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ON THE BETTI NUMBERS OF A LOOP SPACE

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(*communicated by James Stasheff*)

Abstract

Let A be a special homotopy G-algebra over a commutative unital ring $\Bbbk$ such that both $H(A)$ and $\mathrm{Tor}_i^A(\Bbbk,\Bbbk)$ are finitely generated $\Bbbk$-modules for all i, and let $\tau_i(A)$ be the cardinality of a minimal generating set for the $\Bbbk$-module $\mathrm{Tor}_i^A(\Bbbk,\Bbbk)$. Then the set $\{\tau_i(A)\}$ is unbounded if and only if $\tilde{H}(A)$ has two or more algebra generators. When $A = C^*(X;\Bbbk)$ is the simplicial cochain complex of a simply connected finite CW-complex X, there is a similar statement for the "Betti numbers" of the loop space ΩX. This unifies existing proofs over a field $\Bbbk$ of zero or positive characteristic.

To Tornike Kadeishvili and Mamuka Jibladze

1. Introduction

Let Y be a topological space, let $\Bbbk$ be a commutative ring with identity, and assume that the i^{th}-cohomology group $H^i(Y;\Bbbk)$ of Y is finitely generated as a $\Bbbk$-module. We refer to the cardinality of a minimal generating set of $H^i(Y;\Bbbk)$, denoted by $\beta_i(Y)$, as the *generalized* i^{th}*-Betti number of Y.*

Theorem 1. *Let X be a simply connected space. If $H^*(X;\Bbbk)$ is finitely generated as a $\Bbbk$-module and $H^*(\Omega X;\Bbbk)$ has finite type, then the set of generalized i^{th}-Betti numbers $\{\beta_i(\Omega X;\Bbbk)\}$ is unbounded if and only if $\tilde{H}^*(X;\Bbbk)$ has at least two algebra generators.*

Theorem 1 was proved by Sullivan [**11**] over fields of characteristic zero and by McCleary [**8**] over fields of positive characteristic. However, Theorem 1 is a consequence of the following more general algebraic fact: Let $A' = \{A'^i\}, i \geqslant 0$, with $A'^0 = \mathbb{Z}$, $A'^1 = 0$, be a torsion free graded abelian group endowed with a homotopy G-algebra (hga) structure. Then for $A = A' \otimes_{\mathbb{Z}} \Bbbk$ we have the following theorem whose proof appears in Section 4:

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Theorem 2. *Assume that $H^*(A)$ is finitely generated as a $\Bbbk$-module and that $Tor_*^A(\Bbbk,\Bbbk)$ has finite type. Let $\tau_i(A)$ denote the cardinality of a minimal generating set of $Tor_i^A(\Bbbk,\Bbbk)$. Then the set $\{\tau_i(A)\}$ is unbounded if and only if $\tilde{H}(A)$ has at least two algebra generators.*

Let $C^*(X;\Bbbk) = C^*(\mathrm{Sing}^1 X;\Bbbk)/C^{>0}(\mathrm{Sing}\, x;\Bbbk)$ in which $\mathrm{Sing}^1 X \subset \mathrm{Sing} X$ is the Eilenberg 1-subcomplex generated by the singular simplices that send the 1-skeleton of the standard n-simplex Δ^n to the base point x of X. To deduce Theorem 2 from Theorem 1, set $A = C^*(X;\Bbbk)$, and apply Proposition 2 below together with the filtered hga model $(RH(A), d_h) \to A$ of A (a special case of the filtered Hirsch algebra [**9**]). Let BA denote the bar construction of A. When $\tilde{H}(A)$ has at least two algebra generators, we construct two infinite sequences in the filtered model and take all possible $\smile_1$-products of their components to detect a submodule of $H^*(BA)$ at least as large as the polynomial algebra $\Bbbk[x,y]$.

Each of the sequences mentioned above can be thought of as generalizations of an infinite sequence (∞-*implications* of its first component) introduced by Browder [**1**]. Indeed, this work arose after writing down these special sequences in the hga resolution of a commutative graded algebra (cga) over the integers via formulas (3.2)–(3.4) below, at which point we realized that their construction mimics that of Massey symmetric products defined by Kraines [**7**] (see also [**9**]). In general, a sequence formed from Massey symmetric products is closely related to the one obtained from A_∞-operations in an A_∞-algebra defined by Stasheff [**10**] by restricting to the same variables in question. When a differential graded algebra (dga) A is free as a $\Bbbk$-module, the sequence of A_∞-operations on the homology $H(A)$ was constructed by Kadeishvili [**5**].

2. Some preliminaries and conventions

We adopt the notations and terminology of [**9**]. We fix a ground ring $\Bbbk$ with identity, a primary example of which is the integers $\mathbb{Z}$. Let $\mathbb{Z}_\Bbbk \subset \mathbb{Z}$ be the subset defined by

$$\mathbb{Z}_\Bbbk = \{\lambda \in \mathbb{Z} \mid \lambda_\Bbbk : \Bbbk \to \Bbbk,\ \kappa \to \lambda\kappa, \ \text{ is injective}\}.$$

Let $\mu \in \mathbb{Z} \setminus \mathbb{Z}_\Bbbk$ denote the smallest integer such that $\mu\kappa = 0$ for all $\kappa \in \Bbbk$. Thus if $\mu = 0$, $\mathbb{Z}_\Bbbk = \mathbb{Z} \setminus 0$ (e.g. $\Bbbk$ is a field of characteristic zero).

A (positively) graded algebra A is 1-reduced if $A^0 = \Bbbk$ and $A^1 = 0$. For a general definition of an homotopy Gerstenhaber algebra (hga) $(A, d, \cdot, \{E_{p,q}\})_{p\geqslant 0,\, q=0,1}$ see [**3**], [**4**], [**6**]. The defining identities for an hga are the following: Given $k \geqslant 1$,

$$\begin{aligned} dE_{k,1}(a_1,...,a_k;b) \;=\; & \textstyle\sum_{i=1}^{k} (-1)^{\epsilon^a_{i-1}} E_{k,1}(a_1,...,da_i,...,a_k;b) \\ & + (-1)^{\epsilon^a_k} E_{k,1}(a_1,...,a_k;db) \\ & + \textstyle\sum_{i=1}^{k-1} (-1)^{\epsilon^a_i} E_{k-1,1}(a_1,...,a_ia_{i+1},...,a_k;b) \\ & + (-1)^{\epsilon^a_k + |a_k||b|} E_{k-1,1}(a_1,...,a_{k-1};b)\cdot a_k \\ & + (-1)^{|a_1|} a_1 \cdot E_{k-1,1}(a_2,...,a_k;b), \end{aligned} \tag{2.1}$$

$$E_{k,1}(a_1,...,a_k;b\cdot c)$$
$$=\sum_{i=0}^{k}(-1)^{|b|(\epsilon_i^a+\epsilon_k^a)}E_{i,1}(a_1,...,a_i;b)\cdot E_{k-i,1}(a_{i+1},...,a_k;c) \quad (2.2)$$

and

$$\sum_{\substack{k_1+\cdots+k_p=k\\ 1\leqslant p\leqslant k+\ell}}(-1)^{\epsilon}E_{p,1}\left(E_{k_1,\ell_1}(a_1,...,a_{k_1};b_1'),...,E_{k_p,\ell_p}(a_{k-k_p+1},...,a_k;b_p');c\right)$$
$$=E_{k,1}\left(a_1,...,a_k;E_{\ell,1}(b_1,...,b_\ell;c)\right),$$
$$b_i'\in\{1,b_1,..,b_\ell\},\quad \epsilon=\sum_{i=1}^{p}(|b_i'|+1)(\varepsilon_{k_i}^a+\varepsilon_k^a),\ b_i'\neq 1,$$
$$\varepsilon_i^a=|a_1|+\cdots+|a_i|+i. \quad (2.3)$$

A *morphism* $f:A\to A'$ of hga's is a dga map f commuting with all $E_{k,1}$.

Remark 1. *Note that we do not use axiom (2.3) in the sequel.*

Below we review the notion of an hga resolution of a cga as a special Hirsch algebra (the existence of such a resolution is proved in [**9**]). Given a cga H, its hga resolution is a multiplicative resolution

$$\rho:(R^*H^*,d)\to H^*,\quad RH=T(V),\quad V=\langle\mathcal{V}\rangle,$$

endowed with an hga structure

$$E_{k,1}:RH^{\otimes k}\otimes RH\to RH,\quad k\geqslant 1,$$

together with a decomposition of V such that $V^{*,*}=\mathcal{E}^{*,*}\oplus U^{*,*}$, where $\mathcal{E}^{*,*}=\{\mathcal{E}^{<0,*}_{p,q}\}$ is distinguished by an isomorphism of modules

$$E_{k,1}:\otimes_{r=1}^{k}R^{i_r}H^{k_r}\bigotimes V^{j,\ell}\xrightarrow{\approx}\mathcal{E}^{s-k\,,\,t}_{k,1}\subset V^{k-s,t},\quad (s,t)=\left(\sum_{r=1}^{k}i_r+j,\sum_{r=1}^{k}k_r+\ell\right).$$

Furthermore, if H is a $\mathbb{Z}$-algebra, its hga resolution (RH,d) is automatically endowed with two operations $\cup_2$ and $\smile_2$. The first operation $\cup_2$ appears because each cocycle $a\smile_1 a\in\mathcal{E}_{1,1}\cap R^{-1}H^{2j}$, where $a\in R^0H^{2j}$, is killed by some element in $R^{-2}H^{2j}$, denoted by $a\cup_2 a$. The second operation arises from the non-commutativity of $\smile_1$-product in the usual way, and satisfies Steenrod's formula for the $\smile_2$-cochain operation. These two operations are related to each other by the initial relations $a\smile_2 a=2a\cup_2 a$ and $a\smile_2 b=a\cup_2 b$, $a\neq b\in\mathcal{U}$ with $\langle\mathcal{U}\rangle=U$. Note also that $a\smile_2 a=a\cup_2 a=0$ for $a\in U$ of odd degree. In general, $U=\mathcal{T}\oplus\mathcal{N}$, with an element of $\mathcal{T}$ given by $a_1\cup_2\cdots\cup_2 a_n$, $a_i\in U$, $n\geqslant 2$. The action of the resolution differential d on elements of $\mathcal{T}$ such that $da_i=0$ is

$$d(a_1\cup_2\cdots\cup_2 a_n)$$
$$=\sum_{(\mathbf{i};\mathbf{j})}(-1)^{|a_{i_1}|+\cdots+|a_{i_k}|}(a_{i_1}\cup_2\cdots\cup_2 a_{i_k})\smile_1(a_{j_1}\cup_2\cdots\cup_2 a_{j_\ell}), \quad (2.4)$$

where we sum over all unshuffles $(\mathbf{i};\mathbf{j}) = (i_1 < \cdots < i_k\,; j_1 < \cdots < j_\ell)$ of $\underline{n}$ with $(a_{i_1},...,a_{i_k}) = (a_{i'_1},...,a_{i'_k})$ if and only if $\mathbf{i} = \mathbf{i}'$ and $\smile_1$ denotes $E_{1,1}$. In particular, for $a_1 = \cdots = a_n = a = a^{\cup_2 1}$ and $n \geqslant 2$ we get $da^{\cup_2 n} = \sum_{k+\ell=n} a^{\cup_2 k} \smile_1 a^{\cup_2 \ell}$, $k,\ell \geqslant 1$. And in general $d(a \smile_2 b) = nd(a \cup_2 b), n \geqslant 1$.

An hga resolution (RH, d) is *minimal* if

$$d(U) \subset \mathcal{E} + \mathcal{D} + \kappa{\cdot}V$$

where $\mathcal{D}^{*,*} \subset R^*H^*$ denotes the submodule of decomposables $RH^+{\cdot}RH^+$ and $\kappa \in \Bbbk$ is non-invertible; For example, $\kappa \in \mathbb{Z} \setminus \{-1, 1\}$ when $\Bbbk = \mathbb{Z}$ and $\kappa = 0$ when $\Bbbk$ is a field.

Let $K = \{K^j\}_{j\geqslant 3}$ with $K^j = \{a \in \mathcal{V}^{-1,j} \,|\, da = \lambda b,\ \lambda \neq \pm 1,\ b \in \mathcal{V}^{0,j}\}$. Note that a general form of a relation in (minimal) (RH, d) starting by variables $v_i \in K \cup \mathcal{V}^{0,*}$ is

$$du = \sum_{s\geqslant 1} \lambda_s P_s(v_1, ..., v_{r_s}) + \lambda v, \quad \lambda \neq \pm 1,\ \lambda_s \neq 0,\ r_s \geqslant 1,$$

$$u \in \bigcup_{i\geqslant 1} \mathcal{V}^{-i,*},\ \ v \in \bigcup_{i\geqslant 1} \mathcal{V}^{-i,*} \setminus K, \quad (2.5)$$

where $P_s(v_1, ..., v_{r_s})$ is a monomial in $\mathcal{D}^{*,*} \subset R^*H^*$.

Let A be an hga and let $\rho : (RH, d) \to H$ be an hga resolution. A *filtered hga model* of A is an hga quasi-isomorphism

$$f : (RH, d_h) \to (A, d_A)$$

in which

$$d_h = d + h, \quad h = h^2 + \cdots + h^r + \cdots, \quad h^r : R^pH^q \to R^{p+r}H^{q-r+1}.$$

The equality $d_h^2 = 0$ implies the sequence of equalities

$$dh^2 + h^2d = 0, \;\; dh^3 + h^3d = -h^2h^2, \;\; dh^4 + h^4d = -h^2h^3 - h^3h^2, \ldots,$$

and h is referred to as a *perturbation of* d. The map $h^r|_{R^{-r}H} : R^{-r}H \to R^0H$, $r \geqslant 2$, denoted by h^{tr}, is referred to as the *transgressive* component of h. The fact that the perturbation h acts as a derivation on elements of $\mathcal{E}$ implies $h^{tr}|_{\mathcal{E}} = 0$. For the existence of the filtered model see [**9**].

In the sequel, A' denotes a 1-reduced torsion free hga over $\mathbb{Z}$, while A denotes the tensor product hga $A' \otimes_{\mathbb{Z}} \Bbbk$. Denote also $H = H^*(A')$ and $H_\Bbbk = H^*(A)$. Assume (RH, d) is minimal and let $RH_\Bbbk = RH \otimes_{\mathbb{Z}} \Bbbk$; in particular, $RH_\Bbbk = T(V_\Bbbk)$ for $V_\Bbbk = V \otimes_{\mathbb{Z}} \Bbbk$. When $\Bbbk$ is a field of characteristic zero, $\rho \otimes 1 : RH_\Bbbk \to H \otimes_{\mathbb{Z}} \Bbbk = H_\Bbbk$ is an hga resolution of $H_\Bbbk$, which is *not* minimal when $\operatorname{Tor} H \neq 0$. In general, given a filtered model (RH, d_h) of A', we obtain an hga model

$$f \otimes 1 : (RH_\Bbbk, d_h \otimes 1) \to (A, d_A).$$

for (A, d_A). Denote $\bar{V}_\Bbbk = s^{-1}(V_\Bbbk^{>0}) \oplus \Bbbk$ and define the differential $\bar{d}_h$ on $\bar{V}_\Bbbk$ by the restriction of $d + h$ to $V_\Bbbk$ and obtain the cochain complex $(\bar{V}_\Bbbk, \bar{d}_h)$.

Since the map $f \otimes 1$ is in particular a homology isomorphism (by the universal coefficient theorem), the following two propositions follow immediately from the results in [**2**] and the standard isomorphisms $H^*(BA, d_{_{BA}}) \approx Tor^A(\Bbbk,\Bbbk)$ and $H^*(BC^*(X;\Bbbk), d_{_{BC}}) \approx H^*(\Omega X;\Bbbk)$.

Proposition 1. *There are isomorphisms*

$$H^*(\bar{V}_{\Bbbk}, \bar{d}_h) \approx H^*(B(RH_{\Bbbk}), d_{_{B(RH_{\Bbbk})}}) \approx H^*(BA, d_{_{BA}}) \approx Tor^A(\Bbbk,\Bbbk).$$

And for $A = C^*(X;\Bbbk)$ we obtain:

Proposition 2. *There are isomorphisms*

$$H^*(\bar{V}_{\Bbbk}, \bar{d}_h) \approx H^*(BC^*(X;\Bbbk), d_{_{BC}}) \approx H^*(\Omega X;\Bbbk).$$

Given (RH, d) and $x, c \in V$ with $dx, dc \in \mathcal{D} + \lambda V$, $\lambda \neq 1$, let $\eta_{x,c}$ denote an element of $\mathcal{E}_{>1,1}$ such that

$$x \smile_{\mathbf{1}} c := \eta_{x,c} + x \smile_1 c$$

satisfies $d(x \smile_{\mathbf{1}} c) \in \mathcal{D} + \lambda V$. For example, if $dx \in \lambda V$, then $\eta_{x,c} = 0$, and if $dx = \sum_i a_i b_i + \lambda v$ with $da_i, db_i \in \lambda V$, then $\eta_{x,c} = \sum_i (-1)^{|a_i|} E_{2,1}(a_i, b_i\,; c)$. In general, $\eta_{x,c}$ can be found as follows: Let $j : B(RH) \to \overline{RH} \to \bar{V}$ be the canonical projection used by the proof of the first isomorphism in Proposition 1, and choose $y \in B(RH)$ so that $j(y) = \bar{x}$ and $j\mu_E(y;\bar{c}) = \bar{\eta}_{x,c} + \overline{x \smile_1 c}$, where the product $\mu_E : B(RH) \otimes B(RH) \to B(RH)$ is determined by the hga structure on RH.

The following proposition is simple but useful. Let $\mathcal{D}_{\Bbbk} \subset RH$ be a subset defined by $\mathcal{D}_{\Bbbk} = \mathcal{D}$ for $\mu = 0$ and

$$\mathcal{D}_{\Bbbk} = \{u + \lambda v |\, u \in \mathcal{D}, v \in V,\ \lambda \text{ is divisible by } \mu\} \quad \text{for} \quad \mu \geqslant 2.$$

Definition 1. *An element $x \in V$ with $d_h x \in \mathcal{D} + \lambda V$, $\lambda \neq 1$, is* <u>*λ-homologous to zero*</u>*, denoted by $[\bar{x}]_\lambda = 0$, if there are $u, v \in V$ and $z \in \mathcal{D}$ such that*

$$d_h u = x + z + \lambda v;$$

x is <u>*weakly homologous*</u> *to zero when $v = 0$ above.*

Proposition 3. *Let $c \in V$ and $d_h c \in \mathcal{D}_{\Bbbk}$. If $d_h c$ has a summand component $ab \in \mathcal{D}$ such that $a, b \in V$, $d_h a, d_h b \in \mathcal{D}_{\Bbbk}$, both a and b are not weakly homologous to zero, then c is also not weakly homologous to zero.*

Proof. The proof is straightforward using the equality $d_h^2 = 0$. □

In particular, for $\Bbbk = \mathbb{Z}$, under hypotheses of the proposition if $[\bar{a}], [\bar{b}] \neq 0$, then $[\bar{c}] \neq 0$ in $H^*(\bar{V}, \bar{d}_h)$.

Note that over a field $\Bbbk$, Proposition 3 reflects the obvious fact that $x \in H^*(\Omega X;\Bbbk)$ is non-zero whenever some $x' \otimes x'' \neq 0$ in $\Delta x = \sum x' \otimes x''$.

3. Formal ∞-implication sequences

Let x be an element of a Hopf algebra over a finite field. In [**1**], W. Browder introduced the notion of ∞-implications (of an infinite sequence) associated with x in the Hopf algebra. The following can be thought of as a generalization of this: Let $x^{\smile_{\mathbf{1}} p}$ denote the (right most) p^{th}-power of x with respect to $\smile_{\mathbf{1}}$-product with the convention that $x^{\smile_{\mathbf{1}} 1} = x$.

Definition 2. *Let $x \in V^k$, $k \geqslant 2$, $d_h x \in \mathcal{D}_{\Bbbk}$. A sequence $\mathbf{x} = \{x(i)\}_{i \geqslant 0}$ is a formal ∞-implication sequence (f.i.s.) of x if*

(i) $x(0) = x$, $x(i) \in V^{(i+1)k-i}$, *and $x(i)$ is not μ-homologous to zero for all i;*

(ii) Either $x(i) = x^{\smile_1 (i+1)}$ or $x(i)$ is resolved from the following relation in the filtered hga model (RH, d_h) :

$$d_h \mathfrak{b}(i) = x^{\smile_1 (i+1)} + z(i) + \mu' x(i), \quad \mathfrak{b}(i) \in V,\ z(i) \in \mathcal{D},\ \mu' \text{ is divisible by } \mu. \tag{3.1}$$

We are interested in the existence of an f.i.s. for an odd dimensional $x \in V$.

Proposition 4. *Let $x \in V$ be of odd degree with $d_h x \in \mathcal{D}_{\Bbbk}$ such that x is not μ-homologous to zero. For $\mu \geqslant 2$, assume, in addition, there is no relation $d_h u = \mu x$* mod $\mathcal{D}$, *some $u \in V$. Then x has an f.i.s. $\mathbf{x} = \{x(i)\}_{i \geqslant 0}$.*

Proof. Suppose we have constructed $x(i)$ for $0 \leqslant i < n$. If $x^{\smile_1 (n+1)}$ is not μ-homologous to zero, set $x(n) = x^{\smile_1 (n+1)}$; otherwise, there is the relation $d_h u = x^{\smile_1 (n+1)} + z + \mu' v$ for some $u, v \in V$, $z \in \mathcal{D}$ and μ' divisible by μ. Using (2.1)–(2.2) one can easily establish the fact that $dx^{\smile_1 (n+1)}$ contains a summand component of the form $-\sum_{k+\ell=n+1} \binom{n+1}{k} x^{\smile_1 k} x^{\smile_1 \ell}$, $k, \ell \geqslant 1$. We have that $v \neq 0$ in the aforementioned relation since Proposition 3 (applied for $c = x^{\smile_1 (n+1)}$ and $a \cdot b = -\binom{n+1}{k} x^{\smile_1 k} \cdot x^{\smile_1 \ell}$, some k). Clearly, $d_h v = -\frac{1}{\mu'} d\left(x^{\smile_1 (n+1)} + z\right) \in \mathcal{D}$; Assuming μ' to be maximal v is not λ-homologous to zero. Set $x(n) = v$ and $\mathfrak{b}(n) = u$, $z(n) = z$ to obtain (3.1) for $i = n$. □

Thus, for $\mu = 0$ (when $\Bbbk$ is a field of characteristic zero, for example) $\mathbf{x} = \{x^{\smile_1 (n+1)}\}_{n \geqslant 0}$.

Remark 2. *1. The restriction on x in Proposition 4 that no relation $d_h u = \mu x$* mod $\mathcal{D}$ *exists is essential. A counterexample is provided by the exceptional group F_4: Let $A = C^*(BF_4; \mathbb{Z}_3)$ be the cochain complex of the classifying space BF_4. Then we have the relation $du = 3x$ in (RH, d) corresponding to the Bockstein cohomology homomorphism $\delta x_8 = x_9$ on $H^*(BF_4; \mathbb{Z}_3)$ (in the notation of [**13**]), but the element $x(2)$ does not exist (see [**9**] for more details).*

2. Note that if $du = \mu x$ in Proposition 4, but $[u][x] \neq 0 \in H_{\Bbbk}$, then one can modify the proof of the proposition to show that x again has an f.i.s. $\{x(i)\}_{i \geqslant 0}$. Note that in the above example we just have $[u][x] = 0 \in H_{\mathbb{Z}_3} = H^*(BF_4; \mathbb{Z}_3)$.

*3. The existence of ∞-implications of x in [**1**] uses both the $\smile$-product and the Pontrjagin product in the loop space (co)homology. In our case each component of the sequence $\mathbf{x}$ is determined by item (ii) of Definition 2 in which the first case can be thought of as related to the $\smile$-product, and the second with the Pontrjagin product. In particular, primitivity of x required in [**1**] is not issue for the existence of ∞-implications of x.*

In certain cases, a given odd dimensional $b \in V$ rises to an infinite sequence $\mathbf{b} = \{b_i\}_{i \geqslant 0}$ with $b = b_0$ in the hga resolution (RH, d). These sequences are built by explicit formulas and include also the case $du = \lambda b$, i.e., when the hypothesis of

Proposition 4 formally fails (see, for example, Case I of the proof of Proposition 5 below). Namely, we have the following cases:

(i) For $b\in V^{0,*}$ and $[b]^2=0\in H$ (i.e., there exists $b_1\in V^{-1,*}$ with $db_1=b^2$; e.g. $b_1=ab+\frac{\lambda-1}{2}b\smile_1 b$ for $da=\lambda b$ with λ odd, some $a\in V^{-1,*}$), $\mathbf{b}=\{b_i\}_{i\geqslant 0}$ is given by

$$db_n=\sum_{i+j=n-1} b_ib_j \tag{3.2}$$

and satisfies the following relation with $\mathfrak{c}_i\in V$

$$d\mathfrak{c}_n=-(-1)^n((n+1)b_n+b_0\smile_1 b_{n-1})+\sum_{i+j=n-1}(-1)^i\left(\mathfrak{c}_jb_i-b_i\mathfrak{c}_j\right),n\geqslant 1;$$

(ii) For $b\in V^{0,*}$ and $[b]^2\neq 0\in H$ (and $b_1=b\smile_1 b$), $\mathbf{b}=\{b_i\}_{i\geqslant 0}$ is given by

$$db_{2k}=\sum_{i+j=2k-1} b_ib_j,\qquad db_{2k+1}=\sum_{i+j=k}(2b_{2i}b_{2j}+b_{2i-1}b_{2j+1}), \tag{3.3}$$

and satisfies the following relation with $\mathfrak{c}_i\in V$ (below $\mathfrak{c}_1=0$)

$$\begin{aligned}d\mathfrak{c}_{2k}&=-(2k+1)b_{2k}-b_0\smile_1 b_{2k-1}+\sum_{i+j=k}2\left(\mathfrak{c}_{2j-1}b_{2i}-b_{2i}\mathfrak{c}_{2j-1}\right)\\&\qquad-\sum_{i+j=k}\left(\mathfrak{c}_{2j}b_{2i-1}-b_{2i-1}\mathfrak{c}_{2j}\right),\\ d\mathfrak{c}_{2k+1}&=(k+1)b_{2k+1}+b_0\smile_1 b_{2k}+\sum_{i+j=2k}(-1)^i\left(\mathfrak{c}_jb_i-b_i\mathfrak{c}_j\right),\ k\geqslant 1;\end{aligned}$$

(iii) For $b\in V^{-1,*}$ and $db=\mu c$, $\mu\geqslant 2$, $c\in V^{0,*}$ (below $\omega_0:=c$), $\mathbf{b}=\{b_i\}_{i\geqslant 0}$ is given by

$$\begin{aligned}db_n&=\sum_{i+j=n-1}b_ib_j+\mu c_n,\\ c_n&=-\omega_0\smile_1 b_{n-1}-\sum_{\substack{i+j=n-1\\ i\geqslant 1;\, j\geqslant 0}}(-1)^i\omega_i\smile_1 b_j-(-1)^n\omega_n,\ n\geqslant 1\end{aligned} \tag{3.4}$$

and satisfies the following relation with $\mathfrak{c}_i\in V$

$$\begin{aligned}d\mathfrak{c}_1&=2b_1+b_0\smile_1 b_0+\mu\omega_0\cup_2 b_0,\\ d\mathfrak{c}_n&=-(-1)^n((n+1)b_n+b_0\smile_1 b_{n-1})+\sum_{i+j=n-1}(-1)^i\left(\mathfrak{c}_jb_i-b_i\mathfrak{c}_j\right)\\&\qquad+\mu\mathfrak{a}_n,\\ \mathfrak{a}_n&=\sum_{i+j=n-2}(-1)^j\left((\omega_i\cup_2 b_0)\smile_1 b_j+\omega_i\smile_1\mathfrak{c}_{j+1}\right)+\omega_{n-1}\cup_2 b_0,\\ d\omega_k&=\sum_{i+j=k-1}\mu\omega_i\smile_1\omega_j,\quad \omega_k=\mu^k\omega_0^{\cup_2(k+1)},\ k\geqslant 1,\ n\geqslant 2.\end{aligned}$$

For example, in view of Proposition 2, the formulas above are enough to calculate the loop space cohomology algebra with coefficients in $\Bbbk$ for Moore spaces, i.e., the CW-complexes obtained by attaching an $(n+1)$-cell to the n-sphere S^n by a map $S^n \to S^n$ of degree μ.

3.1. Odd dimensional element $l(a)$

Given $m \geqslant 2$, let $H(A)$ be finitely generated as a $\Bbbk$-module with $H^i(A) = 0$ for $i > m$. Let $\mathcal{Z}_\Bbbk$ be the subset of RH defined by

$$\mathcal{Z}_\Bbbk = \mathcal{Z}'_\Bbbk + \mathcal{Z}''_\Bbbk + \mathcal{D}_\Bbbk,$$

$$\mathcal{Z}'_\Bbbk = \{v \in V \mid du = \lambda v, \quad u \in V,\, \lambda \in \mathbb{Z}_\Bbbk\}$$

and

$$\mathcal{Z}''_\Bbbk = \{v \in V \mid v = \lambda u, \quad u \in V,\, \lambda \in \mathbb{Z} \setminus \mathbb{Z}_\Bbbk\}\,.$$

Given $x \in V$ with $d_h x = w \in \mathcal{Z}_\Bbbk$, $w = w' + w'' + z$, define

$$\tilde{x} = \frac{l.c.m.(\lambda''; \mu)}{\lambda''}(\lambda' x - u),\; du = \lambda' w',\; w'' = \lambda'' v'',$$

to obtain $d_h \tilde{x} \in \mathcal{D}_\Bbbk$.

Regarding (2.5), define also the following subsets $K^*_\mu, K^*_0 \subset \mathcal{V}^{-1,*}$ with $K^*_\mu \subset K^*$ as

$$K_\mu = \{a \in K \mid \lambda \text{ is divisible by } \mu\}\,, \quad K_0 = \left\{u \in \mathcal{V}^{-1,*} \setminus \mathcal{E} \mid du \in \mathcal{D}^{0,*}\right\},$$

and assign to a given even dimensional element $a \in V^{0,*} \cup K_\mu$ an odd dimensional element $l(a) \in V$ with $dl(a) \in \mathcal{D}_\Bbbk$ as follows. If $a \in V^{0,*}$, let $l(a) \in K_0$ be an element such that $dl(a) = a^k$, where $k \geqslant 2$ is chosen to be the smallest. If $a \in K_\mu$ with $da = \lambda b$ consider the relation

$$du_1 = -a^2 + \lambda v_1, \quad dv_1 = \frac{1}{\lambda} d(a^2), \quad u_1 \in V^{-3,*},\, v_1 \in V^{-2,*}, \tag{3.5}$$

and the perturbation $hu_1 = h^2 u_1 + h^3 u_1$. When $hu_1 \in \mathcal{Z}_\Bbbk$, set $l(a) = \widetilde{u}_1$, while when $h^3 u_1 \notin \mathcal{Z}_\Bbbk$, consider $\mathfrak{u}_1 = h^3 u_1|_{V^{0,*}}$, the component of $h^3 u_1$ in $V^{0,*}$, and define $l(a)$ as $l(\mathfrak{u}_1)$. When $h^2 u_1 \notin \mathcal{Z}_\Bbbk$, and $h^3 u_1 \in \mathcal{Z}_\Bbbk$, choose the smallest $n > 1$ such that there is the relation

$$du_n = -ah^2 u_{n-1} + \lambda v_n, \quad dv_n = \frac{1}{\lambda} d(ah^2 u_{n-1}), \quad u_n \in V^{-3,*},\, v_n \in V^{-2,*}, \quad \text{with } \; h^2 u_n \in \mathcal{Z}_\Bbbk. \tag{3.6}$$

(The inequality $(n+1)|a| > m$ guarantees the existence of such a relation, since $h^2 u_i \in \mathcal{D} + K_\mu$, while $K^j_\mu = 0$ for $j > m$ in the minimal $V \subset RH$.) Then set $l(a) = \widetilde{u}_n$ for $h^3 u_n \in \mathcal{Z}_\Bbbk$; otherwise, define $l(a)$ as $l(\mathfrak{u}_n)$ for $\mathfrak{u}_n = h^3 u_n|_{V^{0,*}}$.

4. Proof of Theorem 2

The proof of the theorem relies on the two basic propositions below in which the condition that $\tilde{H}(A)$ has at least two algebra generators is treated in two specific cases.

Proposition 5. *Let $H_{\Bbbk}$ be a finitely generated $\Bbbk$-module with $\mu \geqslant 2$. If $\tilde{H}_{\Bbbk}$ has at least two algebra generators and $\tilde{H}_{\mathbb{Q}}$ is either trivial or has a single algebra generator, there are two sequences of odd degree elements $\mathbf{x}_{\Bbbk} = \{x(i)\}_{i\geqslant 0}$ and $\mathbf{y}_{\Bbbk} = \{y(j)\}_{j\geqslant 0}$ in $V_{\Bbbk}$ whose degrees form arithmetic progressions such that all $\bar{x}(i)$, $\bar{y}(j)$ are $\bar{d}_h$-cocycles in $\bar{V}_{\Bbbk}$ and the classes $\left\{[s^{-1}(x(i) \smile_{\mathbf{1}} y(j))]\right\}_{i,j\geqslant 0}$ are linearly independent in $H(\bar{V}_{\Bbbk}, \bar{d}_h)$.*

Proof. The hypotheses of the proposition imply that K_μ defined in subsection 3.1 above is non-empty; also by the restriction on $\tilde{H}_{\mathbb{Q}}$, relation (2.5) reduces to

$$da = \lambda b^m,\ \lambda \neq 0,\ m \geqslant 1,\ (\lambda, m) \neq (1,1),\ b \in \mathcal{V}^{0,*}$$

for $a \in \mathcal{V}^{-1,*}$ to be of the smallest degree.

In the three cases below, we exhibit two odd dimensional elements $x, y \in V \setminus \mathcal{E}$ that fail to be μ-homologous to zero.

Case I. Let $a \in K_\mu$ be of the smallest degree in $K_\mu \cup K_0$ with $da = \lambda b$ and let $|a|$ be even. Consider the element $l(a)$. If it is not λ-homologous to zero, set $x = l(a)$; otherwise, we must have relation (2.5) in which $v_i = a$ for some i and $hu \in \mathcal{Z}_{\Bbbk}$ with $|u| < |l(a)|$, $u \in \bigcup_{i\geqslant 1} \mathcal{V}^{-i,*} \setminus \mathcal{E}$. By (2.5) choose u to be of the smallest degree with $hu \in \mathcal{Z}_{\Bbbk}$, $u \neq u_i, a_1$, where u_i is given by (3.5)–(3.6) and $da_1 = -ab + \lambda b_1$, $db_1 = b^2$. Set $x = \tilde{u}$ for $|u|$ odd. If $|u|$ is even and $u \in \bigcup_{i>1} \mathcal{V}^{-i,*} \setminus \mathcal{E}$ set $x = \tilde{v}$; if $u \in K_0$ and du contains an odd dimensional $v_i \in V^{0,*}$ with $[v_i] \neq 0 \in H_{\mathbb{Q}}$, set $x = v_i$; otherwise, for each monomial $P_s(v_1, ..., v_{r_s})$ choose a variable v_i with a relation $du_i = \mu_i v_i$ (for example, we can choose v_i to be odd dimensional for all s). Let λ be the smallest integer divisible by all μ_i, and replace v_i by $\frac{\lambda}{\mu_i} u_i$ to detect a new relation in (RH, d) given again by (2.5):

$$dw = \sum_{1\leqslant s\leqslant n} \frac{\lambda_s \lambda}{\mu_i} P_s(v_1, ..., v_{i-1}, u_i, v_{i+1} ..., v_{r_s}) + \lambda u, \qquad \lambda_s \in \mathbb{Z}_{\Bbbk},\ w \in \mathcal{V}^{-2,*}.$$

Hence, $|w|$ is odd, and set $x = \widetilde{w}$ for $h^2 w \in \mathcal{Z}_{\Bbbk}$. If $h^2 w \notin \mathcal{Z}_{\Bbbk}$ we have the following two subcases:

(i1) Assume there exists $v \in K_\mu$ with $dv = \lambda h^2 w$. If $[\bar{v}]_\lambda \neq 0$, set $x = v$; otherwise we have a relation $d_h u' = v + z + \lambda' v'$, some $u', v' \in V$, $z \in \mathcal{D}$. Clearly, $h^{tr} v' = -\frac{\lambda}{\lambda'} h^2 w \mod \mathcal{D}$, and set $x = \frac{\lambda}{\lambda'} w + v'$. Note that x is not λ-homologous to zero since the component $\frac{\lambda^2}{\lambda'} u$ in dx.

(i2) Assume $[h^2 w] \neq 0 \in H_{\mathbb{Q}}$. When $r_s > 1$ for all s, choose a variable v_j different from v_i in $P_s(v_1, ..., v_{r_s})$ to form w' entirely analogously to w, and then find x similarly to the above unless $[h^2 w'] \neq 0 \in H_{\mathbb{Q}}$, in which case set $x = \alpha w + \beta w'$, some $\alpha, \beta \in \mathbb{Z}$. When $\mathbf{k} = \{s \in \underline{n} \,|\, r_s = 1 \text{ in } du\} \neq \varnothing$, i.e., $P_s(v_1, ..., v_{r_s}) = v_1^{2m_1+1} := v_s^{2m_s+1}, m_s \geqslant 1$, $|v_s|$ is odd for $s \in \mathbf{k}$ (in particular, μ_s is even, since $[v_s]^2 = 0 \in H$ for μ_s odd; c.f. (3.2)), then

$$du' = \begin{cases} \sum_{s\in\mathbf{k}} \frac{\lambda_s \lambda}{2} (v_s \smile_1 v_s) v_s^{2m_s - 1} \\ + \sum_{s\in\underline{n}\setminus\mathbf{k}} \frac{\lambda_s \lambda}{\mu_j} P_s(v_1, ..., v_{j-1}, u_j, v_{j+1} ..., v_{r_s}) + \lambda u, & \mathbf{k} \neq \underline{n}, \\ \sum_{s\in\underline{n}} \lambda_s (v_s \smile_1 v_s) v_s^{2m_s-1} + 2u, & \mathbf{k} = \underline{n} \end{cases}$$

with $u' \in \mathcal{V}^{-2,*}$, and by considering h^2u' we find x as in item (i1).

To find y, consider b and the associated sequence $\mathbf{b} = \{b_i\}$ given by (3.2) or (3.3). If $hb_i \in \mathcal{Z}_{\Bbbk}$ for all i, set $y = b$ and $\mathbf{y} = \{\tilde{b}_i\}_{i\geqslant 0}$. If $h\mathbf{b} \not\subseteq \mathcal{Z}_{\Bbbk}$, consider the smallest $p > 0$ such that $h^{tr}b_p \notin \mathcal{Z}_{\Bbbk}$. Consider $t_p = h^{tr}b_p|_{V^{0,*}}$, and if $\left[\overline{l(t_p)}\right]_\lambda \neq 0$, set $y = l(t_p)$; if $\left[\overline{l(t_p)}\right]_\lambda = 0$ and $\alpha h^3u_i + \beta h^{tr}b_p = 0$, $\alpha, \beta \in \mathbb{Z}$, for some u_i from (3.5)–(3.6), set $y = \alpha u_i + \beta b_p$; otherwise, we obtain $l(t_p) \in K_0$ different from $l(a)$ above; consequently, we must have another relation in (RH, d) given by (2.5) in which $v_i = t_p$ for some i and $hu \in \mathcal{Z}_{\Bbbk}$ with $|u| < |l(t_p)|$, and then y is found similarly to x.

Case II. Let $a \in K_\mu$ be of the smallest degree in $K_\mu \cup K_0$ with $da = \lambda b$ and let $|a|$ be odd. Set $x = a$. Consider $l(b) \in K_0$, and then y is found as in Case I.

Case III. Let $a \in K_0$ be of smallest degree in $K_\mu \cup K_0$ with $da = \lambda b^m, m \geqslant 2$, and $[b] \neq 0 \in H_{\mathbb{Q}}$. Set

$$x = \begin{cases} b, & |b| \quad \text{is odd} \\ a, & |b| \quad \text{is even.} \end{cases}$$

To find y consider the following two subcases:

(i) Assume $\lambda \in \mathbb{Z}\backslash\mathbb{Z}_{\Bbbk}$. When both $|a|$ and $|b|$ are odd, set $y = a$; otherwise, either $|a|$ or $|b|$ is even, in which case consider $l(\tilde{a})$ or $l(b)$ respectively, and then y is found as in Case I.

(ii) Assume $\lambda \in \mathbb{Z}_{\Bbbk}$. Since $K_\mu \neq \varnothing$, this subcase reduces either to Case I or to Case II.

Finally, having found the elements x and y in Cases I-III, consider the f.i.s. $\mathbf{x}$ and $\mathbf{y}$ in V and the induced sequences $\mathbf{x}_{\Bbbk} = \{x(i)\}_{i\geqslant 0}$ and $\mathbf{y}_{\Bbbk} = \{y(j)\}_{j\geqslant 0}$ in $V_{\Bbbk}$. Then the both sequences $\bar{\mathbf{x}}_{\Bbbk}$ and $\bar{\mathbf{y}}_{\Bbbk}$ consist of $\bar{d}_h$-cocycles in $\bar{V}_{\Bbbk}$ whose degrees form an arithmetic progression respectively. Thus, we obtain that $[\bar{\mathbf{x}}_{\Bbbk}]$, $[\bar{\mathbf{y}}_{\Bbbk}] \subset H(\bar{V}_{\Bbbk}, \bar{d}_h)$ are sequences of non-trivial classes. Moreover, they are linearly independent and $\left\{[s^{-1}(x(i) \smile_{\mathbf{1}} y(j))]\right\}_{i,j\geqslant 0}$ is the sequence of linearly independent classes in $H(\bar{V}_{\Bbbk}, \bar{d}_h)$ as required. □

Before proving the second basic proposition we need the following auxiliary statement. Given a cochain complex (C^*, d) over $\mathbb{Q}$, let $S_C(T) = \sum_{n\geqslant 0}(\dim_{\mathbb{Q}} C^n)T^n$ and $S_{H(C)}(T) = \sum_{n\geqslant 0}(\dim_{\mathbb{Q}} H^n(C))T^n$ be the Poincaré series. As usual, we write $\sum_{n\geqslant 0} a_nT^n \leqslant \sum_{n\geqslant 0} b_nT^n$ if and only if $a_n \leqslant b_n$. The following proposition can be thought of as a modification of Propositions 3 and 4 in [**12**] for the non-commutative case.

Proposition 6. *Given an element $y \in V_{\mathbb{Q}}$ of total degree $K_\mu \geqslant 2$ such that $\bar{d}_h(\bar{y}) = 0$, let $y\bar{V}_{\mathbb{Q}} \subset \bar{V}_{\mathbb{Q}}$ be a subcomplex (additively) generated by the expressions $\{\bar{y} = s^{-1}y,\ s^{-1}(y \smile_1 v)\}_{v\in V_{\mathbb{Q}}}$. Then*

$$S_{H(\bar{V}_{\mathbb{Q}}/y\bar{V}_{\mathbb{Q}})}(T) \leqslant (1 + T^{k-1})S_{H(\bar{V}_{\mathbb{Q}})}(T). \tag{4.1}$$

Proof. Consider the inclusion of cochain complexes $s^k\bar{V}_{\mathbb{Q}} \xrightarrow{\iota} \bar{V}_{\mathbb{Q}}$ defined for $1 \in \mathbb{Q} = (s^k\bar{V}_{\mathbb{Q}})^k$ by $\iota(1) = \bar{y}$, and for $s^k(\bar{v}) \in (s^k\bar{V}_{\mathbb{Q}})^{>k}$, $v \in V_{\mathbb{Q}}^{>1}$, by $\iota(s^k(\bar{v})) = s^{-1}(y \smile_1 v)$.

Then $\iota(s^k\bar{V}_{\mathbb{Q}}) = y\bar{V}_{\mathbb{Q}}$ and there is the short exact sequence of cochain complexes

$$0 \to s^k\bar{V}_{\mathbb{Q}} \xrightarrow{\iota} \bar{V}_{\mathbb{Q}} \to \bar{V}_{\mathbb{Q}}/y\bar{V}_{\mathbb{Q}} \to 0.$$

Consider the induced long exact sequence

$$\cdots \to H^{n-k}(\bar{V}_{\mathbb{Q}}) \xrightarrow{H^n(\iota)} H^n(\bar{V}_{\mathbb{Q}}) \to H^n(\bar{V}_{\mathbb{Q}}/y\bar{V}_{\mathbb{Q}}) \to H^{n-k+1}(\bar{V}_{\mathbb{Q}}) \to \cdots .$$

Let $I = \oplus I_n$, where $I_n = \mathrm{Im}(H^n(\iota))$, $n \geqslant 0$, and form the exact sequence

$$0 \to I_n \to H^n(\bar{V}_{\mathbb{Q}}) \to H^n(\bar{V}_{\mathbb{Q}}/y\bar{V}_{\mathbb{Q}}) \to H^{n-k+1}(\bar{V}_{\mathbb{Q}}) \to I_{n+1} \to 0.$$

Since $I_0 = 0$, we have

$$\sum_{n\geqslant 0}(\dim_{\mathbb{Q}} I_n + \dim_{\mathbb{Q}} I_{n+1})T^n = \frac{(1+T)S_I(T)}{T}.$$

Now apply the Euler-Poincaré lemma for the above exact sequence to obtain the equality

$$\frac{(1+T)S_I(T)}{T} - S_{H(\bar{V}_{\mathbb{Q}})}(T) + S_{H(\bar{V}_{\mathbb{Q}}/y\bar{V}_{\mathbb{Q}})}(T) - T^{k-1}S_{H(\bar{V}_{\mathbb{Q}})}(T) = 0.$$

Consequently,

$$S_{H(\bar{V}_{\mathbb{Q}}/y\bar{V}_{\mathbb{Q}})}(T) = (1+T^{k-1})S_{H(\bar{V}_{\mathbb{Q}})}(T) - \frac{(1+T)S_I(T)}{T},$$

and since $S_I(T) \geqslant 0$, we get (4.1) as required. □

Proposition 7. *Let $H_{\Bbbk}$ be a finitely generated $\Bbbk$-module. If $\tilde{H}_{\mathbb{Q}}$ has at least two algebra generators and $A_{\mathbb{Q}} = A' \otimes_{\mathbb{Z}} \mathbb{Q}$, the set $\left\{\tau_i(A_{\mathbb{Q}}) = \dim_{\mathbb{Q}} Tor_i^{A_{\mathbb{Q}}}(\mathbb{Q},\mathbb{Q})\right\}$ is unbounded.*

Proof. Consider the first two generators $a_i \in V_{\mathbb{Q}}^{-1,*}$ with $da_i \in \mathcal{D}^{0,*}$, $i = 1, 2$. We have two cases:

(i) Both $|a_1|$ and $|a_2|$ are odd. Set $x = a_1$ and $y = a_2$. Then both $\bar{x}$ and $\bar{y}$ are $\bar{d}_h$-cocycles and the classes $[\bar{x}]$ and $[\bar{y}]$ are non-trivial in $H(\bar{V}_{\mathbb{Q}}, \bar{d}_h)$. Consequently, the classes

$$\left\{[s^{-1}\left(x^{\smile_{\mathbf{1}} i} \smile_{\mathbf{1}} y^{\smile_{\mathbf{1}} j}\right)]\right\}_{i,j\geqslant 1} \tag{4.2}$$

are linearly independent in $H(\bar{V}_{\mathbb{Q}}, \bar{d}_h)$.

(ii) Either $|a_1|$ or $|a_2|$ is even. Denote the (smallest) even dimensional generator by a and consider da. Then for a, (2.5) reduces to

$$da = uv, \quad u \in V_{\mathbb{Q}}^{0,2k+1} \text{ and } v \in R^0H_{\mathbb{Q}}^{2\ell}, \text{ some } k, \ell \geqslant 1.$$

There are the following induced relations in $(RH_{\mathbb{Q}}, d)$:

$$db = -u(a + u \smile_1 v) - au, \qquad b \in V_{\mathbb{Q}}^{-2,2(2k+\ell+1)} \quad \text{and}$$
$$dc = -u\left(v \smile_1 a + (u \cup_2 v)v + u(v \cup_2 v)\right) - a^2 + bv, \quad c \in V_{\mathbb{Q}}^{-3,4(k+\ell)+2}.$$

Thus we have $hc = h^2c + h^3c$, and in particular, $dh^2c = h^2b \cdot v$. Consider the following two cases:

(1) Assume $hc \in \mathcal{D}$. Set $x = u$, $y = c$, and obtain linearly independent classes in $H(\bar{V}_{\mathbb{Q}}, \bar{d}_h)$ by formula (4.2).

(2) Assume $hc \notin \mathcal{D}$. Let $(\bar{W}, \bar{d}_W) = (\bar{V}_{\mathbb{Q}}/\bar{C}, \bar{d}_W)$, where $C \subset V_{\mathbb{Q}}$ is a subcomplex (additively) generated by the expressions hc and $hc \smile_1 z$ for $z \in V_{\mathbb{Q}}$. Define x and y as the projections of the elements u and c from $V_{\mathbb{Q}}$ under the quotient map $V_{\mathbb{Q}} \to V_{\mathbb{Q}}/C$, respectively. Then $\bar{x}$ and $\bar{y}$ are $\bar{d}_W$-cocycles in $\bar{W}$. Once again apply formula (4.2) to obtain linearly independent classes in $H(\bar{W}, \bar{d}_W)$. Finally, Proposition 6 implies that $S_{H(\bar{W})}(T) \leqslant S_{H(V_{\mathbb{Q}})}(T)$, and an application of Proposition 1 completes the proof. □

4.1. Proof of Theorem 2

In view of Proposition 1, the proof reduces to the examination of the $\mathbb{k}$-module $H(\bar{V}_{\mathbb{k}}, \bar{d}_h)$. If $\tilde{H}_{\mathbb{k}}$ has a single algebra generator a, then the set $\{\tau_i(A)\}$ is bounded since $\tau_i(A) = 1$. For example, this can be seen from the fact that $H(\bar{V}_{\mathbb{k}}, \bar{d}_h)$ is generated by a single sequence induced by (3.2) or by (3.3), where $x = a$ or $x = l(a)$ for $|a|$ odd or even respectively, and by $\smile_1$-products of its components. If $\tilde{H}_{\mathbb{k}}$ has at least two algebra generators, then the proof follows from Propositions 5 and 7.

References

[1] W. Browder, Torsion in H-spaces, Ann. Math., 74 (1961), 24-51.

[2] Y. Felix, S. Halperin and J.-C. Thomas, Adams' cobar equivalence, Trans. AMS, 329 (1992), 531-549.

[3] M. Gerstenhaber and A.A. Voronov, Higher operations on the Hochschild complex, Functional Analysis and its Applications, 29 (1995), 1–5.

[4] E. Getzler and J.D.S. Jones, Operads, homotopy algebra, and iterated integrals for double loop spaces, preprint (1995).

[5] T. Kadeishvili, On the homology theory of fibre spaces, Russian Math. Survey, 35 (1980), 131-138.

[6] T. Kadeishvili and S. Saneblidze, A cubical model of a fibration, J. Pure and Applied Algebra, 196 (2005), 203-228.

[7] D. Kraines, Massey higher products, Trans. AMS, 124 (1966), 431-449.

[8] J. McCleary, On the mod p Betti numbers of loop spaces, Invent. Math., 87 (1987), 643-654.

[9] S. Saneblidze, Filtered Hirsch algebras, preprint math.AT/0707.2165.

[10] J. D. Stasheff, Homotopy Associativity of H-spaces I, II, Trans. AMS, 108 (1963), 275-312.

[11] D. Sullivan, Diffrential forms and the topology of manifolds, Manifolds-Tokyo 1973, ed. Akio Hattori, Tokyo Univ. Press, 37-51.

[12] M. Vigué-Poirrier and D. Sullivan, The homology theory of the closed geodesic problem, J. Diff. Geom., 11 (1976), 633-644.

[13] H. Toda, Cohomology mod 3 of the classifying space BF_4 of the exceptional group F_4, J. Math. Kyoto Univ., 13-1 (1973), 97-115.

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