *If $f:A\to B$ is a cohomology isomorphism, then $D(f):D(A)\to D(B)$ is a bijection.*

For our purposes the main example of $D(A)$ is the following (cf. [**2**], [**3**])

Example 1. *Fix a graded (abelian) group H_*. Let*

$$\rho:(R_{\geqslant 0}H_q,\partial^R)\to H_q,\ \ \partial^R:R_iH_q\to R_{i-1}H_q,$$

be its free group resolution. Form the bigraded Hom *complex*

$$(\mathcal{R}^{*,*},d^R)=\big(Hom(RH_*,RH_*),d^R\big),\qquad d^R:\mathcal{R}^{s,t}\to\mathcal{R}^{s+1,t};$$

an element $f\in\mathcal{R}^{*,*}$ has bidegree (s,t) if $f:R_jH_q\to R_{j-s}H_{q-t}$. Note also that $\mathcal{R}^{*,*}$ becomes a dga with respect to the composition product.

Given a topological space X, consider the dga

$$(\mathcal{H},\nabla)=(C^*(X;\mathcal{R}),\nabla=d^C+d^R)$$

which is bigraded via $\mathcal{H}^{r,t} = \prod_{r=i+j} C^i(X; \mathcal{R}^{j,t})$. *Thus we get*

$$\mathcal{H} = \{\mathcal{H}^n\}, \qquad \mathcal{H}^n = \prod_{n=r+t} \mathcal{H}^{r,t}, \qquad \nabla : \mathcal{H}^{r,t} \to \mathcal{H}^{r+1,t}.$$

We refer to r *as the* perturbation *degree which is mainly exploited by inductive arguments below. For example, for a twisting cochain* $h \in M(\mathcal{H})$, *we have*

$$h = h^2 + \cdots + h^r + \cdots, \qquad h^r \in \mathcal{H}^{r,1-r},$$

satifying the following sequence of equalities:

$$\nabla(h^2) = 0, \quad \nabla(h^3) = -h^2h^2, \quad \nabla(h^4) = -h^2h^3 - h^3h^2, \dots . \tag{2.3}$$

Define

$$D(X; H_*) = D(\mathcal{H}, \nabla).$$

Then $D(X; H_*)$ *becomes a functor on the category of topological spaces and continuous maps to the category of pointed sets.*

Example 2. *Given two dga's* B^* *and* $C^{*,*}$ *with* $d^B : B^i \to B^{i+1}$ *and* $d_1^C : C^{j,t} \to C^{j+1,t}$, $d_2^C = 0$, *let* $A = B\hat{\otimes}C$. *View* (A, d) *as bigraded via* $A = \{A^{r,t}, d\}$, $A^{r,t} = \prod_{r=i+j} B^i \otimes C^{j,t}$, $d = d^B \otimes 1 + 1 \otimes d_1^C$. *Note also that the dga* $(\mathcal{H}, \nabla)$ *in the previous example can also be viewed as a special case of the above tensor product algebra by setting* $B^* = C^*(X)$ *and* $C^{*,*} = \mathcal{R}^{*,*}$.

3. Predifferential $d(\xi)$ of a fibration

Let $F \to E \xrightarrow{\xi} X$ be a fibration. In [**2**] a unique element of $D(X; H_*(F))$ is naturally assigned to ξ; this element is denoted by $d(\xi)$ and referred to as the *predifferential* of ξ. The naturalness of $d(\xi)$ means that for a map $f : Y \to X$,

$$d(f(\xi)) = D(f)(d(\xi)), \tag{3.1}$$

where $f(\xi)$ denotes the induced fibration on Y.

Originally $d(\xi)$ appeared in homological perturbation theory for measuring the non-freeness of the Brown-Hirsch model: First, in [**11**] G. Hirsch modified E. Brown's twisting tensor product model $(C_*(X) \otimes C_*(F), d_\phi) \to (C_*(E), d_E)$ [**6**], [**8**] by replacing the chains $C_*(F)$ by its homology $H_*(F)$ provided the homology is a free module. In [**2**] the Hirsch model was extended for arbitrary $H_*(F)$ by replacing it by a free module resolution $RH_*(F)$ to obtain $(C_*(X) \otimes RH_*(F), d_h)$ in which $d_h = d_X \otimes 1 + 1 \otimes d_F + - \cap h$ and h is just an element of $M(\mathcal{H})$ in Example 1 with $H_* = H_*(F)$. Furthermore, to an isomorphism $p : (C_*(X) \otimes RH_*(F), d_h) \to (C_*(X) \otimes RH_*(F), d_{h'})$ between two such models answers an equivalence relation $h \sim_p h'$ in $M(\mathcal{H})$, and the class of h in $D(X; H_*(F))$ is identified as $d(\xi)$. More precisely, we recall some basic constructions for the definition of $d(\xi)$ we need for the obstruction theory in question.

For convenience, assume that X is a polyhedron and that $\pi_1(X)$ acts trivially on $H_*(F)$. Then ξ defines the following colocal system of chain complexes over X :

To each simplex $\sigma \in X$ is assigned the singular chain complex $(C_*(F_\sigma), \gamma_\sigma)$ of the space $F_\sigma = \xi^{-1}(\sigma)$:

$$X \ni \sigma \longrightarrow (C_*(F_\sigma), \gamma_\sigma) \subset (C_*(E), d_E),$$

and to a pair $\tau \subset \sigma$ of simplices an induced chain map

$$C_*(F_\tau) \to C_*(F_\sigma).$$

Set $\mathcal{C}_\sigma = \{\mathcal{C}_\sigma^{s,t}\}$, $\mathcal{C}_\sigma^{s,t} = \mathrm{Hom}^{s,t}(R_*H_*(F), C_*(F_\sigma))$ where C_* is regarded as bigraded via $C_{0,*} = C_*, C_{i,*} = 0,\ i \neq 0$, and $f : R_jH_q(F) \to C_{j-s,q-t}(F_\sigma)$ is of bidegree (s,t). Then we obtain a colocal system of cochain complexes $\mathcal{C} = \{\mathcal{C}_\sigma^{*,*}\}$ on X. Define $\mathcal{F}$ as the simplicial cochain complex $C^*(X;\mathcal{C})$ of X with coefficients in the colocal system $\mathcal{C}$. Then

$$\mathcal{F} = \{\mathcal{F}^{i,j,t}\}, \quad \mathcal{F}^{i,j,t} = C^i(X;\mathcal{C}^{j,t}).$$

Furthermore, obtain the bicomplex $\mathcal{F} = \{\mathcal{F}^{r,t}\}$ via

$$\mathcal{F}^{r,t} = \prod_{r=i+j} \mathcal{F}^{i,j,t},\ \delta : \mathcal{F}^{r,t} \to \mathcal{F}^{r+1,t},\ \gamma : \mathcal{F}^{r,t} \to \mathcal{F}^{r,t+1},\ \delta = d^C + \partial^R,\ \gamma = \{\gamma_\sigma\},$$

and finally set

$$\mathcal{F} = \{\mathcal{F}^m\}, \quad \mathcal{F}^m = \prod_{m=r+t} \mathcal{F}^{r,t}.$$

We have a natural dg pairing

$$(\mathcal{F}, \delta + \gamma) \otimes (\mathcal{H}, \nabla) \to (\mathcal{F}, \delta + \gamma)$$

defined by $\smile$ product on $C^*(X;-)$ and the obvious pairing $\mathcal{C}_\sigma \otimes \mathcal{R} \to \mathcal{C}_\sigma$ in coefficients; in particular we have $\gamma(fh) = \gamma(f)h$ for $f \otimes h \in \mathcal{F} \otimes \mathcal{H}$. Denote $\mathcal{R}_\# = Hom(RH_*(F), H_*(F))$ and define

$$(\mathcal{F}_\#, \delta_\#) := (H(\mathcal{F}, \gamma), \delta_\#) = (C^*(X;\mathcal{R}_\#), \delta_\#).$$

Clearly, the above pairing induces the following dg pairing

$$(\mathcal{F}_\#, \delta_\#) \otimes (\mathcal{H}, \nabla) \to (\mathcal{F}_\#, \delta_\#).$$

In other words, this pairing is also defined by $\smile$ product on $C^*(X;-)$ and the pairing $\mathcal{R}_\# \otimes \mathcal{R} \to \mathcal{R}_\#$ in coefficients. Note that ρ induces an epimorphism of chain complexes

$$\rho^* : (\mathcal{H}, \nabla) \to (\mathcal{F}_\#, \delta_\#).$$

In turn, ρ^* induces an isomorphism in cohomology.

Consider the following equation

$$(\delta + \gamma)(f) = fh \tag{3.2}$$

with respect to a pair $(h, f) \in \mathcal{H}^1 \times \mathcal{F}^0$,

$$\begin{aligned} h &= h^2 + \cdots + h^r + \cdots, \quad h^r \in \mathcal{H}^{r,1-r}, \\ f &= f^0 + \cdots + f^r + \cdots, \quad f^r \in \mathcal{F}^{r,-r}, \end{aligned}$$

satisfying the initial conditions:

$$\begin{aligned} &\nabla(h) = -hh \\ &\gamma(f^0) = 0, \qquad [f^0]_\gamma = \rho^*(1) \in \mathcal{F}^{0,0}_{\#}, \quad 1 \in \mathcal{H}. \end{aligned}$$

Let (h, f) be a solution of the above equation. Then $d(\xi) \in D(X; H_*(F))$ is defined as the class of h. Moreover, the transformation of h by (2.1) is extended to pairs (h, f) by the map

$$(M(\mathcal{H}) \times \mathcal{F}^0) \times (G(\mathcal{H}) \times \mathcal{F}^{-1}) \to M(\mathcal{H}) \times \mathcal{F}^0$$

given for $((h, f), (p, s)) \in (M(\mathcal{H}) \times \mathcal{F}^0) \times (G(\mathcal{H}) \times \mathcal{F}^{-1})$ by the formula

$$(h, f) * (p, s) = (h * p\,,\ fp + s(h * p) + (\delta + \gamma)(s))\,. \tag{3.3}$$

We have that a solution (h, f) of the equation exists and is unique up to the above action. Therefore, $d(\xi)$ is well defined.

Note that action (3.3) in particular has a property that if $(\bar{h}, \bar{f}) = (h, f) * (p, s)$ and $h^r = 0$ for $2 \leqslant r \leqslant n$, then in view of (2.2) one gets the equalities

$$\bar{h}^{n+1} = h * (1 + p^n) = h^{n+1} + \nabla(p^n). \tag{3.4}$$

3.1. Fibrations with $d(\xi) = 0$

The main fact of this subsection is the following theorem from [**4**]:

Theorem 5. *Let* $F \to E \xrightarrow{\xi} X$ *be a fibration such that* (X, F) *satisfies* $(1.1)_m$. *If the restriction of* $d(\xi) \in D(X; H_*(F))$ *to* $d(\xi)|_{X^m} \in D(X^m; H_*(F))$ *is zero, then* ξ *has a section on the* m*-skeleton of* X. *The case of* $m = \infty$, *i.e.,* $d(\xi) = 0$, *implies the existence of a section on* X.

Proof. Given a pair $(h, f) \in \mathcal{H} \times \mathcal{F}$, let (h_{tr}, f_{tr}) denote its component that lies in

$$C^*(X; Hom(H_0(F), RH_*(F))) \times C^*(X; Hom(H_0(F), C_*(F_\sigma))).$$

Below (h_{tr}, f_{tr}) is referred to as the *transgressive* component of (h, f). Observe that since $RH_0(F) = H_0(F) = \mathbb{Z}$, we can view (h^{r+1}_{tr}, f^r_{tr}) as a pair of cochains laying in $C^{>r}(X; RH_r(F)) \times C^r(X; C_r(F_\sigma))$. Such an interpretation allows us to identify a section $\chi^r : X^r \to E$ on the r-skeleton $X^r \subset X$ with a cochain, denoted by c^r_χ, in $C^r(X; C_r(F_\sigma))$ via $c^r_\chi(\sigma) = \chi^r|_\sigma : \Delta^r \to F_\sigma \subset E$, $\sigma \subset X^r$ is an r-simplex, $r \geqslant 0$.

The proof of the theorem just consists of choosing a solution (h, f) of (3.2) so that the transgressive component $f_{tr} = \{f^r_{tr}\}_{r \geqslant 0}$ is specified by $f^r_{tr} = c^r_\chi$ with χ a section of ξ. Indeed, since F is path connected, there is a section χ^1 on X^1; consequently, we get the pairs $(0, f^0_{tr}) := (0, c^0_\chi)$ and $(0, f^1_{tr}) := (0, c^1_\chi)$ with $\gamma(f^1_{tr}) = \delta(f^0_{tr})$. Then $\delta(f^1_{tr}) \in C^2(X; C_1(F))$ is a γ-cocycle and $[\delta(f^1_{tr})]_\gamma \in C^2(X; H_1(F))$ becomes the obstruction cocycle $c(\chi^1) \in C^2(X; \pi_1(F))$ for extending of χ^1 on X^2; moreover, one can choose h^2_{tr} to be satisfying $\rho^*(h^2_{tr}) = [\delta(f^1_{tr})]_\gamma$ (since ρ^* is an epimorphism and a weak equivalence).

Suppose by induction that we have constructed a solution (h, f) of (3.2) and a section χ^n on X^n such that $h^r = 0$ for $2 \leqslant r \leqslant n$, $f^n_{tr} = c^n_\chi$ and

$$\rho^*(h^{n+1}_{tr}) = [\delta(f^n_{tr})]_\gamma \in C^{n+1}(X; H_n(F)).$$

In view of (2.3) we have $\nabla(h^{n+1}) = 0$ and from the above equality immediately follows that

$$u^{\#}(c(\chi^n)) = \rho^*(h_{tr}^{n+1})$$

in which $c(\chi^n) \in C^{n+1}(X; \pi_n(F))$ is the obstruction cocycle for extending of χ^n on X^{n+1} and $u^{\#} : C^{n+1}(X; \pi_n(F)) \to C^{n+1}(X; H_n(F))$.

Since $d(\xi)|_{X^m} = 0$, there is $p \in G(\mathcal{H})$ such that $(h * p)|_{X^m} = 0$; in particular, $(h * p)^{n+1} = 0 \in \mathcal{H}^{n+1,-n}$ and in view of (3.4) we establish the equality $h^{n+1} = -\nabla(p^n)$, i.e., $[h^{n+1}] = 0 \in H^*(\mathcal{H}, \nabla)$; in particular, $[h_{tr}^{n+1}] = 0 \in H^{n+1}(X; H_n(F))$. Consequently, $[u^{\#}(c(\chi^n))] = 0 \in H^{n+1}(X; H_n(F))$. Since $(1.1)_n$ is an inclusion induced by $u^{\#}$, $[c(\chi^n)] = 0 \in H^{n+1}(X; \pi_n(F))$. Therefore, we can extend χ^n on X^{n+1} without changing it on X^{n-1} in a standard way. Finally, put $f_{tr}^{n+1} = c_{\chi}^{n+1}$ and choose a ∇-cocycle h_{tr}^{n+2} satisfying $\rho^*(h_{tr}^{n+2}) = [\delta(f_{tr}^{n+1})]_{\gamma}$. The induction step is completed. □

4. Proof of Theorems 1, 2 and 3

First we recall the following application of Theorem 5 ([**4**])

Theorem 6. *Let $f : X \to Y$ be a map such that X is an m-polyhedron and the pair $(X, \Omega Y)$ satisfies $(1.1)_m$. If $0 = D(f) : D(Y; H_*(\Omega Y)) \to D(X; H_*(\Omega Y))$, then f is null homotopic.*

Proof. Let $\Omega Y \to PY \xrightarrow{\pi} Y$ be the path fibration and $f(\pi)$ the induced fibration. It suffices to show that $f(\pi)$ has a section. Indeed, (3.1) together with $D(f) = 0$ implies $d(f(\pi)) = 0$, so Theorem 5 guaranties the existence of the section. □

Now we are ready to prove the theorems stated in the introduction. Note that just below we shall heavily use multiplicative, non-commutative resolutions of cga's that are enriched with $\smile_1$ products. Namely, given a space Z, recall its filtered model $f_Z : (RH(Z), d_h) \to C^*(Z)$ [**24**], [**26**] in which the underlying differential (bi)graded algebra $(RH(Z), d)$ is a non-commutative version of Tate-Jozefiak resolution of the cohomology algebra $H^*(Z)$ ([**28**], [**15**]), while h denotes a perturbation of d similar to [**10**]. Moreover, given a map $X \to Y$, there is a dga map $RH(f) : (RH(Y), d_h) \to (RH(X), d_h)$ (not uniquely defined!) such that the following diagram

$$\begin{array}{ccc} (RH(Y), d_h) & \xrightarrow{RH(f)} & (RH(X), d_h) \\ {\scriptstyle f_Y}\downarrow & & \downarrow{\scriptstyle f_X} \\ C^*(Y) & \xrightarrow{C(f)} & C^*(X) \end{array} \tag{4.1}$$

commutes up to (α, β)-derivation homotopy with $\alpha = C(f) \circ f_Y$ and $\beta = f_X \circ RH(f)$ (see, [**12**], [**24**]).

Proof of Theorem 1. The non-trivial part of the proof is to show that $H(f) = 0$ implies f is null homotopic. In view of Theorem 6 it suffices to show that $D(f) = 0$.

By (4.1) and Proposition 1 we get the commutative diagram of pointed sets

$$\begin{array}{ccc} D(\mathcal{H}_Y) & \stackrel{D(\mathcal{H}(f))}{\longrightarrow} & D(\mathcal{H}_X) \\ {\scriptstyle D(f_Y)}\downarrow & & \downarrow{\scriptstyle D(f_X)} \\ D(Y;H_*(\Omega Y)) & \stackrel{D(f)}{\longrightarrow} & D(X;H_*(\Omega Y)) \end{array}$$

in which

$$\mathcal{H}_X = RH^*(X)\hat{\otimes}Hom(RH_*(\Omega Y)\,,RH_*(\Omega Y)),$$

$$\mathcal{H}_Y = RH^*(Y)\hat{\otimes}Hom(RH_*(\Omega Y)\,,RH_*(\Omega Y))$$

(see Example 2) and the vertical maps are induced by $f_X \otimes 1$ and $f_Y \otimes 1$; these maps are bijections by Theorem 4. Below we need an explicit form of $RH(f)$ to see that $H(f) = 0$ necessarily implies $RH(f)|_{V^{(m)}} = 0$ with $V^{(m)} = \bigoplus_{1\leqslant i+j\leqslant m} V^{i,j}$; hence, the restriction of the map $\mathcal{H}(f) := RH(f) \otimes 1$ to $RH^{(m)} \otimes 1$, $RH^{(m)} = \bigoplus_{1\leqslant i+j\leqslant m} R^iH^j(Y)$, is zero, and, consequently,

$$D(f_X)\circ D(\mathcal{H}(f)) = 0. \tag{4.2}$$

First observe that any multiplicative resolution $(RH, d) = (T(V^{*,*}), d)$, $V = \langle\mathcal{V}\rangle$, of a cga H admits a sequence of multiplicative generators, denoted by

$$a_1 \smile_1 \cdots \smile_1 a_{n+1} \in \mathcal{V}^{-n,*}, \quad a_i \in \mathcal{V}^{0,*}, \quad n \geqslant 1, \tag{4.3}$$

where $a_i \smile_1 a_j = (-1)^{(|a_i|+1)(|a_j|+1)} a_j \smile_1 a_i$ and $a_i \neq a_j$ for $i \neq j$. Furthermore, the expression $ab \smile_1 uv$ also has a sense by means of formally (successively) applying the Hirsch formula

$$c \smile_1 (ab) = (c \smile_1 a)b + (-1)^{|a|(|c|+1)} a(c \smile_1 b). \tag{4.4}$$

The resolution differential d acts on (4.3) by iterative application of the formula

$$d(a \smile_1 b) = da \smile_1 b - (-1)^{|a|} a \smile_1 db + (-1)^{|a|} ab - (-1)^{|a|(|b|+1)} ba.$$

Consequently, we get

$$d(a_1 \smile_1 \cdots \smile_1 a_n) = \sum_{(\mathbf{i};\mathbf{j})} (-1)^{\epsilon} (a_{i_1} \smile_1 \cdots \smile_1 a_{i_k}) \cdot (a_{j_1} \smile_1 \cdots \smile_1 a_{j_\ell})$$

where the summation is over unshuffles $(\mathbf{i};\mathbf{j}) = (i_1 < \cdots < i_k\,; j_1 < \cdots < j_\ell)$ of $\underline{n}$.

In the case of H to be m-relation free with a basis $U^i \subset H^i$, $i \leqslant m$, we have that the minimal multiplicative resolution RH of H can be built by taking $\mathcal{V}$ with $\mathcal{V}^{0,i} \approx \mathcal{U}^i, i \leqslant m$, and $\mathcal{V}^{-n,i}, n > 0$, to be the set consisting of monomials (4.3) for $1 \leqslant i - n \leqslant m$ (compare [**26**]). The verification of the acyclicity in the negative resolution degrees of RH restricted to the range $RH^{(m)}$ is straightforward (see also Remark 1). Regarding the map $RH(f)$, we can choose it on $RH^{(m)}$ as follows. Let $R_0H(f) : R_0H(Y) \to R_0H(X)$ be determined by $H(f)$ in an obvious way and then define $RH(f)$ for $a \in \mathcal{V}^{(m)}$ by

$$RH(f)(a) = \begin{cases} R_0H(f)(a), & a \in \mathcal{V}^{0,*}, \\ R_0H(f)(a_1) \smile_1 \cdots \smile_1 R_0H(f)(a_n), & a = a_1 \smile_1 \cdots \smile_1 a_{n+1}, \\ & a \in \mathcal{V}^{-n,*}, a_i \in \mathcal{V}^{0,*}, n \geqslant 1, \end{cases}$$

and extend to $RH^{(m)}$ multiplicatively. Furthermore, f_X and f_Y are assumed to be preserving the generators of the form (4.3) with respect to the right most association of $\smile_1$ products in question. Since h annihilates monomials (4.3) and the existence of formula (4.4) in a simplicial cochain complex, f_X and f_Y are automatically compatible with the differentials involved. Then the maps α and β in (4.1) also preserve $\smile_1$ products, and become homotopic by an (α,β)-derivation homotopy $s : RH(Y) \to C^*(X)$ defined as follows: choose s on $\mathcal{V}^{0,*}$ by $ds = \alpha - \beta$ and extend on $\mathcal{V}^{-n,*}$ inductively by

$$s(a_0 \smile_1 z_n) = -\alpha(a_0) \smile_1 s(z_n) + s(a_0) \smile_1 \beta(z_n) + s(z_n)s(a_0), \quad n \geqslant 1,$$

in which $z_1 = a_1$ and $z_k = a_1 \smile_1 \cdots \smile_1 a_k$ for $k \geqslant 2$, $a_i \in \mathcal{V}^{0,*}$. Clearly, $H(f) = 0$ implies $RH(f)|_{V^{(m)}} = 0$. Since (4.2), $D(f) = 0$ and so f is null homotopic by Theorem 6. Theorem is proved.

Remark 1. *Let $\mathcal{V}_n^{(m)}$ be a subset of $\mathcal{V}^{(m)}$ consisting of all monomials formed by the $\cdot$ and $\smile_1$ products evaluated on a string of variables $a_1, ..., a_n$. Then there is a bijection of $\mathcal{V}_n^{(m)}$ with the set of all faces of the permutahedron P_n ([**19**], [**27**]) such that the resolution differential d is compatible with the cellular differential of P_n (compare [**16**]). In particular, the monomial $a_1 \smile_1 \cdots \smile_1 a_n$ is assigned to the top cell of P_n, while the monomials $a_{\sigma(1)} \cdots a_{\sigma(n)}, \sigma \in S_n$, to the vertices of P_n (see Fig. 1 for $n = 3$). Thus, the acyclicity of P_n immediately implies the acyclicity of $RH^{(m)}$ in the negative resolution degrees as desired.*

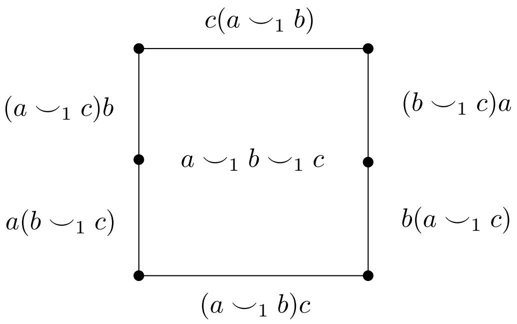

Figure 1. Geometrical interpretation of some syzygies involving $\smile_1$ product as homotopy for commutativity in the resolution RH.

Remark 2. *An example provided by the Hopf map $f : S^3 \to S^2$ shows that the implication $H(f) = 0 \Rightarrow RH(f)|_{V^{(k)}} = 0$, $k < m$ for $RH(f)$ making (4.1) commutative up to (α,β)-derivation homotopy is not true in general. More precisely, let $x \in R^0H^2(S^2)$ and $y \in R^0H^3(S^3)$ with $\rho x \in H^2(S^2)$ and $\rho y \in H^3(S^3)$ to be the generators, and let $x_1 \in R^{-1}H^4(S^2)$ with $dx_1 = x^2$. Then $s(x^2) = \alpha(x)s(x)$ is a cocycle in $C^3(S^3)$ with $d_{S^3}s(x) = \alpha(x)$ (since $\beta = 0$) and $[\alpha(x)s(x)] = \rho y$. Consequently, while $H(f) = 0 = R^0H(f)$, a map $RH(f) : RH(S^2) \to RH(S^3)$ required in (4.1) has a non-trivial component increasing the resolution degree: Namely, $R^{-1}H^4(S^2) \to R^0H^3(S^3)$, $x_1 \to y$.*

Proof of Theorem 2. The conditions that $u_i : \pi_i(\Omega Y) \to H_i(\Omega Y)$ is an inclusion and $\mathrm{Tor}\big(H^{i+1}(X), H_i(\Omega Y)/\pi_i(\Omega Y)\big) = 0$ for $1 \leqslant i < m$, immediately implies $(1.1)_m$. So the theorem follows from Theorem 1.

Proof of Theorem 3. Since the homotopy equivalence $\Omega BG \simeq G$, the conditions of Theorem 2 are satisfied: Indeed, there is the following commutative diagram

$$\begin{array}{ccc} \pi_k(G) & \xrightarrow{u_k} & H_k(G) \\ i_\pi \downarrow & & \downarrow i_H \\ \pi_k(G) \otimes \mathbb{Q} & \xrightarrow{u_k \otimes 1} & H_k(G) \otimes \mathbb{Q} \end{array}$$

where i_π, i_H and $u_k \otimes 1$ are the standard inclusions (the last one is a consequence of a theorem of Milnor-Moore). Consequently, $u_k : \pi_k(\Omega BG) \to H_k(\Omega BG), k < m$, is an inclusion, too. Theorem is proved. □

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Samson Saneblidze
`sane@rmi.acnet.ge`

A. Razmadze Mathematical Institute
Department of Geometry and Topology
M. Aleksidze st., 1
0193 Tbilisi, Georgia