On Derived Categories and Derived Functors

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Abstract: For an abelian category, a category equivalent to its derived category is constructed by means of specific projective (injective) multicomplexes, the so-called homological resolutions.

1. Introduction

The derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ was introduced by Verdier in 1963, see [15] and [16]. It was defined as the localization of the category of unbounded chain complexes with respect to quasi-isomorphisms. The existence of $D(\mathcal{A})$ creates set-theoretical problems. Verdier proved the existence of $D(\mathcal{A})$ only in the case when $\mathcal{A}$ has finite global dimension. Later existence of $D(\mathcal{A})$ was established by Spaltenstein [14] in the case when $\mathcal{A}$ is the category of modules over a ring, or more generally category of modules over a sheaf of rings. The first case was also considered by Hovey in [8]. Recently the existence of $D(\mathcal{A})$ was proved in the case when $\mathcal{A}$ is a Grothendieck category, see for example [1].

In the present paper we prove the existence of $D(\mathcal{A})$ in the case when $\mathcal{A}$ has enough projectives and countable coproducts. By duality the same is true provided $\mathcal{A}$ has enough injectives and countable products.

The essential part of the paper was in fact written about 15 years ago when the author was visiting the Heidelberg University. As it is partially reviewed above in the meantime there appeared various kinds of descriptions of derived categories, however, the decision to write this paper is motivated by reasons mentioned above and continued below: Our approach uses a theory of *multi-complexes* and emphasis a rôle of the homology of differential graded objects (bounded or unbounded) on the additive level; nowadays multicomplexes are considered to be endowed with multiplicative or higher order operations that

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measure certain standard relations up to homotopy (see, for example, [6], [9], [12], [13]). It should be noted that such enriched homological multicomplexes are candidates to be (co)fibrant objects in the appropriated closed model category, since the analogs of Proposition 2 below (compare Proposition 3 in [13]). So that it is expected to use them for homotopy classification problems behind the rational homotopy theory too.

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2. The main result

A chain map between unbounded chain complexes which induces an isomorphism in homology (i.e., a quasi-isomorphism) is not a homotopy equivalence even each complex consists of projective objects of an abelian category $\mathcal{A}$, and also an additive functor, such as Hom and $\otimes$ one, does not preserve quasi-isomorphisms [4]. There are various kinds of restrictions on chain complexes that guarantees quasi-isomorphisms to be homotopy equivalences (see [4], [14], [8], [5]).

Here we consider the other kind of restriction by introducing special projective (injective) chain complexes, *homological multicomplexes*. In particular, by means of these complexes we can describe the derived category $D(\mathcal{A})$ and to construct the derived functor for an additive functor mentioned in the introduction.

Usually the above restrictions are evoked to start inductively from the first non-trivial component of a (bounded) chain complex. In our case, the induction relies on a specific filtration of the total complex of a multicomplex involving all (total) degrees simultaneously (compare [4], [8]). On the other hand, given a chain complex A over $\mathcal{A}$, one considers in the theory of derived category projective (injective) replacements $C \to A$ ($A \to C$) of A, i.e., quasi-isomorphisms with C consisting of projective (injective) components from $\mathcal{A}$. We show that for each chain complex A there is a multicomplex C such that its total complex is quasi-isomorphic to A and $C^{*,j}$ is a projective resolution of the cohomology $H^j(A)$ for each $j \in \mathbb{Z}$. So that among projective replacements of A mentioned above, *the homological resolution C could be chosen small as possible*.

In order to state our main theorem below we choose the language of *projective* objects (the case of injective objects is entirely dual).

Given an abelian category $\mathcal{A}$ with countable coproducts, a *multicomplex*

over $\mathcal{A}$ is a bigraded object $C^{*,*} = \{C^{i,j}\}_{i,j\in\mathbb{Z}}$ together with morphisms

$$d^r : C^{i,j} \longrightarrow C^{i+r,j-r+1}, \qquad r \geq 0,$$

such that

$$\sum_{p+q=n} d^p d^q = 0 \qquad \text{for each} \ \ n \geq 0.$$

The *total* complex of $C^{*,*}$ is the chain complex $(\mathrm{Tot}(C), d^*)$ with

$$\mathrm{Tot}(C)^n = \bigoplus_{i+j=n} C^{i,j} \qquad \text{and} \qquad d^* = d^0 + d^1 + \cdots + d^r + \cdots.$$

In particular, $d^1 d^1 = 0$ when $d^0 = 0$. A multicomplex $(C^{*,*}, d^*)$ is called *homological* if $d^0 = 0$, $C^{i,*} = 0$ for $i > 0$ and $H^i(C^{i,*}, d^1) = 0$ for $i < 0$. A multicomplex $(C^{*,*}, d^*)$ is called *projective* if each $C^{i,j}$ is a projective object of $\mathcal{A}$. A *column (resolution) filtration* of a multicomplex $(C^{*,*}, d^*)$ is a sequence $\{C_{(k)}\}_{k\leq 0}$ with

$$C_{(k)} = \bigoplus_{k\leq i\leq 0} C^{i,*}.$$

A *multicomplex map* $f : A \to B$ between two multicomplexes A and B is a chain map of total degree zero that preserves the column (resolution) filtration, i.e., $f\,\mathrm{Tot}(A)^n \subset \mathrm{Tot}(B)^n$ and $fA_{(k)} \subset B_{(k)}$; so that f has the components $f = f^0 + \cdots + f^i + \cdots$ with $f^i : A^{s,t} \to B^{s+i,t-i}$. A *homotopy* between two maps $f, g : A \to B$ of multicomplexes is a chain homotopy $s : A \to B$ of total degree -1 that lowers the column filtration by 1, i.e., $s\,\mathrm{Tot}(A)^n \subset \mathrm{Tot}(B)^{n-1}$ and $sA_{(k)} \subset B_{(k-1)}$.

Note that, unlike standard bicomplexes, in a homological multicomplex we have no vertical differentials; this fact together with the acyclicity with respect to the horizontal differential d^1 guarantees the spectral sequence arising from the column filtration to be collapsed; in particular, the other components d^r, $r \geq 2$, have no action to change the cohomology non-isomorphically; in other words, when d^r varies in $\mathrm{Hom}(C^{i,j}, C^{i+r,j-r+1})$ for $r \geq 2$ one obtains multicomplexes with isomorphic cohomologies (see Figure 1).

Let $K(\mathcal{A})$ be the category whose objects are chain complexes over $\mathcal{A}$ and morphisms are homotopy classes of maps denoted by $[-,-]$, $K_{\mathcal{M}}(\mathcal{A})$ be the category whose objects are multicomplexes over $\mathcal{A}$ and morphisms are homotopy classes of multicomplex maps denoted by $[-,-]_{\mathcal{M}}$, while $K_{\mathcal{P}}(\mathcal{A})$ be the (sub)category whose objects are homological projective multicomplexes over $\mathcal{A}$ and morphisms are homotopy classes of maps denoted by $[-,-]_{\mathcal{P}}$.

Recall that the *derived category* $D(\mathcal{A})$ of $\mathcal{A}$ is defined as the category obtained from $K(\mathcal{A})$ by inverting the class of quasi-isomorphisms [5], [10], [15], and let $Q : K(\mathcal{A}) \to D(\mathcal{A})$ be the localization functor. Let $\kappa : K_{\mathcal{P}}(\mathcal{A}) \to K(\mathcal{A})$ be the functor defined by the following obvious proposition.

PROPOSITION 1. *Given a multicomlex C and a morphism $\mathfrak{f} : C \to C'$ in $K_{\mathcal{P}}(\mathcal{A})$, the assignments $C \to \mathrm{Tot}(C)$ and $\mathfrak{f} \to [\mathrm{Tot}(f)]$ define a functor $\kappa : K_{\mathcal{P}}(\mathcal{A}) \to K(\mathcal{A})$ for a representative f of $\mathfrak{f}$.*

Proof. First remark that the assignments $C \to \mathrm{Tot}(C)$ and $f \to \mathrm{Tot}(f)$ define a functor from the category of multicomplexes and multicomplex maps to the category of chain complexes and chain maps over $\mathcal{A}$. Now if $f, g : C \to C'$ are two chain homotopic maps of multicomplexes $f \underset{s}{\simeq} g$, then clearly s induces a map $\mathrm{Tot}(s) : \mathrm{Tot}(C) \to \mathrm{Tot}(C')$ such that

$$\mathrm{Tot}(f) \underset{\mathrm{Tot}(s)}{\simeq} \mathrm{Tot}(g) \, .$$

∎

Consider the functor

$$\iota : K_{\mathcal{P}}(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

obtained as the composition $K_{\mathcal{P}}(\mathcal{A}) \xrightarrow{\kappa} K(\mathcal{A}) \xrightarrow{Q} D(\mathcal{A})$. The main statement here is the following

THEOREM 1. *If an abelian category $\mathcal{A}$ has enough projectives and countable coproducts, then the functor $\iota : K_{\mathcal{P}}(\mathcal{A}) \to D(\mathcal{A})$ is an equivalence of categories.*

This theorem relies on the following "Whitehead (or Adams-Hilton) type" proposition that has an independent interest. Given a chain complex A, we consider it as bigraded via $A^{0,*} = A^*$ and $A^{i,*} = 0$ for $i \neq 0$, and then regard $K(\mathcal{A})$ as the subcategory of $K_{\mathcal{M}}(\mathcal{A})$.

PROPOSITION 2. *Let $f : A \to B$ be a quasi-isomorphism in $K_{\mathcal{M}}(\mathcal{A})$ where A or B is a chain complex or a homological multicomlex over $\mathcal{A}$. If C is a homological projective multicomplex then the induced map $f_{\#} : [C, A]_{\mathcal{M}} \to [C, B]_{\mathcal{M}}$ is a bijection.*
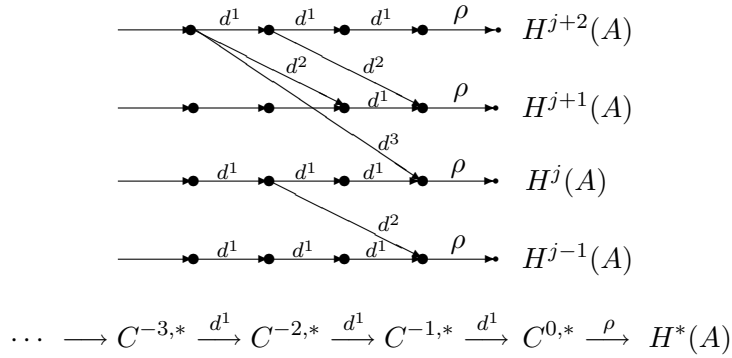
Figure 1. A fragment of a homological resolution.

3. Proof of theorem 1

Given a chain complex (A, d) from $K(\mathcal{A})$, its *homological resolution* is a homological projective multicomlex $(C^{*,*}, d^*)$ with a multicomlex map $\phi : C \to A$ inducing a quasi-isomorphism $\text{Tot}(\phi) : (\text{Tot}(C), d^*) \to (\text{Tot}(A), d) = (A, d)$. In particular, $(C^{*,j}, d^1)$ forms a projective resolution of the object $H^j(A)$ for all $j \in \mathbb{Z}$ (see Figure 1).

PROPOSITION 3. *If an abelian category $\mathcal{A}$ has enough projectives and countable coproducts, then any chain complex A of $K(\mathcal{A})$ has a homological resolution $\phi : C \to A$.*

Proof. First choose a projective resolution $\rho : (C^{*,j}, d^1) \to H^j(A)$ of $H^j(A)$ for each $j \in \mathbb{Z}$ so that $d^1 : C^{i-1,*} \to C^{i,*}$ for $i \le 0$. Consider the epimorphism

$$\rho^0 = \rho|_{C^{0,*}} : C^{0,*} \longrightarrow H^*(A).$$

Factor it through cocycles $C^{0,*} \xrightarrow{\phi'} ZA^* \xrightarrow{\nu} H^*(A)$ and obtain a morphism $\phi^0 : C^{0,*} \xrightarrow{\phi'} ZA^* \hookrightarrow A^*$. Define also a morphism

$$\phi^1 : C^{-1,*} \longrightarrow A^{*-1}$$

by $d\,\phi^1 = \phi^0 d^1$. Assume by induction that we have constructed morphisms

$$d^r : C^{*,*} \longrightarrow C^{*+r,*-r+1} \qquad \text{and} \qquad \phi^r : C^{-r,*} \longrightarrow A^{*-r}$$

for $0 \le r \le n$ (with $d^0 = 0$) such that

$$\sum_{k+\ell \leq n+1} d^k d^\ell = 0 \qquad \text{and} \qquad d\phi^{(n)} = \phi^{(n-1)} d^{(n)} \text{ on } C_{(-n)},$$

where $d^{(n)} = \sum_{1 \leq r \leq n} d^r$ and

$$\phi^{(n)} = \sum_{0 \leq r \leq n} \phi^r : C_{(-n)} \longrightarrow A.$$

Consider the composition $\phi^{(n)} d^{(n)} : C^{-n-1,*} \to A^{*-n}$. Clearly,

$$\phi^{(n)} d^{(n)} : C^{-n-1,*} \longrightarrow ZA^{*-n} (\hookrightarrow A^{*-n}).$$

Form the composition $\nu\phi^{(n)} d^{(n)} : C^{-n-1,*} \to H^{*-n}(A)$ to obtain a morphism $d^{n+1} : C^{-n-1,*} \to C^{0,*-n}$ such that $\rho^0 d^{n+1} = -\nu\phi^{(n)} d^{(n)}$. Since $H^i(C^{i,*}, d^1) = 0$ for $i < 0$, we can extend d^{n+1} on $C^{*,*}$ with $\sum_{k+\ell \leq n+2} d^k d^\ell = 0$. Then $\nu\phi^{(n)} d^{(n+1)} = 0$, and there is a morphism $\phi^{n+1} : C^{-n-1,*} \to A^{*-n-1}$ with $d\,\phi^{(n+1)} = \phi^{(n)} d^{(n+1)}$.

Define

$$d^* = \sum_{r \geq 1} d^r \qquad \text{and} \qquad \phi = \sum_{r \geq 0} \phi^r$$

to obtain the homological resolution $\phi : (C, d^*) \to (A, d)$. ∎

In particular, one can take $d^r = 0$, $r \geq 2$, on $C^{*,*}$ and $\phi = \rho$ when $d_A = 0$.

Note that in the abelian category of modules homological multicomplex resolutions were in fact constructed in [2], [3] (compare [7]).

3.1. PROOF OF PROPOSITION 2. As above the proof uses the induction on the resolution degree of the homological projective multicomplex C. We assume that A and B are chain complexes; the case of homological multicomplexes is similar. First show that $f_\#$ is an epimorphism. Let $\bar{g} : C \to B$. Consider the restriction $\bar{g}^0 = \bar{g}|_{C^{0,*}} : C^{0,*} \to B^*$. Since $\bar{g}$ is chain, $\bar{g}^0$ factors through cocycles $\bar{g}^0 : C^{0,*} \to ZB^*(\hookrightarrow B^*)$. Since $H(f)$ is an isomorphism, we can define $g^0 : C^{0,*} \to ZA^* \hookrightarrow A^*$ such that $\nu f g^0 = \nu\bar{g}^0 : C^{0,*} \to ZB^* \xrightarrow{\nu} H^*(B)$. Obviously, there is $s^0 : C^{0,*} \to B^{*-1}$ with $f g^0 - \bar{g}^0 = ds^0$ and then put $\bar{\bar{g}} = \bar{g} + ds^0 + s^0 d$ to obtain the commutative diagram

$$
\begin{array}{ccc}
C_{(0)}(= C^{0,*}) & \xrightarrow{g^0} & A \\
\downarrow & & \downarrow f \\
C & \xrightarrow{\bar{\bar{g}}} & B.
\end{array}
$$

Assume by induction that we have constructed morphisms

$$g^i : C^{-i,*} \longrightarrow A^{*-i}, \quad 0 \le i \le n, \qquad \text{and} \qquad \tilde{g} : C \longrightarrow B$$

such that $\tilde{g} \simeq \bar{g}$ and the following diagram

$$
\begin{array}{ccc}
C_{(n)} & \xrightarrow{\; g^{(n)} \;} & A \\
\downarrow & & \downarrow{\scriptstyle f} \\
C & \xrightarrow{\; \tilde{g} \;} & B
\end{array}
$$

commutes. Since the above diagram is commutative and $H(f)$ is an isomorphism we can choose $g^{n+1} : C^{-n-1,*} \to A^{*-n-1}$ together with $s^{n+1} : C^{-n-1,*} \to B^{*-n-2}$ such that

$$dg^{n+1} = g^{(n)}d^{(n+1)} \qquad \text{and} \qquad fg^{n+1} - \tilde{g}^{n+1} = ds^{n+1}.$$

Put $\tilde{\tilde{g}} = \tilde{g} + ds^{n+1} + s^{n+1}d$ to obtain the commutative diagram

$$
\begin{array}{ccc}
C_{(n+1)} & \xrightarrow{\; g^{(n+1)} \;} & A \\
\downarrow & & \downarrow{\scriptstyle f} \\
C & \xrightarrow{\; \tilde{\tilde{g}} \;} & B.
\end{array}
$$

Thus, $g = \sum_{i \ge 0} g^i : C \to A$ is a chain map with $fg \simeq \bar{g}$, i.e., $f_\#[g] = [\bar{g}]$.

Now let $g, \bar{h} : C \to A$ be two morphisms such that fg and fh are connected by a chain homotopy $s : C \to B$, i.e., $fg \underset{s}{\simeq} fh$. Clearly, $fg^0 - fh^0 = ds^0$ for $s^0 = s|_{C^{0,*}}$, and, since $H(f)$ is an isomorphism there is $t^0 : C^{0,*} \to A^{*-1}$ with $g^0 - h^0 = dt^0$. Choose t^0 with $ft^0 - s^0 = d\beta^0$ for some $\beta^0 : C^{0,*} \to B^{*-2}$. Put $h' = h + dt^0 + t^0 d$ and $s' = \{s'^k\}_{k \ge 0}$,

$$
s'^k = \begin{cases} 0 & \text{if } k = 0, \\ s^1 + \beta^0 d & \text{if } k = 1, \\ s^k & \text{if } k > 1. \end{cases}
$$

Then $h'^0 = g^0$ and $fg \underset{s'}{\simeq} fh'$.

Assume by induction that we have constructed a morphism $\bar{h} : C \to A$ together with chain homotopy $\bar{s} : C \to B$ such that $\bar{h}^{(n-1)} = g^{(n-1)}$, $\bar{h} \simeq h$ and $fg \underset{\bar{s}}{\simeq} f\bar{h}$ with $\bar{s}^{(n-1)} = 0$. Since $H(f)$ is an isomorphism there is $t^n : C^{-n,*} \to$

A^{*-n-1} such that $g^n - \bar{h}^n = dt^n$. We can choose t^n with $ft^n - \bar{s}^n = d\beta^n$ for some $\beta^n : C^{-n,*} \to B^{*-n-2}$. Put $\bar{\bar{h}} = \bar{h} + dt^n + t^n d$ and $\bar{\bar{s}} = \{\bar{\bar{s}}^k\}_{k \geq 0}$,

$$
\bar{\bar{s}}^k = \begin{cases} 0 & \text{if } 0 \leq k \leq n \,, \\ \bar{s}^{n+1} + \beta^n d & \text{if } k = n+1 \,, \\ \bar{s}^k & \text{if } k > n+1 \,. \end{cases}
$$

Then $\bar{\bar{h}}^{(n+1)} = g^{(n+1)}$ and $fg \underset{\bar{\bar{s}}}{\simeq} f\bar{\bar{h}}$. The induction step is completed.

Finally, we get that $g \simeq h$ as required.

3.2. PROOF OF THEOREM 1. Given a chain complex A, apply Proposition 3 to obtain a resolution $\phi : C \to A$. Given a chain map $f : A \to A'$, consider a diagram

$$
\begin{array}{ccc}
& & C' \\
& & \downarrow{\scriptstyle \phi'} \\
C \xrightarrow{\ \phi\ } A \xrightarrow{\ f\ } A'
\end{array}
$$

and apply Proposition 2 for the quasi-isomorphism ϕ' to obtain a multicomplex map $g : C \to C'$ such that $[\phi'g] = [f\phi]$ in $K_{\mathcal{M}}(\mathcal{A})$. Thus, we get the functor

$$
\varrho : K(\mathcal{A}) \longrightarrow K_{\mathcal{P}}(\mathcal{A})
$$

which to each chain complex assigns its homological resolution. Again by the above propositions we deduce that ϱ transforms quasi-isomorphisms into isomorphisms, so that using the universal property of the localization functor Q we get the functor

$$
\bar{\varrho} : D(\mathcal{A}) \longrightarrow K_{\mathcal{P}}(\mathcal{A})
$$

such that the diagram

$$
\begin{array}{ccc}
K(\mathcal{A}) & \xrightarrow{\ Q\ } & D(\mathcal{A}) \\
{\scriptstyle \varrho} \searrow & & \downarrow{\scriptstyle \bar{\varrho}} \\
& K_{\mathcal{P}}(\mathcal{A}) &
\end{array}
$$

commutes.

Now it is straightforward to check that $\bar{\varrho}$ is an inverse for ι.

3.3. Derived functors. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive covariant functor between abelian categories with enough projectives and countable coproducts. Obviously, we have the induced functor $\mathcal{F} : K_{\mathcal{P}}(\mathcal{A}) \to K_{\mathcal{M}}(\mathcal{B})$. It is easy to verify that the composition

$$D(\mathcal{A}) \xrightarrow{\bar{\varrho}} K_{\mathcal{P}}(\mathcal{A}) \xrightarrow{\mathcal{F}} K_{\mathcal{M}}(\mathcal{B}) \xrightarrow{\kappa} K(\mathcal{B}) \xrightarrow{Q} D(\mathcal{B})$$

is the *left derived functor* in the sense of Verdier

$$LF : D(\mathcal{A}) \longrightarrow D(\mathcal{B}) \,.$$

3.4. The minimality of homological resolutions. Finally, some remarks about the minimality of homological resolutions. For example, given a chain complex (H^*, d) on the category of modules over a principal ideal domain a homological resolution $(C^{*,*}, d^*) \to (H^*, d)$ of (H^*, d) can be chosen to be concentrated in the resolution degrees 0 and -1 with $d^r = 0$ unless $r = 1$. On the other hand, additional structures on $C^{i,j}$ mentioned in the introduction may impose $i < -1$ (cf. [13]): Namely, if $H^* = \mathbb{Z}[x_1, \ldots, x_n], n > 1$, is a polynomial algebra with $d = 0$, then the requirement that $C^{*,*}$ is endowed with a non-commutative multiplication *compatible* with the bigrading imposes the multiplicative generators of the minimal resolution $C^{i,j}$ to be concentrated in resolution degrees $i \geq -n + 1$, while the resolution lengthes of groups $C^{*,j}$ for $j \geq 0$ are unbounded. If one introduces a non-commutative operation $\smile_1$ on $C^{*,*}$ that measures the non-commutativity of the above multiplication and is compatible with the bigrading, then even the multiplicative generators can not be no longer chosen to be bounded by the resolution degree and so on.

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