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THE BIDERIVATIVE AND A_∞-BIALGEBRAS

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(*communicated by James Stasheff*)

Abstract

An A_∞-bialgebra is a DGM H equipped with structurally compatible operations $\left\{\omega^{j,i}: H^{\otimes i} \to H^{\otimes j}\right\}$ such that $\left(H, \omega^{1,i}\right)$ is an A_∞-algebra and $\left(H, \omega^{j,1}\right)$ is an A_∞-coalgebra. Structural compatibility is controlled by the biderivative operator Bd, defined in terms of two kinds of cup products on certain cochain algebras of pemutahedra over the universal PROP $U = End\,(TH)$.

To Jim Stasheff on the occasion of his 68th birthday.

1. Introduction

In his seminal papers of 1963, J. Stasheff [**22**] introduced the notion of an A_∞-algebra, which is (roughly speaking) a DGA in which the associative law holds up to homotopy. Since then, A_∞-algebras have assumed their rightful place as fundamental structures in algebra [**12**], [**19**], topology [**5**], [**10**], [**23**], and mathematical physics [**6**], [**7**], [**13**], [**14**], [**27**], [**28**]. Furthermore, his idea carries over to homotopy versions of coalgebras [**15**], [**21**], [**25**] and Lie algebras [**9**], and one can deform a classical DG algebra, coalgebra or Lie algebra to the corresponding homotopy version in a standard way.

This paper introduces the notion of an A_∞-bialgebra, which is a DGM H equipped with "structurally compatible" operations $\left\{\omega^{j,i}: H^{\otimes i} \to H^{\otimes j}\right\}_{i,j\geqslant 1}$ such that $\left(H, \omega^{1,i}\right)_{i\geqslant 1}$ is an A_∞-algebra and $\left(H, \omega^{j,1}\right)_{j\geqslant 1}$ is an A_∞-coalgebra. The main result of this project, the proof of which appears in the sequel [**18**], is the fact that over a field, the homology of every A_∞-bialgebra inherits an A_∞-bialgebra structure. In particular, the Hopf algebra structure on a classical Hopf algebra extends to an A_∞-bialgebra structure and the A_∞-bialgebra structure on the homology of a loop space specializes to the A_∞-(co)algebra structures observed by Gugenheim [**4**] and Kadeishvili [**5**]. Thus loop space homology provides a primary family of examples. In

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fact, one can introduce an A_∞-bialgebra structure on the double cobar construction of H.-J. Baues [**1**].

The problem that motivated this project was to classify rational loop spaces that share a fixed Pontryagin algebra. This problem was considered by the second author in the mid 1990's as a deformation problem in some large (but unknown) rational category containing DG Hopf algebras. And it was immediately clear that if such a category exists, it contains objects with rich higher order structure that specializes to simultaneous A_∞-algebra and A_∞-coalgebra structures. Evidence of this was presented by the second author at Jim Stasheff's schriftfest (June 1996) in a talk entitled, "In search of higher homotopy Hopf algebras" [**26**]. Given the perspective of this project, we conjecture that there exists a deformation theory for A_∞-bialgebras in which the infinitesimal deformations of classical DG bialgebra's observed in that talk approximate A_∞-bialgebras to first order. Shortly thereafter, the first author used perturbation methods to solve this classification problem [**15**]. The fact that A_∞-bialgebras appear implicitly in this solution led to the collaboration in this project.

Given a DGM H, let $U = End(TH)$ be the associated universal PROP. We construct internal and external cup products on $C^*(P;\mathbf{U})$, the cellular chains of permutahedra $P = \sqcup_{n\geqslant 1} P_n$ with coefficients in a certain submodule $\mathbf{U} \subset TTU$. The first is defined for every polytope and in particular for each P_n; the second is defined globally on $C^*(P;\mathbf{U})$ and depends heavily on the representation of faces of permutahedra as leveled trees (see our prequel [**17**], for example). These cup products give rise to a biderivative operator Bd on $\mathbf{U}$ with the following property: Given $\omega \in \mathbf{U}$, there is a unique element $d_\omega \in \mathbf{U}$ fixed by the action of Bd that bimultiplicatively extends ω. We define a (non-bilinear) operation $\circledcirc$ on $\mathbf{U}$ in terms of Bd and use it to define the notion of an A_∞-bialgebra. The paper is organized as follows: Cup products are constructed in Section 2, the biderivative is defined in Section 3 and A_∞-bialgebras are defined in Section 4.

2. Cochain Algebras Over the Universal PROP

Let R be a commutative ring with identity and let H be an R-free DGM of finite type. For $x, y \in \mathbb{N}$, let $U_{y,x} = Hom(H^{\otimes x}, H^{\otimes y})$ and view $U_H = End(TH)$ as the bigraded module

$$U_{*,*} = \bigoplus_{x,y\in\mathbb{N}} U_{y,x}.$$

Given matrices $X = [x_{ij}]$ and $Y = [y_{ij}] \in \mathbb{N}^{q\times p}$, consider the module

$$\begin{aligned} U_{Y,X} &= (U_{y_{11},x_{11}} \otimes \cdots \otimes U_{y_{1p},x_{1p}}) \otimes \cdots \otimes (U_{y_{q1},x_{q1}} \otimes \cdots \otimes U_{y_{qp},x_{qp}}) \\ &\subset \left(U^{\otimes p}\right)^{\otimes q} \subset TTU. \end{aligned}$$

Represent a monomial $A \in U_{Y,X}$ as the $q \times p$ matrix $[A] = \left[\theta_{y_{ij},x_{ij}}\right]$ with rows thought of as elements of $U^{\otimes p} \subset TU$. We refer to A as a $q \times p$ monomial; we often abuse notation and write A when we mean $[A]$. Note that

$$\bigoplus_{X,Y\in\mathbb{N}^{q\times p}} U_{Y,X} = (U^{\otimes p})^{\otimes q};$$

in Subsection 2.1 below we construct the "upsilon product" on the module

$$M = \bigoplus_{\substack{X,Y\in\mathbb{N}^{q\times p}\\ p,q\geqslant 1}} U_{Y,X} = \bigoplus_{p,q\geqslant 1} (U^{\otimes p})^{\otimes q}.$$

In particular, given $\mathbf{x} = (x_1,\ldots,x_p) \in \mathbb{N}^p$ and $\mathbf{y} = (y_1,\ldots,y_q) \in \mathbb{N}^q$, set $X = (x_{ij} = x_j)_{1\leqslant i\leqslant q}$, $Y = (y_{ij} = y_i)_{1\leqslant j\leqslant p}$ and $\mathbf{U}_{\mathbf{x}}^{\mathbf{y}} = U_{Y,X}$. A monomial $A \in \mathbf{U}_{\mathbf{x}}^{\mathbf{y}}$ is represented by a $q\times p$ matrix

$$A = \begin{bmatrix} \theta_{y_1,x_1} & \cdots & \theta_{y_1,x_p} \\ \vdots & & \vdots \\ \theta_{y_q,x_1} & \cdots & \theta_{y_q,x_p} \end{bmatrix}.$$

We refer to the vectors $\mathbf{x}$ and $\mathbf{y}$ as the *coderivation* and *derivation leaf sequences* of A, respectively (see Subsection 2.3). Note that for $a,b\in\mathbb{N}$, monomials in $\mathbf{U}_a^{\mathbf{y}}$ and $\mathbf{U}_{\mathbf{x}}^b$ appear as $q\times 1$ and $1\times p$ matrices. Let

$$\mathbf{U} = \bigoplus_{\substack{\mathbf{x}\times\mathbf{y}\in\mathbb{N}^p\times\mathbb{N}^q\\ p,q\geqslant 1}} \mathbf{U}_{\mathbf{x}}^{\mathbf{y}}.$$

We graphically represent a monomial $A = \left[\theta_{y_j,x_i}\right] \in \mathbf{U}_{\mathbf{x}}^{\mathbf{y}}$ two ways. First as a matrix of "double corollas" in which entry θ_{y_j,x_i} is pictured as two corollas joined at the root–one opening downward with x_i inputs and one opening upward with y_j outputs–and second as an arrow in the positive integer lattice $\mathbb{N}^2$ (see Figure 1). The arrow representation is motivated by the fact that A can be thought of as an operator on $\mathbb{N}^2$. Since H has finite type, A admits a representation as a map

$$A : \left(H^{\otimes x_1}\otimes\cdots\otimes H^{\otimes x_p}\right)^{\otimes q} \to \left(H^{\otimes y_1}\right)^{\otimes p}\otimes\cdots\otimes\left(H^{\otimes y_q}\right)^{\otimes p}.$$

For $\mathbf{u} = (u_1,\ldots,u_k)\in\mathbb{N}^k$, let $|\mathbf{u}| = u_1+\cdots+u_k$ and identify $(s,t)\in\mathbb{N}^2$ with the module $(H^{\otimes s})^{\otimes t}$. Let $\sigma_{s,t} : (H^{\otimes s})^{\otimes t} \overset{\approx}{\to} (H^{\otimes t})^{\otimes s}$ be the canonical permutation of tensor factors and identify a $q\times p$ monomial $A\in\mathbf{U}_{\mathbf{x}}^{\mathbf{y}}$ with the operator $\left(\sigma_{y_1,p}\otimes\cdots\otimes\sigma_{y_q,p}\right)\circ A$ on $\mathbb{N}^2$, i.e., the composition

$$\left(H^{\otimes|\mathbf{x}|}\right)^{\otimes q} \approx \left(H^{\otimes x_1}\otimes\cdots\otimes H^{\otimes x_p}\right)^{\otimes q} \overset{A}{\to} \left(H^{\otimes y_1}\right)^{\otimes p}\otimes\cdots\otimes\left(H^{\otimes y_q}\right)^{\otimes p}$$

$$\overset{\sigma_{y_1,p}\otimes\cdots\otimes\sigma_{y_q,p}}{\longrightarrow} \left(H^{\otimes p}\right)^{\otimes y_1}\otimes\cdots\otimes\left(H^{\otimes p}\right)^{\otimes y_q} \approx \left(H^{\otimes p}\right)^{\otimes|\mathbf{y}|},$$

where $\approx$ denotes the canonical isomorphism that changes filtration. Thus we represent A as an arrow from $(|\mathbf{x}|, q)$ to $(p, |\mathbf{y}|)$. In particular, a monomial $A\in\mathbf{U}_a^b$ "transgresses" from $(a,1)$ to $(1,b)$.

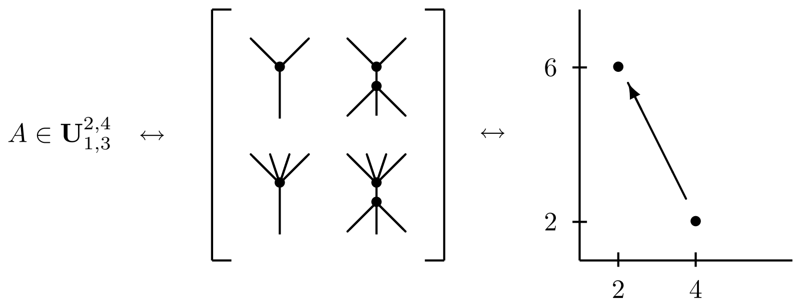

Figure 1. Graphical representations of a typical monomial.

2.1. Products on U

We begin by defining dual associative cross products on $\mathbf{U}$. Given a pair of monomials $A \otimes B \in \mathbf{U}^{\mathbf{y}}_{\mathbf{v}} \otimes \mathbf{U}^{\mathbf{u}}_{\mathbf{x}}$, define the *wedge* and *cech cross products* by

$$A \overset{\wedge}{\times} B = \begin{cases} A \otimes B, & \text{if } \mathbf{v} = \mathbf{x}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad A \overset{\vee}{\times} B = \begin{cases} A \otimes B, & \text{if } \mathbf{u} = \mathbf{y}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathbf{U}^{\mathbf{y}}_{\mathbf{x}} \overset{\wedge}{\times} \mathbf{U}^{\mathbf{u}}_{\mathbf{x}} \subseteq \mathbf{U}^{\mathbf{y},\mathbf{u}}_{\mathbf{x}}$ and $\mathbf{U}^{\mathbf{y}}_{\mathbf{v}} \overset{\vee}{\times} \mathbf{U}^{\mathbf{y}}_{\mathbf{x}} \subseteq \mathbf{U}^{\mathbf{y}}_{\mathbf{v},\mathbf{x}}$; denote $\hat{\mathbf{U}} = \left(\mathbf{U}, \overset{\wedge}{\times}\right)$ and $\check{\mathbf{U}} = \left(\mathbf{U}, \overset{\vee}{\times}\right)$.

Non-zero cross products create block matrices:

$$A \overset{\wedge}{\times} B = \left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] \quad \text{and} \quad A \overset{\vee}{\times} B = [A\ B].$$

In terms of arrows, $A \overset{\wedge}{\times} B \in \mathbf{U}^{\mathbf{y},\mathbf{u}}_{\mathbf{x}}$ runs from the vertical $x = |\mathbf{x}|$ to vertical $x = p$ in $\mathbb{N}^2$ and $A \overset{\vee}{\times} B \in \mathbf{U}^{\mathbf{y}}_{\mathbf{v},\mathbf{x}}$ runs from horizontal $y = q$ to $y = |\mathbf{y}|$. Thus an $n \times 1$ monomial $A^{\overset{\wedge}{\times} n} \in \mathbf{U}^{b \cdots b}_{a}$ initiates at (a, n) and terminates at $(1, nb)$; a $1 \times n$ monomial $A^{\overset{\vee}{\times} n} \in \mathbf{U}^{b}_{a \cdots a}$ initiates at $(na, 1)$ and terminates at (n, b).

We also define a composition product on $\mathbf{U}$.

Definition 1. *A monomial pair* $A^{q \times s} \otimes B^{t \times p} = [\theta_{y_{k\ell}, v_{k\ell}}] \otimes \left[\eta_{u_{ij}, x_{ij}}\right] \in M \otimes M$ *is a*

(i) <u>*Transverse Pair*</u> *(TP) if* $s = t = 1$, $u_{1,j} = q$ *and* $v_{k,1} = p$ *for all* j, k, *i.e., setting* $x_j = x_{1,j}$ *and* $y_k = y_{k,1}$ *gives*

$$A \otimes B = \begin{bmatrix} \theta_{y_1,p} \\ \vdots \\ \theta_{y_q,p} \end{bmatrix} \otimes \begin{bmatrix} \eta_{q,x_1} & \cdots & \eta_{q,x_p} \end{bmatrix} \in \mathbf{U}^{\mathbf{y}}_{p} \otimes \mathbf{U}^{q}_{\mathbf{x}}.$$

(ii) <u>*Block Transverse Pair*</u> *(BTP) if there exist* $t \times s$ *block decompositions* $A = [A'_{k'\ell}]$ *and* $B = \left[B'_{ij'}\right]$ *such that* $A'_{i\ell} \otimes B'_{i\ell}$ *is a TP for all* i, ℓ.

Note that BTP block decomposition is unique; furthermore, $A \otimes B \in \mathbf{U}^{\mathbf{y}}_{\mathbf{v}} \otimes \mathbf{U}^{\mathbf{u}}_{\mathbf{x}}$ is a BTP if and only if $\mathbf{y} \in \mathbb{N}^{|\mathbf{u}|}$ and $\mathbf{x} \in \mathbb{N}^{|\mathbf{v}|}$ if and only if the initial point of arrow A and the terminal point of arrow B coincide.

Example 1. *A pairing of monomials* $A^{4\times 2}\otimes B^{2\times 3}\in \mathbf{U}_{2,1}^{1,5,4,3}\otimes \mathbf{U}_{1,2,3}^{3,1}$ *is a* 2×2 *BTP per the block decompositions*

$$\left(\begin{array}{c|c} \theta_{1,2} & \theta_{1,1} \\ \theta_{5,2} & \theta_{5,1} \\ \theta_{4,2} & \theta_{4,1} \\ \hline \theta_{3,2} & \theta_{3,1} \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} \eta_{3,1} & \eta_{3,2} & \eta_{3,3} \\ \hline \eta_{1,1} & \eta_{1,2} & \eta_{1,3} \end{array}\right).$$

As arrows, A initializes at $(6,2)$ *and terminates at* $(3,4)$; *B initializes at* $(3,4)$ *and terminates at* $(2,13)$.

When $\mathbf{x}\times\mathbf{y}\in\mathbb{N}^p\times\mathbb{N}^q$, every pair of monomials $A\otimes B\in\mathbf{U}_p^{\mathbf{y}}\otimes\mathbf{U}_{\mathbf{x}}^q$ is a TP. Define a mapping

$$\gamma:\mathbf{U}_p^{\mathbf{y}}\otimes\mathbf{U}_{\mathbf{x}}^q\to\mathbf{U}_{|\mathbf{x}|}^{|\mathbf{y}|}$$

by the composition

$$\mathbf{U}_p^{\mathbf{y}}\otimes\mathbf{U}_{\mathbf{x}}^q\overset{\iota_q\otimes\iota_p}{\longrightarrow}\mathbf{U}_{pq}^{|\mathbf{y}|}\otimes\mathbf{U}_{|\mathbf{x}|}^{qp}\overset{1\otimes\sigma_{q,p}^*}{\longrightarrow}\mathbf{U}_{pq}^{|\mathbf{y}|}\otimes\mathbf{U}_{|\mathbf{x}|}^{pq}\overset{\circ}{\to}\mathbf{U}_{|\mathbf{x}|}^{|\mathbf{y}|},$$

where ι_p and ι_q are the canonical isomorphisms. Then for $A=[\theta_{y_k,p}]\in\mathbf{U}_p^{\mathbf{y}}$ and $B=\left[\eta_{q,x_j}\right]\in\mathbf{U}_{\mathbf{x}}^q$, we have

$$\gamma\left(A\otimes B\right)=\left(\theta_{y_1,p}\otimes\cdots\otimes\theta_{y_q,p}\right)\sigma_{q,p}\left(\eta_{q,x_1}\otimes\cdots\otimes\eta_{q,x_p}\right);$$

denote this expression either by $A\cdot B$ or $\gamma(\theta_{y_1,p},\ldots,\theta_{y_q,p};\eta_{q,x_1},\ldots,\eta_{q,x_p})$. The γ-product on matrices of double corollas is typically a matrix of non-planar graphs (see Figure 2). Note that γ agrees with the composition product on the universal preCROC [**20**].

More generally, if $A\otimes B$ is a BTP with block decompositions $A=[A'_{i\ell}]$ and $B=[B'_{i\ell}]$, define $\gamma\left(A\otimes B\right)_{i\ell}=\gamma\left(A'_{i\ell}\otimes B'_{i\ell}\right)$. Then γ sends $A^{q\times s}\otimes B^{t\times p}\in\mathbf{U}_{\mathbf{v}}^{\mathbf{y}}\otimes\mathbf{U}_{\mathbf{x}}^{\mathbf{u}}$ to a $t\times s$ monomial in $\mathbf{U}_{\mathbf{x}'}^{\mathbf{y}'}$, where $\mathbf{x}'$ and $\mathbf{y}'$ are obtained from $\mathbf{x}$ and $\mathbf{y}$ by summing s and t successive coordinate substrings: The length of the i^{th} substring of $\mathbf{x}$ is the length of the row matrices in the i^{th} column of B'; the length of the ℓ^{th} substring of $\mathbf{y}$ is the length of the column matrices in the ℓ^{th} row of A'. In any case, $\gamma\left(A\otimes B\right)$ is expressed as an arrow from the initial point of B to the terminal point of A.

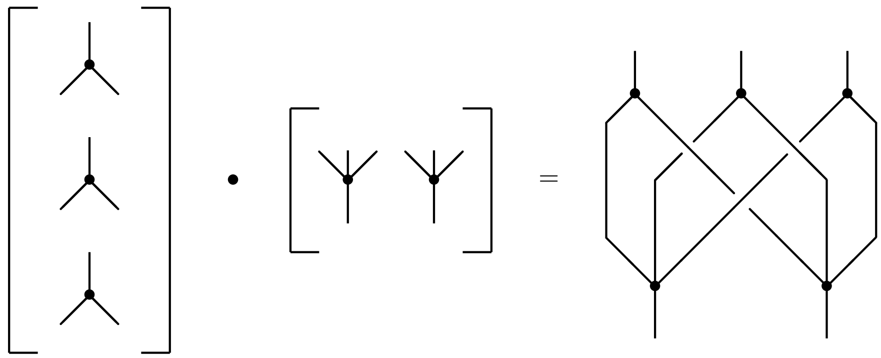

Figure 2. The γ-product as a non-planar graph.

Define the *upsilon product* $\Upsilon : M \otimes M \to M$ on matrices $A, B \in M$ by

$$\Upsilon(A \otimes B)_{i\ell} = \begin{cases} \gamma\left(A'_{i\ell} \otimes B'_{i\ell}\right), & \text{if } A \otimes B \text{ is a BTP} \\ 0, & \text{otherwise} \end{cases}$$

and let $A \cdot B = \Upsilon(A \otimes B)$. Note that Υ restricts to an associative product on $\mathbf{U}$.

Example 2. *The γ-product of the 2×2 BTP $A^{4\times 2} \otimes B^{2\times 3}$ in Example 1 is the following 2×2 monomial in $\mathbf{U}_{3,3}^{10,3}$:*

$$\begin{pmatrix} \theta_{1,2} & \theta_{1,1} \\ \theta_{5,2} & \theta_{5,1} \\ \theta_{4,2} & \theta_{4,1} \\ \theta_{3,2} & \theta_{3,1} \end{pmatrix} \cdot \begin{pmatrix} \eta_{3,1} & \eta_{3,2} & \eta_{3,3} \\ \eta_{1,1} & \eta_{1,2} & \eta_{1,3} \end{pmatrix} = \begin{pmatrix} \gamma(\theta_{1,2}, \theta_{5,2}, \theta_{4,2}; \eta_{3,1}, \eta_{3,2}) & \gamma(\theta_{1,1}, \theta_{5,1}, \theta_{4,1}; \eta_{3,3}) \\ \gamma(\theta_{3,2}; \eta_{1,1}, \eta_{1,2}) & \gamma(\theta_{3,1}; \eta_{1,3}) \end{pmatrix}.$$

The row matrices in successive columns of the block decomposition of B have respective lengths 2 and 1; thus $\mathbf{x}' = (3,3)$ is obtained from $\mathbf{x} = (1,2,3)$. Similarly, the column matrices in successive rows of the block decomposition of A have respective lengths 3 and 1; thus $\mathbf{y}' = (10,3)$ is obtained from $\mathbf{y} = (1,5,4,3)$. Finally, the map $A \cdot B : \left(H^{\otimes 3} \otimes H^{\otimes 3}\right)^{\otimes 2} \to \left(H^{\otimes 10}\right)^{\otimes 2} \otimes \left(H^{\otimes 3}\right)^{\otimes 2}$ is expressed as an arrow initializing at $(6,2)$ and terminating at $(2,13)$.

2.2. Cup products on $C^*(P, \mathbf{U})$

Let $C_*(X)$ denote the cellular chains on a polytope X and assume that $C_*(X)$ comes equipped with a diagonal $\Delta_X : C_*(X) \to C_*(X) \otimes C_*(X)$. Let G be a module (graded or ungraded); if G is graded, ignore the grading and view G as a graded module concentrated in degree zero. The cellular k-cochains on X with coefficients in G is the graded module

$$C^k(X;G) = Hom^{-k}(C_*(X), G).$$

When G is a DGA with multiplication μ, the diagonal Δ_X induces a DGA structure on $C^*(X;G)$ with cup product

$$f \smile g = \mu(f \otimes g)\Delta_X.$$

Unless explicitly indicated otherwise, non-associative cup products with multiple factors are parenthesized on the extreme left, i.e., $f \smile g \smile h = (f \smile g) \smile h$.

In our prequel [**17**] we constructed an explicit non-coassociative non-cocommutative diagonal Δ_P on the cellular chains of permutahedra $C_*(P_n)$ for each $n \geqslant 1$. Thus we immediately obtain non-associative, non-commutative DGA's $C^*(P_n;\hat{\mathbf{U}})$ and $C^*(P_n;\check{\mathbf{U}})$ with respective wedge and cech cup products $\wedge$ and $\vee$. Of course, summing over all n gives wedge and cech cup products on $C^*(P;\mathbf{U})$.

The modules $C^*(P;\hat{\mathbf{U}})$ and $C^*(P;\check{\mathbf{U}})$ are equipped with second cup products $\wedge_\ell$ and $\vee_\ell$, which arise from the Υ-product on $\mathbf{U}$ together with the "level coproduct." Recall that m-faces of P_{n+1} are indexed by PLT's with $n+2$ leaves, $n-m+1$ levels and root in level $n-m+1$ (see [**11**] or [**17**], for example). The *level coproduct*

$$\Delta_\ell : C_*(P) \to C_*(P) \otimes C_*(P)$$

vanishes on $e^n \subset P_{n+1}$ and is defined on proper m-faces e^m as follows: For each k, prune the tree of e^m between levels k and $k+1$ and sequentially number the stalks or trees removed from left-to-right. Let e'_k denote the pruned tree; let e''_k denote the tree obtained by attaching all stalks and trees removed during pruning to a common root (see Figure 4). Then

$$\Delta_\ell(e^m) = \sum_{1 \leqslant k \leqslant n-m} e'_k \otimes e''_k.$$

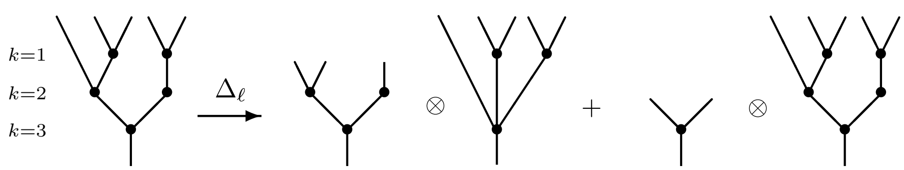

Figure 3: $\Delta_\ell(24|1|3) = 1|2 \otimes 24|13 + 1 \otimes 24|1|3$.

Obviously, Δ_ℓ is non-counital, non-cocommutative and non-coassociative; in fact, it fails to be a chain map. Fortunately this is not an obstruction to lifting the γ-product on $\mathbf{U}$ to a $\smile_\ell$-product on $C^*(P;\mathbf{U})$ since we restrict to certain canonically associative subalgebras of $\mathbf{U}$. For $\varphi, \varphi' \in C^*(P;\hat{\mathbf{M}})$ define $\varphi \wedge_\ell \varphi' = \varphi \smile_\ell \varphi'$ and for $\psi, \psi' \in C^*(P;\check{\mathbf{M}})$ define $\psi \vee_\ell \psi' = \psi' \smile_\ell \psi$. Some typical $\wedge_\ell$-products appear in Example 3 below.

2.3. Leaf sequences

Let T be a PLT with at least 2 leaves. Prune T immediately below the first level, trimming off k stalks and corollas. Number them sequentially from left-to-right and let n_j denote the number of leaves in the j^{th} corolla (if T is a corolla, $k=1$ and the pruned tree is a stalk). The *leaf sequence* of T is the vector $(n_1, \ldots, n_k) \in \mathbb{N}^k$.

Given integers n and k with $1 \leqslant k \leqslant n+1$, let $\mathbf{n} = (n_1, \ldots, n_k) \in \{\mathbf{x} \in \mathbb{N}^k \mid |\mathbf{x}| = n+2\}$. When $k=1$, $e_{\mathbf{n}}$ denotes the $(n+2)$-leaf corolla. Otherwise, $e_{\mathbf{n}}$ denotes the 2-levelled tree with leaf sequence $\mathbf{n}$. Now consider the DGA $\mathbf{U}$ with its γ-product. Given a codim 0 or 1 face $e_{\mathbf{n}} \subset P$ and a cochain $\varphi \in C^*(P; \mathbf{U})$, let $\varphi_{\mathbf{n}} = \varphi(e_{\mathbf{n}})$.

Example 3. *Let $\varphi \in C^0(P; \overset{\wedge}{\mathbf{U}})$ and $\bar{\varphi} \in C^1(P; \overset{\wedge}{\mathbf{U}})$. When $n=1$, the proper faces of P_2 are its vertices $1|2$ and $2|1$ with $\Delta_\ell(1|2) = 1 \otimes 1|2$ and $\Delta_\ell(2|1) = 1 \otimes 2|1$. Evaluating $\wedge_\ell$-squares on vertices gives the compositions*

$$\varphi^2(1|2) = \varphi_2\varphi_{21} \qquad \varphi^2(2|1) = \varphi_2\varphi_{12}.$$

When $n=2$, the proper faces of P_3 are its edges and vertices (see Figure 4). Evaluating quadratic and cubic $\wedge_\ell$-products on edges and vertices gives

$$\begin{array}{lll} \bar{\varphi}^2(1|23) &=& \bar{\varphi}_3\bar{\varphi}_{211} \\ \bar{\varphi}^2(2|13) &=& \bar{\varphi}_3\bar{\varphi}_{121} \\ \bar{\varphi}^2(3|12) &=& \bar{\varphi}_3\bar{\varphi}_{112} \\ \varphi\bar{\varphi}(12|3) &=& \varphi_2\bar{\varphi}_{31} \\ \varphi\bar{\varphi}(13|2) &=& \varphi_2\bar{\varphi}_{22} \\ \varphi\bar{\varphi}(23|1) &=& \varphi_2\bar{\varphi}_{13} \end{array} \qquad \begin{array}{lll} \varphi^2\bar{\varphi}(1|2|3) &=& \varphi_2\varphi_{21}\bar{\varphi}_{211} \\ \varphi^2\bar{\varphi}(1|3|2) &=& \varphi_2\varphi_{12}\bar{\varphi}_{211} \\ \varphi^2\bar{\varphi}(2|1|3) &=& \varphi_2\varphi_{21}\bar{\varphi}_{121} \\ \varphi^2\bar{\varphi}(2|3|1) &=& \varphi_2\varphi_{12}\bar{\varphi}_{121} \\ \varphi^2\bar{\varphi}(3|1|2) &=& \varphi_2\varphi_{21}\bar{\varphi}_{112} \\ \varphi^2\bar{\varphi}(3|2|1) &=& \varphi_2\varphi_{12}\bar{\varphi}_{112}. \end{array}$$

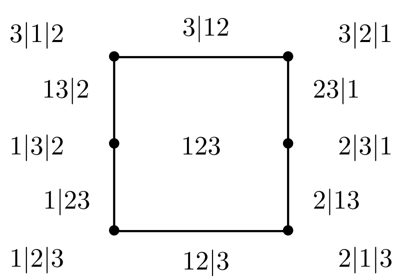

Figure 4: The permutahedron P_3.

3. The biderivative

The definition of the biderivative operator $Bd : \mathbf{U} \to \mathbf{U}$ requires some notational preliminaries. Let $\mathbf{x}_i(r) = (1, \ldots, r, \ldots, 1)$ with $r \geqslant 1$ in the i^{th} position; the subscript i will be suppressed unless we need its precise value; in particular, let $\mathbf{1}^k = \mathbf{x}(1) \in \mathbb{N}^k$. Again, we often suppress the superscript and write $\mathbf{1}$ when the context is clear. Let

$$\mathbf{U}_0 = \bigoplus_{\mathbf{x},\mathbf{y}=\mathbf{1}} \mathbf{U}_{\mathbf{x}}^{\mathbf{y}} \quad \text{and} \quad \mathbf{U}_+ = \mathbf{U}/\mathbf{U}_0 = \bigoplus_{\mathbf{x}\neq\mathbf{1} \text{ or } \mathbf{y}\neq\mathbf{1}} \mathbf{U}_{\mathbf{x}}^{\mathbf{y}};$$

also denote the submodules

$$\begin{array}{lclclcl} \mathbf{U}_{u_0} & = & \bigoplus_{\mathbf{x}\in\mathbb{N}^p;\ |\mathbf{x}|>p\geqslant 1} \mathbf{U}_{\mathbf{x}}^{1} & \qquad & \mathbf{U}_{v_0} & = & \bigoplus_{\mathbf{y}\in\mathbb{N}^q;\ |\mathbf{y}|>q\geqslant 1} \mathbf{U}_{1}^{\mathbf{y}} \\ \mathbf{U}_{u} & = & \bigoplus_{\substack{\mathbf{x}\in\mathbb{N}^p;\ |\mathbf{x}|>1 \\ p,q\geqslant 1}} \mathbf{U}_{\mathbf{x}}^{q} & & \mathbf{U}_{v} & = & \bigoplus_{\substack{\mathbf{y}\in\mathbb{N}^q;\ |\mathbf{y}|>1 \\ p,q\geqslant 1}} \mathbf{U}_{p}^{\mathbf{y}} \end{array}$$

and note that

$$\mathbf{U}_{u\cap v} = \mathbf{U}_u \cap \mathbf{U}_v = \bigoplus_{p,q\geqslant 2} \mathbf{U}_p^q.$$

Monomials in $\mathbf{U}_u$ and $\mathbf{U}_v$ are respectively row and column matrices. In terms of arrows, $\mathbf{U}_0$ consists of all arrows of length zero; $\mathbf{U}_+$ consists of all arrows of positive length. Arrows in $\mathbf{U}_u$ initiate on the x-axis at $(|\mathbf{x}|, 1)$, $|\mathbf{x}| > 1$, and terminate in the region $x \leqslant |\mathbf{x}|$; in particular, arrows in $\mathbf{U}_{u_0}$ lie on the x-axis and terminate at $(p, 1)$. Arrows in $\mathbf{U}_v$ initiate in the region $y \leqslant |\mathbf{y}|$ and terminate at $(1, |\mathbf{y}|)$, $|\mathbf{y}| > 1$; in particular, arrows in $\mathbf{U}_{v_0}$ lie on the y-axis and initiate at $(1, q)$. Thus arrows in $\mathbf{U}_{u\cap v}$ "transgress" from the x to the y axis.

3.1. The non-linear operator BD

Recall that $\mathbf{n}$ is a leaf sequence if and only if $\mathbf{n} \neq \mathbf{1}$; when this occurs, $e_{\mathbf{n}}$ is a face of $P_{|\mathbf{n}|-1}$ in dimension $|\mathbf{n}| - 2$ or $|\mathbf{n}| - 3$. Let

$$\hat{T}op(\mathbf{U}_+) \subset C^*(P; \bigoplus\nolimits_{\mathbf{x}\neq\mathbf{1} \text{ or } \mathbf{y}\neq\mathbf{1}} \mathbf{U}_{\mathbf{x}}^{\mathbf{y}})$$

be the submodule supported on $e_{\mathbf{x}}$ when $\mathbf{x} \neq \mathbf{1}$ or on $e_{\mathbf{y}}$ otherwise. Dually, let

$$\check{T}op(\mathbf{U}_+) \subset C^*(P; \bigoplus\nolimits_{\mathbf{x}\neq\mathbf{1} \text{ or } \mathbf{y}\neq\mathbf{1}} \mathbf{U}_{\mathbf{x}}^{\mathbf{y}})$$

be the submodule supported on $e_{\mathbf{y}}$ when $\mathbf{y} \neq \mathbf{1}$ or on $e_{\mathbf{x}}$ otherwise. When $\mathbf{x}, \mathbf{y} \neq \mathbf{1}$, a monomial $A \in \mathbf{U}_{\mathbf{x}}^{\mathbf{y}}$ is identified with the cochains $\varphi_A \in \hat{T}op(\mathbf{U}_+)$ and $\psi_A \in \check{T}op(\mathbf{U}_+)$ respectively supported on the codim 0 or 1 faces of P with leaf sequences $\mathbf{x}$ and $\mathbf{y}$ (see Figure 5). Let

$$\hat{\pi} : C^*(P; \mathbf{U}_+) \to \hat{T}op(\mathbf{U}_+) \quad \text{and} \quad \check{\pi} : C^*(P; \mathbf{U}_+) \to \check{T}op(\mathbf{U}_+)$$

be the canonical projections.

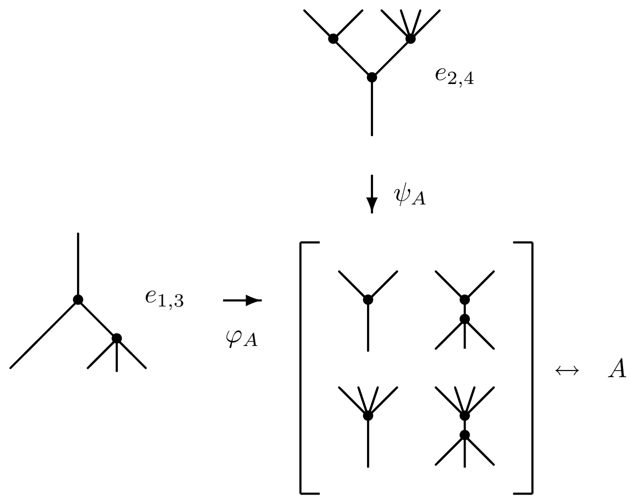

Figure 5: The monomial A is identified with φ_A and ψ_A.

For $\mathbf{x}_i(n), \mathbf{y}_i(n) \in \mathbb{N}^q$, let

$$[\theta_{1,n}]_i^\vee = [Id \ \cdots \theta_{1,n} \cdots \ Id] \in \mathbf{U}^1_{\mathbf{x}_i(n)} \subset \mathbf{U}_{u_0}$$

and

$$[\theta_{n,1}]_i^\wedge = \begin{bmatrix} Id \\ \vdots \\ \theta_{n,1} \\ \vdots \\ Id \end{bmatrix} \in \mathbf{U}_1^{\mathbf{y}_i(n)} \subset \mathbf{U}_{v_0}.$$

Given $\phi \in C^*(P; \mathbf{U}_+)$ and $n \geqslant 2$, consider the top dimensional cell $e_n \subseteq P_{n-1}$ and components $\phi_{1,n}(e_n) \in \mathbf{U}^1_n \subset \mathbf{U}_{u_0}$ and $\phi_{n,1}(e_n) \in \mathbf{U}^n_1 \subset \mathbf{U}_{v_0}$ of $\phi(e_n)$. The *coderivation cochain of* ϕ is the global cochain $\phi^c \in \hat{T}op(\mathbf{U}_+)$ given by

$$\phi^c(e_\mathbf{x}) = \begin{cases} [\phi_{1,n}(e_n)]_i^\vee, & \text{if } \mathbf{x} = \mathbf{x}_i(n),\ 1 \leqslant i \leqslant q,\ n \geqslant 2 \\ 0, & \text{otherwise.} \end{cases}$$

Dually, the *derivation cochain of* ϕ is the cochain $\phi^a \in \check{T}op(\mathbf{U}_+)$ given by

$$\phi^a(e_\mathbf{y}) = \begin{cases} [\phi_{n,1}(e_n)]_i^\wedge, & \text{if } \mathbf{y} = \mathbf{y}_i(n),\ 1 \leqslant i \leqslant q,\ n \geqslant 2 \\ 0, & \text{otherwise.} \end{cases}$$

Thus ϕ^c is supported on the union of the $e_{\mathbf{x}_i(n)}$'s and takes the value

$$\phi^c C_*(P) = \sum_{\substack{1 \leqslant i \leqslant q \\ q \geqslant 1}} [Id \cdots \underbrace{\phi_{u_0}(e_n)}_{i^{th}} \cdots Id]^{1\times q} \in \mathbf{U}_{u_0},$$

and dually for ϕ^a.

Finally, define an operator $\tau : C^*(P;\mathbf{U}_+) \to C^*(P;\mathbf{U}_+)$ on a cochain $\xi \in C^*(P;\mathbf{U}_\mathbf{x}^\mathbf{y})$ by

$$\tau(\xi)(e) = \begin{cases} \xi(e_\mathbf{x}), & \text{if } e = e_\mathbf{y};\ \mathbf{x},\mathbf{y} \neq \mathbf{1}, \\ \xi(e_\mathbf{y}), & \text{if } e = e_\mathbf{x};\ \mathbf{x},\mathbf{y} \neq \mathbf{1}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that τ is involutory on $\hat{T}op(\mathbf{U}_+) \cap \check{T}op(\mathbf{U}_+)$.

We are ready to define the non-linear operator BD. First define operators

$$\hat{B}D : C^*(P;\hat{\mathbf{U}}_+) \to C^*(P;\hat{\mathbf{U}}_+) \text{ and } \check{B}D : C^*(P;\check{\mathbf{U}}_+) \to C^*(P;\check{\mathbf{U}}_+)$$

by

$$\hat{B}D(\varphi) = \hat{\varphi} \text{ and } \check{B}D(\psi) = \check{\psi},$$

where

$$\begin{aligned} \hat{\varphi} &= \xi_u + \xi_u \wedge \xi_u + \cdots + \xi_u^n + \cdots \\ \xi &= \varphi + \varphi \wedge_\ell \varphi + \cdots + \varphi^n + \cdots \end{aligned} \quad \text{and} \quad \begin{aligned} \check{\psi} &= \zeta_v + \zeta_v \vee \zeta_v + \cdots + \zeta_v^n + \cdots \\ \zeta &= \psi + \psi \vee_\ell \psi + \cdots + \psi^n + \cdots . \end{aligned}$$

Then define

$$BD : C^*(P;\mathbf{U}_+) \times C^*(P;\mathbf{U}_+) \to C^*(P;\mathbf{U}_+) \times C^*(P;\mathbf{U}_+)$$

on a pair $\varphi \times \psi$ by

$$BD(\varphi \times \psi) = (\hat{\pi} \circ \hat{B}D)(\varphi^c + \tau\psi) \times (\check{\pi} \circ \check{B}D)(\psi^a + \tau\varphi).$$

Theorem 1. *Given* $\sum_{(m,n)\in\mathbb{N}^2\smallsetminus\mathbf{1}} \theta_{n,m} \in U$, *there is a unique fixed point*

$$\varphi \times \psi = BD(\varphi \times \psi) \tag{3.1}$$

such that

$$\begin{array}{rcll} \varphi_{u_0}(e_m) &=& \theta_{1,m}, & m \geqslant 2 \\ \varphi_{u\cap v}(e_m) &=& \sum_{n\geqslant 2} \theta_{n,m}, & m \geqslant 2 \\ \psi_{v_0}(e_n) &=& \theta_{n,1}, & n \geqslant 2 \\ \psi_{u\cap v}(e_n) &=& \sum_{m\geqslant 2} \theta_{n,m}, & n \geqslant 2. \end{array} \tag{3.2}$$

Before proving this theorem, we remark that the existence of a fixed point $\varphi \times \psi$ for BD is a deep generalization of the following classical fact: If a map h is (co)multiplicative (or a (co)derivation), restricting h to generators and (co)extending as a (co)algebra map (or as a (co)derivation) recovers h. These classical (co)multiplicative or (f,g)-(co)derivation extension procedures appear here as restrictions (3.1) to P_1 (a point) or to P_2 (an interval). Restricting (3.1) to a general permutahedron P_n gives a new extension procedure whose connection with the classical ones is maintained by the compatibility of the canonical cellular projection $P_n \to I^{n-1}$ with diagonals. Let us proceed with a proof of Theorem 1.

Proof. Define $BD^{(1)} = BD$ and $BD^{(n+1)} = BD \circ BD^{(n)},\ n \geqslant 1$. Let

$$\hat{F}_n\mathbf{U} = \bigoplus_{n<|\mathbf{x}|} \mathbf{U}_{\mathbf{x}}^{\mathbf{y}} \text{ and } \overset{\vee}{F}_n\mathbf{U} = \bigoplus_{n<|\mathbf{y}|} \mathbf{U}_{\mathbf{x}}^{\mathbf{y}}.$$

A straightforward check shows that for each $n \geqslant 1$,

$$BD^{(n+1)} = BD^{(n)} \text{ modulo } \hat{F}_n C^*(P;\mathbf{U}) \times \overset{\vee}{F}_n C^*(P;\mathbf{U}).$$

So define

$$D = \varinjlim BD^{(n)}.$$

Clearly, $BD \circ D = D$.

Let $\varphi_{u\cap v} \in \hat{T}op(\mathbf{U}_+)$ and $\psi_{u\cap v} \in \overset{\vee}{T}op(\mathbf{U}_+)$ be the two cochains uniquely defined by (3.2) and supported on the appropriate faces. Then

$$\varphi \times \psi = D((\varphi^c + \varphi_{u\cap v}) \times (\psi^a + \psi_{u\cap v}))$$

is the (unique) solution of (3.1). □

3.2. The biderivative operator on U

Let $\widetilde{Bd} : \mathbf{U}_+ \times \mathbf{U}_+ \to \mathbf{U}_+ \times \mathbf{U}_+$ be the operator given by the composition

$$\begin{array}{ccc} \mathbf{U}_+ \times \mathbf{U}_+ & \xrightarrow{\widetilde{Bd}} & \mathbf{U}_+ \times \mathbf{U}_+ \\ \| & & \| \\ \hat{T}op(\mathbf{U}_+) \times \overset{\vee}{T}op(\mathbf{U}_+) & \xrightarrow[BD]{} & \hat{T}op(\mathbf{U}_+) \times \overset{\vee}{T}op(\mathbf{U}_+), \end{array}$$

where the vertical maps are canonical identification bijections and BD is its restriction to $\hat{T}op(\mathbf{U}_+) \times \overset{\vee}{T}op(\mathbf{U}_+)$. For $A \in \mathbf{U}_+$, let $A_1 \times A_2 = \widetilde{Bd}\,(A \times A)$ and define operators $\hat{B}d, \overset{\vee}{B}d : \mathbf{U}_+ \to \mathbf{U}_+$ by

$$\hat{B}d(A) = A_1 \quad \text{and} \quad \overset{\vee}{B}d(A) = A_2.$$

Given an operator $F : \mathbf{U} \to \mathbf{U}$ and a submodule $\mathbf{U}_\epsilon \subset \mathbf{U}$, denote the composition of F with the projection $\mathbf{U} \to \mathbf{U}_\epsilon$ by F_ϵ. Define the operator $Bd_+ : \mathbf{U}_+ \to \mathbf{U}_+$ as the sum

$$Bd_+ = Id_{u\cap v} + \hat{B}d_{u_0 \oplus v} + \overset{\vee}{B}d_{u \oplus v_0}.$$

Note that $\hat{B}d_{u_0}(\theta)$ is the cofree coextension of $\theta \in U_{1,*}$ as a coderivation of $T^c H$; dually, $\overset{\vee}{B}d_{v_0}(\eta)$ is the free extension of $\eta \in U_{*,1}$ as a derivation of $T^a H$.

On the other hand, observe that $U \cap \mathbf{U}_0 = U_{1,1}$. Given $A \in U_{1,1}$, $1 \leqslant i \leqslant q$ and $1 \leqslant j \leqslant p$, let $A_{ij}^{q\times p} = (a_{k\ell}) \in \mathbf{U}_{\mathbf{1}^p}^{\mathbf{1}^q}$ be the $q \times p$ monomial such that

$$a_{k\ell} = \begin{cases} A, & \text{if } (k,\ell) = (i,j), \\ Id, & \text{otherwise.} \end{cases}$$

Define $Bd_0 : U_{1,1} \to TTU_{1,1}$ by

$$Bd_0(A) = \sum_{\substack{1 \leqslant i \leqslant q,\ 1 \leqslant j \leqslant p \\ p,q \geqslant 1}} A_{ij}^{q \times p}.$$

Then $Bd_0\,(A)$ is the free linear extension of A as a (co)derivation of TTH.

We establish the following fundamental notion:

Definition 2. *The biderivative operator*

$$Bd : \mathbf{U} \to \mathbf{U}$$

associated with the universal PROP U is the sum

$$Bd = Bd_0 + Bd_+ : \mathbf{U}_0 \oplus \mathbf{U}_+ \to \mathbf{U}_0 \oplus \mathbf{U}_+.$$

An element $A \in \mathbf{U}$ is a biderivative if $A = Bd(A)$.

Restating Theorem 1 in these terms we have:

Proposition 1. *Every element $\omega = \sum_{i,j \geqslant 1} \omega_{j,i} \in U$ has a unique biderivative $d_\omega \in TTU$.*

Thus the biderivative can be viewed as a non-linear map $d_- : U \to TTU$.

3.3. The $\circledcirc$-product on U

The biderivative operator allows us to extend Gerstenhaber's (co)operation [**3**] $\circ : U_{*,1} \oplus U_{1,*} \to U$ to a (non-bilinear) operation

$$\circledcirc : U \times U \to U \tag{3.3}$$

defined for $\theta \times \eta \in U \times U$ by the composition

$$\circledcirc : U \times U \overset{d_\theta \times d_\eta}{\longrightarrow} \mathbf{U} \times \mathbf{U} \overset{\Upsilon}{\longrightarrow} \mathbf{U} \overset{pr}{\to} U,$$

where the last map is the canonical projection. The following is now obvious:

Proposition 2. *The $\circledcirc$ operation (3.3) acts bilinearly only on the submodule $U_{*,1} \oplus U_{1,*}$.*

Remark 1. *The bilinear part of the $\circledcirc$ operation, i.e., its restriction to $U_{*,1} \oplus U_{1,*}$, is completely determined by the associahedra K (rather than the permutahedra) and induces the cellular projection $P_n \to K_{n+1}$ due to A. Tonks [**24**].*

Example 4. *Throughout this example the symbol "1" denotes the identity. Consider a DGM (H,d) together with maps $\mu = \theta_{1,2}$, $\theta = \theta_{2,2}$, $\Delta = \theta_{2,1} \in End(TH)$. Let us compute the biderivative of $\omega = d + \mu + \theta + \Delta$ and its $\circledcirc$-square. Consider the pair of cochains $\varphi \times \psi \in \hat{T}op\,(\mathbf{U}_+) \times \overset{\vee}{T}op\,(\mathbf{U}_+)$ supported on $e_2 \times e_2$ such that $\varphi\,(e_2) = \mu + \theta$ and $\psi\,(e_2) = \theta + \Delta$. Then*

$$\varphi^c\,(e_2 + e_{21} + e_{12} + \cdots) = \mu + [\mu\ 1] + [1\ \mu] + \cdots \in \mathbf{U}_{u_0},$$

$$\psi^a\,(e_2 + e_{21} + e_{12} + \cdots) = \Delta + \begin{bmatrix}\Delta \\ 1\end{bmatrix} + \begin{bmatrix}1 \\ \Delta\end{bmatrix} + \cdots \in \mathbf{U}_{v_0} \quad \textit{and}$$

$$\tau\,(\varphi)\,(e_2) = \varphi_{u \cap v}\,(e_2) = \theta = \psi_{u \cap v}\,(e_2) = \tau\,(\psi)\,(e_2)$$

Set $\alpha = \varphi^c + \tau\psi$ *and* $\beta = \psi^a + \tau\varphi$*; then*

$$(\alpha \wedge_\ell \alpha)(C_*P) = (\mu + \theta)(\mu \otimes 1 + 1 \otimes \mu) + \cdots \quad \text{and}$$
$$(\beta \vee_\ell \beta)(C_*P) = (\Delta \otimes 1 + 1 \otimes \Delta)(\theta + \Delta) + \cdots$$

Furthermore, the projections $\alpha_u = \alpha$ *and* $\beta_v = \beta$ *so that*

$$\xi_u = \alpha + \alpha \wedge_\ell \alpha + \cdots \quad \text{and} \quad \zeta_v = \beta + \beta \vee_\ell \beta + \cdots .$$

Then $BD(\varphi \times \psi) = \hat{\varphi} \times \check{\psi}$*, where*

$$\hat{\varphi} = \xi_u + \xi_u \wedge \xi_u + \cdots \quad \text{and} \quad \check{\psi} = \zeta_v + \zeta_v \vee \zeta_v + \cdots .$$

Now $\hat{\varphi}_{u_0 \oplus v}(C_*P) = \mu + \mu(\mu \otimes 1 + 1 \otimes \mu) + \theta + \left[\begin{smallmatrix}\theta\\ \theta\end{smallmatrix}\right] + \left[\begin{smallmatrix}\theta\\ \mu\end{smallmatrix}\right] + \left[\begin{smallmatrix}\mu\\ \theta\end{smallmatrix}\right] + \left[\begin{smallmatrix}\mu\\ \mu\end{smallmatrix}\right] + \cdots$ *and*

$\check{\psi}_{u \oplus v_0}(C_*P) = \theta + [\theta\ \theta] + [\Delta\ \theta] + [\theta\ \Delta] + [\Delta\ \Delta] + \Delta + (\Delta \otimes 1 + 1 \otimes \Delta)\Delta + \cdots$

so that $Bd_+(\omega) = \theta + \left(\hat{\varphi}_{u_0 \oplus v} + \check{\psi}_{u \oplus v_0}\right)(C_*P)$. *Finally, we adjoin the linear extension of the differential* d *in* $TTU_{1,1}$ *and obtain*

$$\begin{aligned} d_\omega = {} & d + [d\ 1] + [1\ d] + \cdots + \left[\begin{smallmatrix}d\\ 1\end{smallmatrix}\right] + \left[\begin{smallmatrix}1\\ d\end{smallmatrix}\right] + \cdots + \mu + \Delta + \cdots \\ & + \theta + \mu(\mu \otimes 1 + 1 \otimes \mu) + \cdots + (\Delta \otimes 1 + 1 \otimes \Delta)\Delta + \cdots \\ & + \left[\begin{smallmatrix}\theta\\ \theta\end{smallmatrix}\right] + \left[\begin{smallmatrix}\theta\\ \mu\end{smallmatrix}\right] + \left[\begin{smallmatrix}\mu\\ \theta\end{smallmatrix}\right] + \left[\begin{smallmatrix}\mu\\ \mu\end{smallmatrix}\right] + \cdots + [\theta\ \theta] + [\Delta\ \theta] + [\theta\ \Delta] + [\Delta\ \Delta] + \cdots . \end{aligned}$$

Then (up to sign),

$$\begin{aligned} \omega \circledcirc \omega = {} & \left(\left[\begin{smallmatrix}d\\ 1\end{smallmatrix}\right] + \left[\begin{smallmatrix}1\\ d\end{smallmatrix}\right]\right) \cdot \theta + \theta \cdot ([d\ 1] + [1\ d]) + \Delta \cdot \mu + \left[\begin{smallmatrix}\mu\\ \mu\end{smallmatrix}\right] \cdot [\Delta\ \Delta] + \\ & + \left[\begin{smallmatrix}\mu\\ \mu\end{smallmatrix}\right] \cdot ([\Delta\ \theta] + [\theta\ \Delta]) + \theta \cdot ([1\ \mu] + [\mu\ 1]) + \\ & + \left(\left[\begin{smallmatrix}\theta\\ \mu\end{smallmatrix}\right] + \left[\begin{smallmatrix}\mu\\ \theta\end{smallmatrix}\right]\right) \cdot [\Delta\ \Delta] + \left(\left[\begin{smallmatrix}\Delta\\ 1\end{smallmatrix}\right] + \left[\begin{smallmatrix}1\\ \Delta\end{smallmatrix}\right]\right) \cdot \theta + \cdots . \end{aligned}$$

Some low dimensional relations implied by $\omega \circledcirc \omega = 0$ *are (up to sign):*

$$\begin{aligned} (d \otimes 1 + 1 \otimes d)\theta + \theta(d \otimes 1 + 1 \otimes d) &= \Delta\mu - (\mu \otimes \mu)\sigma_{2,2}(\Delta \otimes \Delta) \\ (\mu \otimes \mu)\sigma_{2,2}(\Delta \otimes \theta + \theta \otimes \Delta) &= \theta(\mu \otimes 1 + 1 \otimes \mu) \\ (\mu \otimes \theta + \theta \otimes \mu)\sigma_{2,2}(\Delta \otimes \Delta) &= (\Delta \otimes 1 + 1 \otimes \Delta)\theta. \end{aligned}$$

In fact, if $\omega \circledcirc \omega = 0$ *then* (H, ω) *is an "*A_∞*-bialgebra."*

4. A_∞-bialgebras

In this section we define the notion of an A_∞-bialgebra. Our approach extends the definition of an A_∞-(co)algebra in terms of Gerstenhaber's (co)operation. Roughly speaking, an A_∞-bialgebra is a graded R-module H equipped with compatible A_∞-algebra and A_∞-coalgebra structures. Structural compatibility of the operations in an A_∞-bialgebra is determined by the $\circledcirc$ operation (3.3). Before stating the definition, we mention three natural settings in which A_∞-bialgebras appear (details appear in the sequel [**18**]).

(1) Let X be a space and let $C_*(X)$ denote the simplicial singular chain complex of X. Although Adams' cobar construction $\Omega C_*(X)$ is a (strictly coassociative) DG

Hopf algebra [**1**], [**2**], [**8**], it seems impossible to introduce a strictly coassociative coproduct on the double cobar construction $\Omega^2 C_*(X)$. Instead there is an A_∞-coalgebra structure on $\Omega^2 C_*(X)$ that is compatible with the product and endows $\Omega^2 C_*(X)$ with an A_∞-bialgebra structure.

(2) If H is a graded bialgebra and $\rho : RH \longrightarrow H$ is a (bigraded) multiplicative resolution, it is difficult to introduce a strictly coassociative coproduct on RH in such a way that ρ is a map of bialgebras. However, there exists an A_∞-bialgebra structure on RH such that ρ is a morphism of A_∞-bialgebras.

(3) If A is any DG bialgebra, its homology $H(A)$ has a canonical A_∞-bialgebra structure.

The definition of an A_∞-bialgebra H uses the $\circledcirc$-operation on U_H to mimic the definition of an A_∞-algebra.

Definition 3. *An* $\underline{A_\infty\text{-bialgebra}}$ *is a graded* R*-module* H *equipped with operations*

$$\{\omega^{j,i} \in Hom^{i+j-3}(H^{\otimes i}, H^{\otimes j})\}_{i,j\geqslant 1}$$

such that $\omega = \sum_{i,j\geqslant 1} \omega^{j,i} \in U$ *satisfies* $\omega \circledcirc \omega = 0$.

Here are some of the first structural relations among the operations in an A_∞-bialgebra:

$$\begin{aligned}
d\omega^{2,2} = \;& \omega^{2,1}\omega^{1,2} - \left(\omega^{1,2}\otimes\omega^{1,2}\right)\sigma_{2,2}(\omega^{2,1}\otimes\omega^{2,1})\\[1ex]
d\omega^{3,2} = \;& \omega^{3,1}\omega^{1,2} + (\omega^{2,1}\otimes 1 - 1\otimes\omega^{2,1})\omega^{2,2}\\
& -(\omega^{1,2}\otimes\omega^{1,2}\otimes\omega^{1,2})\sigma_{3,2}\left[\omega^{3,1}\otimes(1\otimes\omega^{2,1})\omega^{2,1} + (\omega^{2,1}\otimes 1)\omega^{2,1}\otimes\omega^{3,1}\right]\\
& + \left[(\omega^{2,2}\otimes\omega^{1,2} - \omega^{1,2}\otimes\omega^{2,2})\right]\sigma_{2,2}(\omega^{2,1}\otimes\omega^{2,1})\\[1ex]
d\omega^{2,3} = \;& -\omega^{2,1}\omega^{1,3} + \omega^{2,2}(1\otimes\omega^{1,2} - \omega^{1,2}\otimes 1)\\
& +[\omega^{1,3}\otimes\omega^{1,3}(1\otimes\omega^{1,3}) + \omega^{1,3}(\omega^{1,2}\otimes 1)\otimes\omega^{1,3}]\sigma_{2,3}(\omega^{2,1}\otimes\omega^{2,1}\otimes\omega^{2,1})\\
& +(\omega^{1,2}\otimes\omega^{1,2})\sigma_{2,2}(\omega^{2,1}\otimes\omega^{2,2} - \omega^{2,2}\otimes\omega^{2,1}).
\end{aligned}$$

Example 5. *The structure of an* A_∞*-bialgebra whose initial data consists of a strictly coassociative coproduct* $\Delta : H \to H^{\otimes 2}$ *together with* A_∞*-algebra operations* $m_i : H^{\otimes i} \to H$, $i \geqslant 2$, *is determined as in Example 4 but with* $\varphi(e_i) = m_i$, $\psi(e_2) = \Delta$. *This time the action of* τ *is trivial since all initial maps lie in* $\mathbf{U}_{u_0 \oplus v_0}$ *and we obtain the following structure relation for each* $i \geqslant 2$*:*

$$(\xi_u \wedge \xi_u)(e_i) \cdot [\underbrace{\Delta \;\cdots\; \Delta}_{i \text{ factors}}] = \Delta \cdot m_i.$$

Indeed, the classical bialgebra relation appears when $i = 2$.

We conclude with a statement of our main theorem (the definition of an A_∞-bialgebra morphism appears in the sequel [**18**]).

Theorem 2. *Let* A *be an* A_∞*-bialgebra; if the ground ring* R *is not a field, assume that the homology* $H = H(A)$ *is torsion-free. Then* H *inherits a canonical bialgebra structure that extends to an* A_∞*-bialgebra structure* $\{\omega^{j,i}\}_{i,j\geqslant 1}$ *with* $\omega^{1,1} = 0$.

Furthermore, there is a map of A_∞-bialgebras

$$F = \{F^{j,i}\}_{i,j\geqslant 1} : H \Longrightarrow A,$$

with $F^{j,i} \in Hom^{i+j-2}(H^{\otimes i}, A^{\otimes j})$, such that $F^{1,1} : H \to A$ is a map of DGM's inducing an isomorphism on homology.

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