Ann. Funct. Anal. 7 (2016), no. 2, 217–231
http://dx.doi.org/10.1215/20088752-3334839
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

SCREEN TYPE PROBLEMS FOR ANISOTROPIC PSEUDO-MAXWELL'S EQUATIONS

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Dedicated to Professor Anthony To-Ming Lau

Communicated by Gradimir V. Milovanovic

Abstract. We investigate screen-type boundary value problems for anisotropic pseudo-Maxwell's equations. It is shown that the problems with tangent traces are well posed in tangent Sobolev spaces. The unique solvability result is then obtained by using a potential method and the coercivity result of Costabel for the bilinear form associated with the pseudo-Maxwell's equation.

1. Introduction

The study of boundary value problems in electromagnetism naturally leads us to the pseudo-Maxwell's equations with inherited tangent boundary conditions, which are in some sense nonstandard for the system of elliptic equations (see works of Buffa, Costabel, Christiansen, Dauge, Hazard, Lenoir, Mitrea, Nicaise and others). Indeed, let us consider the time-harmonic Maxwell equations in an anisotropic domain $\Omega \subset \mathbb{R}^3$,

$$\begin{cases} \operatorname{curl} \mathbf{H} + i\omega\varepsilon\mathbf{E} = 0, \\ \operatorname{curl} \mathbf{E} - i\omega\mu\mathbf{H} = 0. \end{cases}$$

Here, the vector-functions $\mathbf{E} = (E_1, E_2, E_3)^\top$ and $\mathbf{H} = (H_1, H_2, H_3)^\top$ denote the scattered electric and magnetic fields, ω is the frequency, and the corresponding

Received Mar. 31, 2015; Accepted May 12, 2015.
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2010 *Mathematics Subject Classification.* Primary 35J25; Secondary 35C15, 47F05.
Keywords. screens, pseudo-Maxwell equations, integral representation, potential theory.

relative dielectric permittivity ε and the relative magnetic permeability μ are real-valued, constant matrices

$$\varepsilon = [\varepsilon_{jk}]_{3\times 3}, \qquad \mu = [\mu_{jk}]_{3\times 3}, \tag{1.1}$$

which are symmetric and positive definite

$$\langle \varepsilon\xi, \xi\rangle \geq c|\xi|^2, \qquad \langle \mu\xi, \xi\rangle \geq d|\xi|^2, \quad \forall \xi \in \mathbb{R}^3,$$

for some positive constants $c > 0$, $d > 0$, where $\langle \eta, \xi\rangle := \sum_{j=1}^{3} \eta_j \overline{\xi}_j, \eta, \xi \in \mathbb{C}^3$. The electric field $\mathbf{E}$ satisfies the second-order equation

$$\operatorname{curl} \mu^{-1} \operatorname{curl} \mathbf{E} - \omega^2 \varepsilon \mathbf{E} = 0 \quad \text{in } \Omega.$$

Since also

$$\operatorname{div} \varepsilon \mathbf{E} = 0 \quad \text{in } \Omega \tag{1.2}$$

holds, $\mathbf{E}$ satisfies

$$\operatorname{curl} \mu^{-1} \operatorname{curl} \mathbf{E} - s\varepsilon \operatorname{grad} \operatorname{div}(\varepsilon \mathbf{E}) - \omega^2 \varepsilon \mathbf{E} = 0 \quad \text{in } \Omega, \tag{1.3}$$

where s is a positive real number. The typical boundary condition for the “electric” boundary value problems is

$$\boldsymbol{\nu} \times \mathbf{E} = \mathbf{f} \quad \text{on } \mathcal{S} := \partial\Omega, \tag{1.4}$$

where $\boldsymbol{\nu}$ is the unit normal vector field to $\mathcal{S}$. Note that $\mathbf{E}$ is a solution of the elliptic equation (1.3), and in order to get the Shapiro–Lopatinsky conditions we complete (1.4) by transferring (1.2) on the boundary, and thus we have

$$\operatorname{div} \varepsilon \mathbf{E} = 0 \quad \text{on } \mathcal{S}.$$

In the present article, we consider the case when $\Omega = \mathbb{R}^3 \setminus \overline{\mathcal{C}}$, where $\mathcal{C} \subset \mathbb{R}^3$ denotes a screen which is a compact, orientable, and nonself-intersecting surface where the boundary and the frequency parameter ω is nonzero and complex-valued (i.e., $\operatorname{Im} \omega \neq 0$).

Let us mention that due to the presence of tangent boundary conditions (1.4), the use of potential methods for investigation is complicated; these types of problems are mostly studied by variational methods. This fact leads us to overcome the difficulties and investigate the well-posedness of the screen-type boundary value problems for pseudo-Maxwell equations

$$\boldsymbol{A}(D)\boldsymbol{U} := \operatorname{curl} \mu^{-1} \operatorname{curl} \boldsymbol{U} - s\varepsilon \operatorname{grad} \operatorname{div}(\varepsilon \boldsymbol{U}) - \omega^2 \varepsilon \boldsymbol{U} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{C}} \tag{1.5}$$

with the help of the potential method and tools of pseudodifferential equations. Moreover, it is worth mentioning the importance of the method presented here. Indeed, having the integral representation formula for solutions allow us to use the boundary element methods for numerical approximations of the elliptic boundary value problems.

2. Formulation of the problems

For the remainder of this paper, unless stated otherwise, Ω denotes either a bounded $\Omega^+ \subset \mathbb{R}^3$ or an unbounded $\Omega^- := \mathbb{R}^3 \backslash \overline{\Omega^+}$ domain with smooth, nonself-intersecting boundary $\mathcal{S} := \partial\Omega^+$ and where $\boldsymbol{\nu}$ is the outer unit normal vector field to $\mathcal{S}$. Whenever necessary, we will specify the case.

By $\mathcal{C}$ we denote a subsurface of $\mathcal{S}$ (a screen) with boundary $\partial\mathcal{C}$, which has two faces $\mathcal{C}^-$ and $\mathcal{C}^+$ and which inherits the orientation from $\mathcal{S}$: $\mathcal{C}^+$ borders (the inner domain Ω^+) and $\mathcal{C}^-$ borders (the outer domain Ω^-). The unbounded domain with a screen configuration is denoted by

$$\mathbb{R}^3_{\mathcal{C}} := \mathbb{R}^3 \setminus \overline{\mathcal{C}}.$$

Let $H^r(\Omega)$ and $H^r(\mathcal{S})$ be Bessel potential spaces. The space $\widetilde{H}^r(\mathcal{C})$ comprises those functions $\varphi \in H^r(\mathcal{S})$ which are supported in $\overline{\mathcal{C}}$ (functions with the "vanishing traces on the boundary"). (For detailed definitions and properties of these spaces, see [10], [11], [14]). Note that for the vector analogues, we use $\mathbb{H}$ letter (i.e., we write $\mathbb{H}^r(\Omega)$, $\mathbb{H}^r(\mathcal{S})$, and $\widetilde{\mathbb{H}}^r(\mathcal{C})$, respectively). Moreover, we use the boldface letters for vector-functions, in contrast to scalar functions, which are denoted by nonboldface letters.

Let us note that since $\mathcal{S}$ is smooth, the Dirichlet trace $\gamma\boldsymbol{U}$, the tangential (Dirichlet) traces $\gamma_\tau\boldsymbol{U} = \gamma(\boldsymbol{\nu}\times\boldsymbol{U})$ and $\gamma_\pi\boldsymbol{U} = \gamma[(\boldsymbol{\nu}\times\boldsymbol{U})\times\boldsymbol{\nu}]$, and the normal (Dirichlet) traces $\gamma_n\boldsymbol{U} = \langle\boldsymbol{\nu}, \gamma\boldsymbol{U}\rangle$ (i.e., $\gamma_n\boldsymbol{U} = \boldsymbol{\nu}\cdot\gamma\boldsymbol{U}$) are well defined for the elements of $\mathbb{H}^1(\Omega)$ and that $\gamma_\tau\boldsymbol{U}, \gamma_\pi\boldsymbol{U}$ belong to the Sobolev space $\mathbb{H}^{1/2}_t(\mathcal{S})$ of tangential vector fields of order $1/2$ on the surface $\mathcal{S}$, while $\gamma_n\boldsymbol{U} \in H^{1/2}(\mathcal{S})$ and $\gamma\boldsymbol{U} \in \mathbb{H}^{1/2}(\mathcal{S})$.

Let us define the proper linear subspace

$$\mathbb{H}^r_{\varepsilon\boldsymbol{\nu},0}(\mathcal{S}) := \big\{\boldsymbol{U} \in \mathbb{H}^r(\mathcal{S}) : \langle\varepsilon\boldsymbol{\nu}, \boldsymbol{U}\rangle = 0\big\}$$

of $\mathbb{H}^r(\mathcal{S})$. For a constant diagonal matrix $\varepsilon = \varepsilon_0 I_3$, the space $\mathbb{H}^r_{\varepsilon\boldsymbol{\nu},0}(\mathcal{S}) = \mathbb{H}^r_{\boldsymbol{\nu},0}(\mathcal{S})$ coincides with the space of tangent vector fields. The operator

$$\pi^\perp_{\varepsilon\boldsymbol{\nu}}\boldsymbol{U} := \frac{\varepsilon\boldsymbol{\nu}}{|\varepsilon\boldsymbol{\nu}|} \times \boldsymbol{U} \times \frac{\varepsilon\boldsymbol{\nu}}{|\varepsilon\boldsymbol{\nu}|} = \boldsymbol{U} - \Big\langle \frac{\varepsilon\boldsymbol{\nu}}{|\varepsilon\boldsymbol{\nu}|}, \boldsymbol{U}\Big\rangle \frac{\varepsilon\boldsymbol{\nu}}{|\varepsilon\boldsymbol{\nu}|} = \Big(I - \frac{(\varepsilon\boldsymbol{\nu})(\varepsilon\boldsymbol{\nu})^\top}{|\varepsilon\boldsymbol{\nu}|^2}\Big)\boldsymbol{U},$$

which is actually a multiplication by a 3×3 matrix function, is a projection onto the subspace $\pi^\perp_{\varepsilon\boldsymbol{\nu}}\mathbb{H}^r(\mathcal{S}) = \mathbb{H}^r_{\varepsilon\boldsymbol{\nu},0}(\mathcal{S})$.

It is easy to see that the operator

$$\pi^\perp_{\varepsilon\boldsymbol{\nu}} : \mathbb{H}^r_{\boldsymbol{\nu},0}(\mathcal{S}) \to \mathbb{H}^r_{\varepsilon\boldsymbol{\nu},0}(\mathcal{S})$$

is continuous and invertible for all $r \in \mathbb{R}$; the inverse mapping is given by the formula

$$(\pi^\perp_{\varepsilon\boldsymbol{\nu}})^{-1}\boldsymbol{U} = \boldsymbol{U} - \frac{\langle\boldsymbol{\nu}, \boldsymbol{U}\rangle}{\langle\boldsymbol{\nu}, \varepsilon\boldsymbol{\nu}\rangle}\varepsilon\boldsymbol{\nu}, \quad \boldsymbol{U} \in \mathbb{H}^r_{\varepsilon\boldsymbol{\nu},0}(\mathcal{S})$$

and we have

$$\pi^\perp_{\varepsilon\boldsymbol{\nu}}\boldsymbol{U} - \frac{\langle\boldsymbol{\nu}, \pi^\perp_{\varepsilon\boldsymbol{\nu}}\boldsymbol{U}\rangle}{\langle\boldsymbol{\nu}, \varepsilon\boldsymbol{\nu}\rangle}\varepsilon\boldsymbol{\nu} = \boldsymbol{U} \quad \text{for all } \boldsymbol{U} \in \mathbb{H}^r_{\boldsymbol{\nu},0}(\mathcal{S}).$$

We introduce another projection

$$\pi_{\boldsymbol{\nu}}\boldsymbol{U} := \langle \boldsymbol{\nu}, \boldsymbol{U}\rangle \boldsymbol{\nu},$$

which is $\pi_{\boldsymbol{\nu}}\boldsymbol{U} = \boldsymbol{U} - \pi^{\perp}_{\boldsymbol{\nu}}\boldsymbol{U}$.

We also use the spaces

$$\begin{aligned}
\mathbb{H}^1_{\boldsymbol{\nu}\times,0}(\mathbb{R}^3_{\mathcal{C}}) &:= \big\{\boldsymbol{U} \in \mathbb{H}^1(\mathbb{R}^3_{\mathcal{C}}) : \gamma^{\pm}_{\tau}\boldsymbol{U} = 0 \text{ on } \mathcal{C}\big\},\\
\mathbb{H}^{\pm 1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S}) &:= \big\{g\boldsymbol{\nu} \in \mathbb{H}^{\pm 1/2}(\mathcal{S}) : g \in H^{\pm 1/2}(\mathcal{S})\big\},\\
\widetilde{\mathbb{H}}^{\pm 1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}) &:= \big\{g\boldsymbol{\nu} \in \mathbb{H}^{\pm 1/2}(\mathcal{S}) : g \in \widetilde{H}^{\pm 1/2}(\mathcal{C})\big\},\\
\mathbb{H}^{\pm 1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}) &:= \big\{g\boldsymbol{\nu} \in \mathbb{H}^{\pm 1/2}(\mathcal{C}) : g \in H^{\pm 1/2}(\mathcal{C})\big\}.
\end{aligned}$$

For our purposes we need to define traces $\gamma_\tau(\mu^{-1}\operatorname{curl}\boldsymbol{U})$ and $\gamma(\operatorname{div}(\varepsilon\boldsymbol{U}))$. First, by applying Green's formulae to the operator $\boldsymbol{A}(D)$ in (1.5), we obtain

$$\begin{aligned}
\big(\boldsymbol{A}(D)\boldsymbol{U}, \boldsymbol{V}\big)_{\Omega^+} = (\boldsymbol{\nu} \times \mu^{-1}\operatorname{curl}\boldsymbol{U}, \boldsymbol{V}_\pi)_{\mathcal{S}} - \big(s\operatorname{div}(\varepsilon\boldsymbol{U}), \langle\varepsilon\boldsymbol{\nu}, \boldsymbol{V}\rangle\big)_{\mathcal{S}}\\
+ \boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U}, \boldsymbol{V})_{\Omega^+} - \omega^2(\varepsilon\boldsymbol{U}, \boldsymbol{V})_{\Omega^+},
\end{aligned} \tag{2.1}$$

where $\boldsymbol{U}$ is a smooth vector, $\boldsymbol{a}_{\varepsilon,\mu}$ is the natural bilinear differential form associated with the Green formula

$$\boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U}, \boldsymbol{V})_{\Omega} := (\mu^{-1}\operatorname{curl}\boldsymbol{U}, \operatorname{curl}\boldsymbol{V})_{\Omega} + s\big(\operatorname{div}(\varepsilon\boldsymbol{U}), \operatorname{div}(\varepsilon\boldsymbol{V})\big)_{\Omega}, \tag{2.2}$$

and $\boldsymbol{V}_\pi = \pi_{\boldsymbol{\nu}}\boldsymbol{V} := \boldsymbol{V} - \langle\boldsymbol{\nu}, \boldsymbol{V}\rangle\boldsymbol{\nu}$.

Now we can simultaneously define the traces $s\operatorname{div}(\varepsilon\boldsymbol{U})|_{\mathcal{S}}$ and $\boldsymbol{\nu} \times \mu^{-1}\operatorname{curl}\boldsymbol{U}|_{\mathcal{S}}$

$$s\operatorname{div}(\varepsilon\boldsymbol{U}) \in H^{-1/2}(\mathcal{S}), \qquad \boldsymbol{\nu} \times \mu^{-1}\operatorname{curl}\boldsymbol{U} \in \mathbb{H}^{-1/2}(\mathcal{S}) \tag{2.3}$$

by duality for any $\boldsymbol{U} \in \mathbb{H}^1(\Omega^+)$ such that $\boldsymbol{A}(D)\boldsymbol{U} \in \mathbb{L}_2(\Omega^+)$ as in

$$\begin{aligned}
&\big(s\operatorname{div}(\varepsilon\boldsymbol{U}), \gamma\varphi\big)_{\mathcal{S}} - (\boldsymbol{\nu} \times \mu^{-1}\operatorname{curl}\boldsymbol{U}, \gamma\boldsymbol{\Psi})_{\mathcal{S}}\\
&\quad := \boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U}, \boldsymbol{V})_{\Omega^+} - \big(\boldsymbol{A}(D)\boldsymbol{U}, \boldsymbol{V}\big)_{\Omega^+} - \omega^2(\varepsilon\boldsymbol{U}, \boldsymbol{V})_{\Omega^+},
\end{aligned} \tag{2.4}$$

where $\varphi \in H^1(\Omega^+)$ and $\boldsymbol{\Psi} \in \mathbb{H}^1(\Omega^+)$ are arbitrarily chosen, while $\boldsymbol{V}$ is defined as

$$\boldsymbol{V} = \frac{\varphi - \langle\varepsilon\boldsymbol{\nu}, \Psi\rangle}{\langle\varepsilon\boldsymbol{\nu}, \boldsymbol{\nu}\rangle}\boldsymbol{\nu} + \boldsymbol{\Psi}.$$

Indeed, we have $\langle\varepsilon\boldsymbol{\nu}, \boldsymbol{V}\rangle = \varphi$ and $\boldsymbol{V}_\pi = \boldsymbol{V} - \langle\boldsymbol{\nu}, \boldsymbol{V}\rangle\boldsymbol{\nu} = \boldsymbol{\Psi}$. Then formula (2.4) coincides with (2.1) and, since $\gamma\varphi \in H^{1/2}(\mathcal{S})$ and $\gamma\boldsymbol{\Psi} \in \mathbb{H}^{1/2}(\mathcal{S})$ are arbitrary, the claimed inclusions (2.3) follow from (2.4) by the duality.

Note that we can rewrite Green's formula (2.1) in the following form:

$$\begin{aligned}
\big(\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}, \boldsymbol{V}\big)_{\mathcal{S}} &= \boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U}, \boldsymbol{V})_{\Omega^+} - \big(\boldsymbol{A}(D)\boldsymbol{U}, \boldsymbol{V}\big)_{\Omega^+} - \omega^2(\varepsilon\boldsymbol{U}, \boldsymbol{V})_{\Omega^+},\\
\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U} &:= s\operatorname{div}(\varepsilon\boldsymbol{U})\varepsilon\boldsymbol{\nu} - \boldsymbol{\nu} \times \mu^{-1}\operatorname{curl}\boldsymbol{U},
\end{aligned} \tag{2.5}$$

where $\boldsymbol{T}(D,\boldsymbol{\nu})$ is the Neumann's boundary operator and $\boldsymbol{V} \in \mathbb{H}^1(\Omega^+)$ is arbitrary. The formula (2.5) allows us to prove the mapping property of the Neumann's operator

$$\boldsymbol{T}(D,\boldsymbol{\nu}) : \mathbb{H}^1(\Omega^+) \to \mathbb{H}^{-1/2}(\mathcal{S}) \tag{2.6}$$

by duality for any $\boldsymbol{U} \in \mathbb{H}^1(\Omega^+)$ such that $\boldsymbol{A}(D)\boldsymbol{U} \in \mathbb{L}_2(\Omega^+)$.

Theorem 2.1 (See [4, Lemma 2.1])**.** *The operator in (1.5)*

$$\boldsymbol{A}(D)\boldsymbol{U} := \operatorname{curl}\mu^{-1}\operatorname{curl}\boldsymbol{U} - s\varepsilon\operatorname{grad}\operatorname{div}(\varepsilon\boldsymbol{U}) - \omega^2\varepsilon\boldsymbol{U}$$

is elliptic, is self-adjoint, and has a positive definite principal symbol $\mathcal{A}_{\rm pr}(\xi)$*, that is,*

$$\big\langle \mathcal{A}_{\rm pr}(\xi)\eta, \eta\big\rangle \geq c|\xi|^2|\eta|^2, \quad c = \text{const} > 0, \forall\xi \in \mathbb{R}^3, \forall\eta \in \mathbb{C}^3.$$

Now we are ready to formulate screen-type *boundary value problems* (BVP) for anisotropic pseudo-Maxwell equations where "hybrid" boundary conditions are formed by combining the tangent Dirichlet traces with the $\operatorname{div}(\varepsilon\boldsymbol{U})|_{\mathcal{C}^\pm}$ traces motivated by the natural boundary conditions for the Maxwell equations (see [1]–[3], [8]).

Problem H_π. Find $\boldsymbol{U} \in \mathbb{H}^1(\mathbb{R}^3_{\mathcal{C}})$ such that

$$\begin{cases} \boldsymbol{A}(D)\boldsymbol{U} = 0 & \text{in } \mathbb{R}^3_{\mathcal{C}}, \\ \gamma^\pm_\pi(\boldsymbol{U}) = \mathbf{f}^\pm & \text{on } \mathcal{C}, \\ \gamma^\pm(\operatorname{div}(\varepsilon\boldsymbol{U})) = g^\pm & \text{on } \mathcal{C}, \end{cases}$$

where the given data $\mathbf{f}^\pm$, $g^\pm$ satisfy the conditions

$$\mathbf{f}^\pm \in \mathbb{H}^{1/2}_t(\mathcal{C}), \quad \mathbf{f}^+ - \mathbf{f}^- \in r_{\mathcal{C}}\widetilde{\mathbb{H}}^{1/2}_t(\mathcal{C}),$$

and

$$g^\pm \in H^{-1/2}(\mathcal{C}), \quad g^+ - g^- \in r_{\mathcal{C}}\widetilde{H}^{-1/2}_t(\mathcal{C}).$$

Problem H_τ. Find $\boldsymbol{U} \in \mathbb{H}^1(\mathbb{R}^3_{\mathcal{C}})$ such that

$$\begin{cases} \boldsymbol{A}(D)\boldsymbol{U} = 0 & \text{in } \mathbb{R}^3_{\mathcal{C}}, \\ \gamma^\pm_\tau(\boldsymbol{U}) = \mathbf{f}^\pm & \text{on } \mathcal{C}, \\ \gamma^\pm(\operatorname{div}(\varepsilon\boldsymbol{U})) = g^\pm & \text{on } \mathcal{C}, \end{cases}$$

where the given data $\mathbf{f}^\pm$, $g^\pm$ satisfy the conditions

$$\mathbf{f}^\pm \in \mathbb{H}^{1/2}_t(\mathcal{C}), \quad \mathbf{f}^+ - \mathbf{f}^- \in r_{\mathcal{C}}\widetilde{\mathbb{H}}^{1/2}_t(\mathcal{C}),$$

and

$$g^\pm \in H^{-1/2}(\mathcal{C}), \quad g^+ - g^- \in r_{\mathcal{C}}\widetilde{H}^{-1/2}_t(\mathcal{C}).$$

Note that since we consider smooth screens, there is a connection between the traces γ_τ and γ_π established by the geometric operation $\boldsymbol{\nu} \times \cdot$ which is in fact a rotation, and therefore from the uniqueness, existence, and regularity results for the *Problem* H_π we get the same results for the *Problem* H_τ and vice versa. Moreover, it turns out that we can easily reduce the investigation of the *Problem* H_τ as well as of the *Problem* H_π to the investigation of the following Neumann boundary value problem.

The Neumann boundary value problem N. Find $\boldsymbol{U} \in \mathbb{H}^1_{\boldsymbol{\nu}\times,0}(\mathbb{R}^3_{\mathcal{C}})$ such that

$$\begin{cases} \boldsymbol{A}(D)\boldsymbol{U} = 0 & \text{in } \mathbb{R}^3_{\mathcal{C}}, \\ \gamma^\pm_{\mathcal{C}}(\pi_{\boldsymbol{\nu}}\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}) = g^\pm\boldsymbol{\nu} & \text{on } \mathcal{C}, \end{cases} \tag{2.7}$$

where the given data $g^{\pm}$ satisfy the conditions

$$g^{\pm} \in H^{-1/2}(\mathcal{C}), \quad g^{+} - g^{-} \in r_{\mathcal{C}}\widetilde{H}^{-1/2}(\mathcal{C}). \tag{2.8}$$

Indeed, if $\boldsymbol{U}$ is a solution of the *Problem* H_{τ} and $\boldsymbol{U}_1$ is a solution of the Dirichlet boundary value problem (see [5])

$$\begin{cases} \boldsymbol{A}(D)\boldsymbol{U}_1 = 0 & \text{in } \mathbb{R}^3_{\mathcal{C}}, \\ \gamma^{\pm}(\boldsymbol{U}_1) = \mathbf{f}^{\pm} \times \boldsymbol{\nu} & \text{on } \mathcal{C}, \end{cases}$$

then $\boldsymbol{U}_2 := \boldsymbol{U} - \boldsymbol{U}_1 \in \mathbb{H}^1_{\boldsymbol{\nu}\times,0}(\mathbb{R}^3_{\mathcal{C}})$. Clearly, we obtain

$$\gamma_{\tau}\boldsymbol{U}_2 = \gamma_{\tau}\boldsymbol{U} - \gamma_{\tau}\boldsymbol{U}_1 = \mathbf{f}^{\pm} - \gamma_{\tau}\mathbf{h}^{\pm} = \mathbf{f}^{\pm} - \gamma_{\pi}\mathbf{f}^{\pm} = \mathbf{f}^{\pm} - \mathbf{f}^{\pm} = 0$$

since for all $\mathbf{f}^{\pm} \in \mathbb{H}^{1/2}_t(\mathcal{C})$ we have $\gamma_{\pi}\mathbf{f}^{\pm} = \mathbf{f}$. Thus the unique solvability result of the *Problem* H_{τ} will follow from the corresponding results for the *Problem* N with respect to $\boldsymbol{U}_2 \in \mathbb{H}^1_{\boldsymbol{\nu}\times,0}(\mathbb{R}^3_{\mathcal{C}})$ satisfying the boundary conditions

$$\gamma^{\pm}_{\mathcal{C}}\big(\pi_{\boldsymbol{\nu}}\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}_2\big) = \mathbf{g}^{\pm},$$

where $\mathbf{g}^{\pm} = s\langle\varepsilon\boldsymbol{\nu},\boldsymbol{\nu}\rangle(g^{\pm} - \gamma^{\pm}(\operatorname{div}(\varepsilon\boldsymbol{U}_1)))\boldsymbol{\nu}$. Therefore it remains for us to study the *Problem* N.

3. Vector potentials and the uniqueness of a solution

Let us consider the *single layer* and *double layer* potential operators

$$\mathbf{V}\boldsymbol{U}(x) := \oint_{\mathcal{S}} \mathbf{F}_{\boldsymbol{A}}(x-\tau)\boldsymbol{U}(\tau)\,dS,$$
$$\mathbf{W}\boldsymbol{U}(x) := \oint_{\mathcal{S}} \Big[\big(\boldsymbol{T}\big(D,\boldsymbol{\nu}(\tau)\big)\mathbf{F}_{\boldsymbol{A}}\big)(x-\tau)\Big]^{\top}\boldsymbol{U}(\tau)\,dS, \quad x \in \Omega,$$

related to pseudo-Maxwell equations in (1.5), where by $\mathbf{F}_{\boldsymbol{A}}(x)$ we denote the fundamental solution (see [10], [4]) of the elliptic operator $\boldsymbol{A}(D)$ in (1.5). Obviously,

$$\boldsymbol{A}(D)\mathbf{V}\boldsymbol{U}(x) = \boldsymbol{A}(D)\mathbf{W}\boldsymbol{U}(x) = 0, \quad \forall\boldsymbol{U} \in \mathbb{L}_1(\mathcal{S}), \forall x \in \Omega.$$

For the next Propositions 3.1–3.4 and for their proofs, see [9], [12].

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a domain with the smooth boundary $\mathcal{S} = \partial\Omega$. The potential operators above map continuously the spaces*

$$\mathbf{V} : \mathbb{H}^r(\mathcal{S}) \to \mathbb{H}^{r+3/2}(\Omega), \qquad \mathbf{W} : \mathbb{H}^r(\mathcal{S}) \to \mathbb{H}^{r+1/2}(\Omega), \quad \forall r \in \mathbb{R}.$$

The direct values $\mathbf{V}_{-\mathbf{1}}$, $\mathbf{W}_{\mathbf{0}}$ and $\mathbf{V}_{+\mathbf{1}}$ of the operators $\mathbf{V}$, $\mathbf{W}$ and $\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{W}$ are pseudodifferential operators of order -1, 0, and 1, respectively, and map continuously the spaces

$$\mathbf{V}_{-\mathbf{1}} : \mathbb{H}^r(\mathcal{S}) \to \mathbb{H}^{r+1}(\mathcal{S}), \qquad \mathbf{W}_{\mathbf{0}} : \mathbb{H}^r(\mathcal{S}) \to \mathbb{H}^r(\mathcal{S}),$$
$$\mathbf{V}_{+\mathbf{1}} : \mathbb{H}^r(\mathcal{S}) \to \mathbb{H}^{r-1}(\mathcal{S}),$$

for all $r \in \mathbb{R}$.

Proposition 3.2. *The potential operators on an open, compact, smooth surface $\mathcal{C} \subset \mathbb{R}^3$ have the following mapping properties:*

$$\mathbf{V} : \widetilde{\mathbb{H}}^r(\mathcal{C}) \to \mathbb{H}^{r+3/2}(\mathbb{R}^3_{\mathcal{C}}), \qquad \mathbf{W} : \widetilde{\mathbb{H}}^r(\mathcal{C}) \to \mathbb{H}^{r+1/2}(\mathbb{R}^3_{\mathcal{C}}), \quad \forall r \in \mathbb{R}.$$

The direct values $\mathbf{V_{-1}}$, $\mathbf{W_0}$, and $\mathbf{V_{+1}}$ of the potential operators $\mathbf{V}$, $\mathbf{W}$, and $\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{W}$ are pseudodifferential operators of order -1, 0, and 1, respectively, and have the following mapping properties:

$$\mathbf{V_{-1}} : \widetilde{\mathbb{H}}^r(\mathcal{C}) \to \mathbb{H}^{r+1}(\mathcal{C}), \qquad \mathbf{W_0} : \widetilde{\mathbb{H}}^r(\mathcal{C}) \to \mathbb{H}^{r}(\mathcal{C}), \qquad \mathbf{V_{+1}} : \widetilde{\mathbb{H}}^r(\mathcal{C}) \to \mathbb{H}^{r-1}(\mathcal{C}),$$

for all $r \in \mathbb{R}$.

Proposition 3.3. *For the traces of potential operators, we have the following Plemelji formulae:*

$$\begin{aligned} (\gamma_{\mathcal{S}^-}\mathbf{V}\boldsymbol{U})(x) &= (\gamma_{\mathcal{S}^+}\mathbf{V}\boldsymbol{U})(x) = \mathbf{V_{-1}}\boldsymbol{U}(x), \\ \big(\gamma_{\mathcal{S}^\pm}\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{V}\boldsymbol{U}\big)(x) &= \mp\frac{1}{2}\boldsymbol{U}(x) + (\mathbf{W_0})^*(x,D)\boldsymbol{U}(x), \\ (\gamma_{\mathcal{S}^\pm}\mathbf{W}\boldsymbol{U})(x) &= \pm\frac{1}{2}\boldsymbol{U}(x) + \mathbf{W_0}(x,D)\boldsymbol{U}(x), \\ \big(\gamma_{\mathcal{S}^-}\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{W}\boldsymbol{U}\big)(x) &= \big(\gamma_{\mathcal{S}^+}\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{W}\boldsymbol{U}\big)(x) \\ &= \mathbf{V_{+1}}\boldsymbol{U}(x), \quad x \in \mathcal{S}, \boldsymbol{U} \in \mathbb{H}^s_p(\mathcal{S}), \end{aligned} \tag{3.1}$$

where $(\mathbf{W_0})^*(x,D)$ is the adjoint to the pseudodifferential operator $\mathbf{W_0}(x,D)$, the direct value of the potential operator $\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{V}$ on the boundary $\mathcal{S}$.

Proposition 3.4. *Let the boundary $\mathcal{S} = \partial\Omega^\pm$ be a compact smooth surface. Solutions to pseudo-Maxwell equations with anisotropic coefficients ε and μ are represented as*

$$\boldsymbol{U}(x) = \pm\mathbf{W}(\gamma_{\mathcal{S}^\pm}\boldsymbol{U})(x) \mp \mathbf{V}\big(\gamma_{\mathcal{S}^\pm}\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}\big)(x), \quad x \in \Omega^\pm, \tag{3.2}$$

where $\gamma_{\mathcal{S}^\pm}\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{\Psi}$ is the Neumann trace operator (see (2.6)) and $\gamma_{\mathcal{S}^\pm}\boldsymbol{\Psi}$ is the Dirichlet trace operator. If $\mathcal{C} \subset \mathbb{R}^3$ is an open compact smooth surface, then a solution to pseudo-Maxwell equations with anisotropic coefficients ε and μ is represented as

$$\begin{aligned} \boldsymbol{U}(x) &= \mathbf{W}\big([\boldsymbol{U}]\big)(x) - \mathbf{V}\big(\big[\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}\big]\big)(x), \quad x \in \mathbb{R}^3_{\mathcal{C}}, \\ [\boldsymbol{U}] &:= \gamma_{\mathcal{C}^+}\boldsymbol{U} - \gamma_{\mathcal{C}^-}\boldsymbol{U}, \qquad \big[\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}\big] := \gamma_{\mathcal{C}^+}\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U} - \gamma_{\mathcal{C}^-}\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}. \end{aligned}$$

As a consequence of the representation formula (3.2), we derive the following.

Corollary 3.5. *For a complex-valued frequency, a solution $\boldsymbol{U} \in \mathbb{H}^1(\mathbb{R}^3_{\mathcal{C}})$ to the screen-type boundary value problems for pseudo-Maxwell equations decays at infinity exponentially, that is,*

$$\boldsymbol{U}(x) = \mathcal{O}(e^{-\alpha|x|}) \quad \text{as } |x| \to \infty \text{ provided that } \operatorname{Im}\omega \neq 0 \tag{3.3}$$

for some $\alpha > 0$.

Indeed, since $\operatorname{Im}\omega \neq 0$, the symbol $\mathcal{A}(\xi)$ does not vanish on $\mathbb{R}^3$ (see Theorem 2.1). Then there is a unique tempered fundamental solution given by Fourier transformation

$$\mathbf{F}_{\boldsymbol{A}}(x) := \mathcal{F}^{-1}_{\xi\to x}\big[\mathcal{A}^{-1}(\xi)\big],$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform and $\mathcal{A}(\xi)$ is the full symbol of the operator $\boldsymbol{A}(D)$. Now $\boldsymbol{U} \in \mathbb{H}^1(\mathbb{R}^3_{\mathcal{C}})$ can be seen as a tempered distribution on $\mathbb{R}^3$, and since $\boldsymbol{A}(D)\boldsymbol{U} =: \mathbf{G}$ for some $\mathbf{G}$ from the space $\mathcal{E}'(\mathbb{R}^3)$ of compactly supported distributions, we have $\mathcal{A}(\xi)\mathcal{F}\boldsymbol{U}(\xi) = \mathcal{F}\mathbf{G}(\xi)$, and hence

$$\boldsymbol{U}(x) = \int_{\mathbb{R}^3} \mathbf{F}_{\boldsymbol{A}}(x-y)\mathbf{G}(y)\,dy. \tag{3.4}$$

On the other hand, any distribution $\boldsymbol{U}$ given by (3.4) decays exponentially at infinity for any $\mathbf{G} \in \mathcal{E}'(\mathbb{R}^3)$. Thus (3.3) holds true. For more details, see [10], [7].

Now we are ready to establish the uniqueness result for the screen-type Neumann problem (2.7), (2.8).

Theorem 3.6. *The screen-type Neumann boundary value problems (2.7) and (2.8) has at most one solution.*

Proof. The proof is standard and uses Green's formula (see (2.1)–(2.5)). Let R be a sufficiently large positive number and let $B(R)$ be the ball that is centered at the origin with radius R. Set $\Omega_R := \mathbb{R}^3_{\mathcal{C}} \cap B(R)$. Note that the domain Ω_R has a piecewise smooth boundary S_R including both sides of $\mathcal{C}$.

Let $\boldsymbol{U}$ be a solution of the homogeneous problem. Then applying Green's formula for $\mathbf{V} = \boldsymbol{U}$ in Ω_R and passing to the limit $R \to \infty$, taking into account the estimate $\boldsymbol{U}(x) = O(e^{-\alpha|x|})$ as $|x| \to \infty$ for $\alpha > 0$, we get

$$\boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U},\boldsymbol{U})_{\mathbb{R}^3} - \omega^2\boldsymbol{(}\varepsilon\boldsymbol{U},\boldsymbol{U}\boldsymbol{)}_{\mathbb{R}^3} = 0.$$

Since ε and μ^{-1} are real, positive definite, constant matrices, with $s > 0$ and $\operatorname{Im}\omega \neq 0$, it follows that

$$\boldsymbol{(}\varepsilon\boldsymbol{U},\boldsymbol{U}\boldsymbol{)}_{\mathbb{R}^3} = 0$$

and therefore that $\boldsymbol{U} \equiv 0$ in $\mathbb{R}^3_{\mathcal{C}}$. □

4. Existence of a solution

To prove the existence of a solution, we derive and investigate equivalent boundary pseudodifferential equations for the screen-type Neumann problem in $\Omega^\pm$.

Consider the potential operator

$$\mathbf{P\Phi}(x) := \mathbf{V}(\mathbf{V_{-1}})^{-1}\mathbf{\Phi}(x), \quad \mathbf{\Phi} \in \mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S}), x \in \Omega, \tag{4.1}$$

where $\Omega = \Omega^\pm$. Note that $\boldsymbol{U} = \mathbf{P\Phi} = \mathbf{V\Psi}$, $\mathbf{\Psi} := (\mathbf{V_{-1}})^{-1}\mathbf{\Phi}$, satisfies the basic equation in (1.5) in $\Omega^\pm$.

By introducing $\boldsymbol{U} = \mathbf{P\Phi}$ from (4.1) into $\gamma^\pm_{\mathcal{S}}(\pi_{\boldsymbol{\nu}}\boldsymbol{T}(D,\boldsymbol{\nu})\boldsymbol{U}) = \mathbf{g}^\pm$ on $\mathcal{S}$, where $\mathbf{g}^\pm = g^\pm\boldsymbol{\nu} \in \mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})$, and then using Plemelji's formulae (3.1), we derive the following boundary pseudodifferential equations

$$\mathcal{P}_{\tau,\pm}\mathbf{\Phi} = \mp\gamma_{\mathcal{S}^\pm}\pi_{\boldsymbol{\nu}}\boldsymbol{T}(D,\boldsymbol{\nu})\mathbf{V}(\mathbf{V_{-1}})^{-1}\mathbf{\Phi} = \mp\mathbf{g}^\pm,$$

where

$$\mathcal{P}_{\tau,\pm} := \pi_{\boldsymbol{\nu}}\Big(\frac{1}{2}I \mp (\mathbf{W_0})^*\Big)(\mathbf{V_{-1}})^{-1} \tag{4.2}$$

are the *modified Poincaré–Steklov* pseudodifferential operators of order 1.

Lemma 4.1. *The operators*

$$\mathcal{P}_{\tau,\pm} : \mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S}) \to \mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S}) \tag{4.3}$$

are coercive

$$\operatorname{Re}\big(\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi}, \boldsymbol{\Phi}\big)_{\mathcal{S}} \geq c_0\big\|\boldsymbol{\Phi} \mid \mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|^2 - c_1\big\|\boldsymbol{\Phi} \mid \mathbb{L}^{2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|^2 \tag{4.4}$$

for some positive constants c_0, c_1, and all $\boldsymbol{\Phi} \in \mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})$. Moreover, the operators have the trivial kernels (i.e., $\operatorname{Ker}\mathcal{P}_{\tau,\pm} = \{0\}$*) and are invertible. If the frequency is purely imaginary $\omega = i\beta \neq 0$, $\beta \in \mathbb{R}$, then the operators $\mathcal{P}_{\tau,\pm}$ are positive definite*

$$\big(\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi}, \boldsymbol{\Phi}\big)_{\mathcal{S}} \geq M_{\pm}\big\|\boldsymbol{\Phi} \mid \mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\| \tag{4.5}$$

for some positive constants $M_{\pm}$.

Proof. By introducing $\widetilde{\boldsymbol{U}} = \mathbf{V}(\mathbf{V_{-1}})^{-1}\boldsymbol{\Phi}$ into the Green formula (2.5), we find out that

$$\begin{aligned}\big(\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi}, \boldsymbol{\Phi}\big)_{\mathcal{S}} = \big(\mu^{-1}\operatorname{curl}\widetilde{\boldsymbol{U}}, \operatorname{curl}\widetilde{\boldsymbol{U}}\big)_{\Omega^\pm} + s\big(\operatorname{div}(\varepsilon\widetilde{\boldsymbol{U}}), \operatorname{div}(\varepsilon\widetilde{\boldsymbol{U}})\big)_{\Omega^\pm} \\ - \omega^2\big(\varepsilon\widetilde{\boldsymbol{U}}, \widetilde{\boldsymbol{U}}\big)_{\Omega^\pm}.\end{aligned} \tag{4.6}$$

Since $\gamma\widetilde{\boldsymbol{U}} = \boldsymbol{\Phi} \in \mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})$, due to [6] (see also [4, Corollary 3.9]) and Corollary 3.5, the forms

$$\boldsymbol{a}_{\varepsilon,\mu}(\widetilde{\boldsymbol{U}}, \widetilde{\boldsymbol{U}})_{\Omega^\pm} = \big(\mu^{-1}\operatorname{curl}\widetilde{\boldsymbol{U}}, \operatorname{curl}\widetilde{\boldsymbol{U}}\big)_{\Omega^\pm} + s\big(\operatorname{div}(\varepsilon\widetilde{\boldsymbol{U}}), \operatorname{div}(\varepsilon\widetilde{\boldsymbol{U}})\big)_{\Omega^\pm}$$

are coercive; the inequality

$$\boldsymbol{a}_{\varepsilon,\mu}(\widetilde{\boldsymbol{U}}, \widetilde{\boldsymbol{U}})_{\Omega^\pm} \geq c_2\big\|\widetilde{\boldsymbol{U}} \mid \mathbb{H}^{1}_{\boldsymbol{\nu}\times,0}(\Omega^\pm)\big\|^2 - c_3\big\|\widetilde{\boldsymbol{U}} \mid \mathbb{L}^{2}_{\boldsymbol{\nu}\times,0}(\Omega^\pm)\big\|^2$$

holds for $\widetilde{\boldsymbol{U}} = \mathbf{V}(\mathbf{V_{-1}})^{-1}\boldsymbol{\Phi}$, $\boldsymbol{\Phi} \in \mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})$ and some $c_2 > 0$, $c_3 > 0$. From (4.6) we then obtain

$$\operatorname{Re}\big(\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi}, \boldsymbol{\Phi}\big)_{\mathcal{S}} \geq c_2\big\|\widetilde{\boldsymbol{U}} \mid \mathbb{H}^{1}_{\boldsymbol{\nu}\times,0}(\Omega^\pm)\big\|^2 - c_4\big\|\widetilde{\boldsymbol{U}} \mid \mathbb{L}^{2}_{\boldsymbol{\nu}\times,0}(\Omega^\pm)\big\|^2$$

for some $c_4 > 0$. Further, invoking the trace theorem (see [14, Section 2.9]) and the continuity property of the operator $\mathbf{P} = \mathbf{V}(\mathbf{V_{-1}})^{-1}$, we can easily derive the following inequalities:

$$\begin{aligned}\big\|\widetilde{\boldsymbol{U}} \mid \mathbb{H}^{1}_{\boldsymbol{\nu}\times,0}(\Omega^\pm)\big\| \geq c_5\big\|\gamma\widetilde{\boldsymbol{U}} \mid \mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|, \quad c_5 > 0, \\ \big\|\mathbf{V}(\mathbf{V_{-1}})^{-1}\boldsymbol{\Phi} \mid \mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\Omega^\pm)\big\| \leq c_6\big\|\boldsymbol{\Phi} \mid \mathbb{L}^{2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|, \quad c_6 > 0.\end{aligned} \tag{4.7}$$

Applying the inequalities (4.7), we get the estimate with suitable positive constants

$$\begin{aligned}\operatorname{Re}\,(\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi},\boldsymbol{\Phi})_{\mathcal{S}} &\geq c_2\big\|\widetilde{\boldsymbol{U}}\mid\mathbb{H}^1_{\boldsymbol{\nu}\times,0}(\Omega^\pm)\big\|^2 - c_4\big\|\widetilde{\boldsymbol{U}}\mid\mathbb{L}^2_{\boldsymbol{\nu}\times,0}(\Omega^\pm)\big\|^2\\ &\geq c_7\big\|\gamma\widetilde{\boldsymbol{U}}\mid\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|^2 - c_8\big\|\widetilde{\boldsymbol{U}}\mid\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\Omega^\pm)\big\|^2\\ &= c_7\big\|\boldsymbol{\Phi}\mid\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|^2 - c_8\big\|\mathbf{V}(\mathbf{V}_{-\mathbf{1}})^{-1}\boldsymbol{\Phi}\mid\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\Omega^\pm)\big\|^2\\ &\geq c_0\big\|\boldsymbol{\Phi}\mid\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|^2 - c_1\big\|\boldsymbol{\Phi}\mid\mathbb{L}^2_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|^2\end{aligned}$$

for $\boldsymbol{\Phi}\in\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})$. Thus the operator (4.3) is coercive and, therefore, is Fredholm with the index zero. Moreover, it is invertible since it has a trivial kernel. Indeed, for $\operatorname{Im}\omega\neq 0$ equating in (4.6) the imaginary part to 0, we get $(\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi},\boldsymbol{\Phi})_{\mathcal{S}}=0$, which implies that

$$0=(\varepsilon\widetilde{\boldsymbol{U}},\widetilde{\boldsymbol{U}})_{\Omega^\pm}\geq c\big\|\widetilde{\boldsymbol{U}}\mid\mathbb{L}^2(\Omega^\pm)\big\|^2 \Longrightarrow \widetilde{\boldsymbol{U}}\equiv 0 \quad \text{in } \Omega^\pm.$$

Therefore $\gamma^\pm_{\mathcal{S}}\widetilde{\boldsymbol{U}}=\boldsymbol{\Phi}\equiv 0$ on $\mathcal{S}$.

If $\omega=i\beta$, then $\mathcal{P}_{\tau,\pm}$ is positive definite:

$$\operatorname{Re}\,(\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi},\boldsymbol{\Phi})_{\mathcal{S}}=(\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi},\boldsymbol{\Phi})_{\mathcal{S}}=\boldsymbol{a}_{\varepsilon,\mu}(\widetilde{\boldsymbol{U}},\widetilde{\boldsymbol{U}})_{\Omega^\pm}+\beta^2(\varepsilon\widetilde{\boldsymbol{U}},\widetilde{\boldsymbol{U}})_{\Omega^\pm}>0.$$

If $\widetilde{\boldsymbol{U}}\neq 0$ in $\Omega^\pm$ (see (4.6)) then, therefore, $\boldsymbol{\Phi}\neq 0$ on $\mathcal{S}$; moreover, $\mathcal{P}_{\tau,\pm}$ is coercive so that

$$(\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi},\boldsymbol{\Phi})_{\mathcal{S}}=\operatorname{Re}\,(\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi},\boldsymbol{\Phi})_{\mathcal{S}}\geq c_0\big\|\boldsymbol{\Phi}\mid\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|^2 - c_1\big\|\boldsymbol{\Phi}\mid\mathbb{L}^2_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|^2$$

for all $\boldsymbol{\Phi}\in\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})$ (see (4.4)). The positive definiteness (4.5) is a consequence of these two properties (see [13, Exercise 2.17]). □

Corollary 4.2. *The operators $\mathcal{P}_{\tau,\pm}$ are invertible in the following space settings:*

$$\mathcal{P}_{\tau,\pm}:\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\to\mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S}),\quad \forall r\in\mathbb{R}.$$

The inverse operators

$$\mathcal{P}^{-1}_{\tau,\pm}:\mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\to\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S}),$$

are coercive so that

$$\operatorname{Re}\,(\mathcal{P}^{-1}_{\tau,\pm}\boldsymbol{\Psi},\boldsymbol{\Psi})_{\mathcal{S}}\geq m_0\big\|\boldsymbol{\Psi}\mid\mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|^2 - m_1\big\|\boldsymbol{\Psi}\mid\mathbb{H}^{-1}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|^2$$

for some positive constants m_0, m_1, and all $\boldsymbol{\Psi}\in\mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})$. If the frequency is purely imaginary $\omega=i\beta\neq 0$, $\beta\in\mathbb{R}$, the operators $\mathcal{P}^{-1}_{\tau,\pm}$ are positive definite so that

$$(\mathcal{P}^{-1}_{\tau,\pm}\boldsymbol{\Psi},\boldsymbol{\Psi})_{\mathcal{S}}\geq M_\pm\big\|\boldsymbol{\Psi}\mid\mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|$$

for some positive constants $M_\pm$.

Proof. The proof follows from Lemma 4.1 if we introduce $\Phi=\mathcal{P}^{-1}_{\tau,\pm}\boldsymbol{\Psi}$ and recall that, due to the invertibility of the operators in (4.3), the estimates

$$\frac{1}{m}\big\|\boldsymbol{\Psi}\mid\mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|^2\leq\big\|\mathcal{P}^{-1}_{\pm}\boldsymbol{\Psi}\mid\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|\leq m\big\|\boldsymbol{\Psi}\mid\mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|$$

hold for some $m>0$. □

Now let us prove analogues of Lemma 4.1 and Corollary 4.2, for a subsurface $\mathcal{C}\subset\mathcal{S}$ with the boundary $\partial\mathcal{C}\neq\emptyset$.

Lemma 4.3. *For an open subsurface $\mathcal{C}\subset\mathcal{S}$, the operators*

$$r_{\mathcal{C}}\mathcal{P}_{\tau,\pm}:\widetilde{\mathbb{H}}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C})\to\mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}) \tag{4.8}$$

are coercive with suitable positive constants c_0, c_1,

$$\operatorname{Re}\big(r_{\mathcal{C}}\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi},\boldsymbol{\Phi}\big)_{\mathcal{C}}\geq c_0\big\|\boldsymbol{\Phi}\mid\widetilde{\mathbb{H}}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C})\big\|^2-c_1\big\|\boldsymbol{\Phi}\mid\mathbb{L}^2_{\boldsymbol{\nu}\times,0}(\mathcal{C})\big\|^2 \tag{4.9}$$

for $\boldsymbol{\Phi}\in\widetilde{\mathbb{H}}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C})$. Moreover, the operators have the zero kernels $\operatorname{Ker} r_{\mathcal{C}}\mathcal{P}_{\tau,\pm}=\{0\}$ and are invertible. If the frequency is purely imaginary and $\omega=i\beta\neq0$, $\beta\in\mathbb{R}$, the operators $r_{\mathcal{C}}\mathcal{P}_{\tau,\pm}$ are positive definite with a suitable positive constants $M_\pm$:

$$\big(r_{\mathcal{C}}\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi},\boldsymbol{\Phi}\big)_{\mathcal{C}}\geq M_\pm\big\|\boldsymbol{\Phi}\mid\widetilde{\mathbb{H}}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C})\big\|. \tag{4.10}$$

Proof. Using the continuity of the embedding $\widetilde{\mathbb{H}}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C})\subset\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})$ and the proved coercivity (4.4), we have

$$\begin{aligned}\operatorname{Re}\big(r_{\mathcal{C}}\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi},\boldsymbol{\Phi}\big)_{\mathcal{C}}&=\operatorname{Re}\big(\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi},\boldsymbol{\Phi}\big)_{\mathcal{S}}\\&\geq c_0\big\|\boldsymbol{\Phi}\mid\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})\big\|^2-c_1\big\|\boldsymbol{\Phi}\mid\mathbb{L}_{2,\boldsymbol{\nu}\times,0}(\mathcal{C})\big\|^2\\&=c_0\big\|\boldsymbol{\Phi}\mid\widetilde{\mathbb{H}}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C})\big\|^2-c_1\big\|\boldsymbol{\Phi}\mid\mathbb{L}_{2,\boldsymbol{\nu}\times,0}(\mathcal{C})\big\|^2,\end{aligned}$$

for all $\boldsymbol{\Phi}\in\widetilde{\mathbb{H}}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C})$. The coercivity (4.9) is proved. Since $r_{\mathcal{C}}\mathcal{P}_{\tau,\pm}$ is coercive, it is Fredholm and has vanishing index (i.e., $\operatorname{Ind}(r_{\mathcal{C}}\mathcal{P}_{\tau,\pm})=0$).

Thus, to prove the invertibility, we just need to check that the kernel of the operator in (4.8) is trivial (i.e., $\operatorname{Ker} r_{\mathcal{C}}\mathcal{P}_{\tau,\pm}=\{0\}$).

For this purpose we apply the equality

$$\begin{gathered}\big(\mathcal{P}_{\tau,\pm}\boldsymbol{\Psi},\boldsymbol{\Psi}\big)_{\mathcal{S}}=\big(\mu^{-1}\operatorname{curl}\mathbf{F},\operatorname{curl}\mathbf{F}\big)_{\Omega^\pm}+s\big(\operatorname{div}(\varepsilon\mathbf{F}),\operatorname{div}(\varepsilon\mathbf{F})\big)_{\Omega^\pm}-\omega^2\big(\varepsilon\mathbf{F},\mathbf{F}\big)_{\Omega^\pm},\\ \mathbf{F}=\mathbf{V}(\mathbf{V}_{-\mathbf{1}})^{-1}\boldsymbol{\Psi},\qquad\gamma\mathbf{F}=\boldsymbol{\Psi}\in\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S}),\end{gathered}$$

proved for a surface $\mathcal{S}$ without boundary. By introducing in the above equality the vector $\boldsymbol{\Psi}=\boldsymbol{\Phi}\in\widetilde{\mathbb{H}}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C})\subset\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})$, we get

$$\begin{aligned}\big(r_{\mathcal{C}}\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi},\boldsymbol{\Phi}\big)_{\mathcal{C}}=\big(\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi},\boldsymbol{\Phi}\big)_{\mathcal{S}}=\big(\mu^{-1}\operatorname{curl}\mathbf{F},\operatorname{curl}\mathbf{F}\big)_{\Omega^\pm}&\\+s\big(\operatorname{div}(\varepsilon\mathbf{F}),\operatorname{div}(\varepsilon\mathbf{F})\big)_{\Omega^\pm}-\omega^2\big(\varepsilon\mathbf{F},\mathbf{F}\big)_{\Omega^\pm}&\end{aligned} \tag{4.11}$$

for all $\mathbf{F}=\mathbf{V}(\mathbf{V}_{-\mathbf{1}})^{-1}\boldsymbol{\Phi}$. Since ω is complex-valued, from (4.11) it follows that the equality $r_{\mathcal{C}}\mathcal{P}_{\tau,\pm}\boldsymbol{\Phi}=0$ implies that $\mathbf{F}\equiv0$ and, consequently, that $\boldsymbol{\Phi}=0$, and therefore we get $\operatorname{Ker} r_{\mathcal{C}}\mathcal{P}_{\tau,\pm}=\{0\}$.

The positive definiteness of (4.10) follows from the positive definiteness of (4.5) as in the case of coerciveness. □

Corollary 4.4. *The operators $r_{\mathcal{C}}\mathcal{P}_{\tau,\pm}$ are invertible in the following space setting:*

$$r_{\mathcal{C}}\mathcal{P}_{\pm}:\widetilde{\mathbb{H}}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C})\to\mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}).$$

Let us look for a solution to the screen-type problem (2.7) in the form

$$\boldsymbol{U}(x)=\begin{cases}\mathbf{V}(\mathbf{V_{-1}})^{-1}\mathbf{\Phi}^{+}(x), & x\in\Omega^{+},\\ \mathbf{V}(\mathbf{V_{-1}})^{-1}\mathbf{\Phi}^{-}(x), & x\in\Omega^{-}\ \text{for some}\ \mathbf{\Phi}^{\pm}\in\mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S}).\end{cases}$$

Then $\boldsymbol{U}$ satisfies the basic differential equation from BVP (2.7) in the domains $\Omega^{\pm}$ and, due to the mapping properties of $\mathbf{V}$, we have $\boldsymbol{U}\in\mathbb{H}^{1}_{\boldsymbol{\nu}\times,0}(\mathbb{R}^3_{\mathcal{C}})$. Further we need to satisfy the boundary conditions (see (2.6))

$$r_{\mathcal{C}}\gamma_{\mathcal{S}^{\pm}}\big(\pi_{\boldsymbol{\nu}}\mathbf{T}(D,\boldsymbol{\nu})\boldsymbol{U}\big)=\mathbf{g}^{\pm}\quad\text{on }\mathcal{C}. \tag{4.12}$$

Due to the Plemelji formulae (3.1), equation (4.12) acquires the form

$$r_{\mathcal{C}}\mathcal{P}_{\tau,\pm}\mathbf{\Phi}^{\pm}=\mp\mathbf{g}^{\pm}\quad\text{on }\mathcal{C}, \tag{4.13}$$

where $\mathcal{P}_{\tau,\pm}$ are the *modified Poincaré–Steklov* pseudodifferential operators of order 1, defined in (4.2).

Let $\ell\mathbf{g}^{+}\in\mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})$ be a fixed extension of the function $\mathbf{g}^{+}\in\mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C})$ up to the entire closed surface $\mathcal{S}$, and let $\ell_0(\mathbf{g}^{+}-\mathbf{g}^{-})\in\mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})$ be an extension by zero of the function $\mathbf{g}^{+}-\mathbf{g}^{-}\in r_{\mathcal{C}}\widetilde{\mathbb{H}}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C})$. Then $\ell\mathbf{g}^{-}:=\ell\mathbf{g}^{+}-\ell_0(\mathbf{g}^{+}-\mathbf{g}^{-})\in\mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})$ is an extension of the function $\mathbf{g}^{-}\in\mathbb{H}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C})$, that is,

$$r_{\mathcal{C}}\ell\mathbf{g}^{-}=\mathbf{g}^{+}-(\mathbf{g}^{+}-\mathbf{g}^{-})=\mathbf{g}^{-}$$

and

$$r_{\mathcal{C}^c}\ell\mathbf{g}^{+}=r_{\mathcal{C}^c}\ell\mathbf{g}^{-}.$$

Using (4.13), we write the boundary conditions on $\mathcal{S}$ as

$$\mathcal{P}_{\tau,\pm}\mathbf{\Phi}^{\pm}=\mp(\ell\mathbf{g}^{\pm}+\mathbf{\Psi}^{\pm}),$$

where the functions $\mathbf{\Psi}^{\pm}\in\widetilde{\mathbb{H}}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c)$ are unknown.

Due to Lemma 4.1 and Corollary 4.2, we then obtain

$$\mathbf{\Phi}^{\pm}=\mp\mathcal{P}^{-1}_{\tau,\pm}\ell\mathbf{g}^{\pm}\mp\mathcal{P}^{-1}_{\tau,\pm}\mathbf{\Psi}^{\pm}. \tag{4.14}$$

From the ellipticity of the differential operator $\boldsymbol{A}(D)$ it follows that a generalized solution to the equation (1.5) is analytic in $\mathbb{R}^3_{\mathcal{C}}$ and, therefore, the continuity conditions

$$\begin{cases}r_{\mathcal{C}^c}\gamma_{\mathcal{S}^+}\boldsymbol{U}-r_{\mathcal{C}^c}\gamma_{\mathcal{S}^-}\boldsymbol{U}=0,\\ r_{\mathcal{C}^c}\gamma_{\mathcal{S}^+}(\mathbf{T}(D,\boldsymbol{\nu})\boldsymbol{U})-r_{\mathcal{C}^c}\gamma_{\mathcal{S}^-}(\mathbf{T}(D,\boldsymbol{\nu})\boldsymbol{U})=0\end{cases}$$

hold across the complementary surface $\mathcal{C}^c$.

Then taking into the account (4.14), we obtain the following system of equations with respect to the unknown functions $\mathbf{\Psi}^{\pm}$:

$$\begin{cases}r_{\mathcal{C}^c}\mathcal{P}^{-1}_{\tau,+}\mathbf{\Psi}^{+}+r_{\mathcal{C}^c}\mathcal{P}^{-1}_{-}\mathbf{\Psi}^{-}=-r_{\mathcal{C}^c}\mathcal{P}^{-1}_{\tau,+}\ell\mathbf{g}^{+}-r_{\mathcal{C}^c}\mathcal{P}^{-1}_{\tau,-}\ell\mathbf{g}^{-},\\ r_{\mathcal{C}^c}\mathbf{\Psi}^{+}-r_{\mathcal{C}^c}\mathbf{\Psi}^{-}=0.\end{cases} \tag{4.15}$$

The last equation in (4.15) implies that

$$\mathbf{\Psi}:=\mathbf{\Psi}^{+}=\mathbf{\Psi}^{-}\in\widetilde{\mathbb{H}}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c),$$

and we obtain an equivalent pseudodifferential operator to the BVP (2.7)

$$r_{\mathcal{C}^c}\mathfrak{B}(D)\mathbf{\Psi} = \mathbf{F},$$

where

$$\mathfrak{B}(D) := \mathcal{P}_{\tau,+}^{-1} + \mathcal{P}_{\tau,-}^{-1}, \tag{4.16}$$

and where

$$\mathbf{F} := -r_{\mathcal{C}^c}\mathcal{P}_{\tau,+}^{-1}\ell\mathbf{g}^+ - r_{\mathcal{C}^c}\mathcal{P}_{\tau,-}^{-1}\ell\mathbf{g}^- = -r_{\mathcal{C}^c}\mathfrak{B}(D)\ell\mathbf{g}^+ + r_{\mathcal{C}^c}\mathcal{P}_{\tau,-}^{-1}\ell_0(\mathbf{g}^+ - \mathbf{g}^-).$$

Lemma 4.5. *The operator*

$$r_{\mathcal{C}^c}\mathfrak{B}(D) : \widetilde{\mathbb{H}}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c) \to \mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c) \tag{4.17}$$

is coercive,

$$\begin{aligned}&\operatorname{Re}\big(r_{\mathcal{C}^c}\mathfrak{B}(D)\mathbf{\Psi},\mathbf{\Psi}\big)_{\mathcal{C}^c}\\ &\quad\geq C_1\big\|\mathbf{\Psi} \mid \widetilde{\mathbb{H}}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c)\big\| - C_2\big\|\mathbf{\Psi} \mid \widetilde{\mathbb{H}}^{-1}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c)\big\|, \quad \forall\mathbf{\Psi}\in\widetilde{\mathbb{H}}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c),\end{aligned} \tag{4.18}$$

and invertible. Moreover, if the frequency is purely imaginary $\omega = i\beta \neq 0$, $\beta\in\mathbb{R}$, then the operator $\mathfrak{B}(D)$ is positive definite and the inequality

$$\big(r_{\mathcal{C}^c}\mathfrak{B}(D)\mathbf{\Psi},\mathbf{\Psi}\big)_{\mathcal{C}^c} \geq M_0\big\|\mathbf{\Psi} \mid \widetilde{\mathbb{H}}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c)\big\|, \quad \forall\mathbf{\Psi}\in\widetilde{\mathbb{H}}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c) \tag{4.19}$$

holds for some constant $M_0 > 0$.

Proof. Similar to Lemma 4.3, the coercivity (4.18), and the positive definiteness (4.19) of the operator $r_{\mathcal{C}^c}\mathfrak{B}(D)$, we obtain corresponding results for the "nonrestricted" operator $\mathfrak{B}(D)$ in (4.16), which follow immediately from similar properties of the summands $\mathcal{P}_{\tau,\pm}^{-1}$ established in Corollary 4.2.

From the coercivity (4.18), it follows that the operator in (4.17) is Fredholm and has trivial index (i.e., $\operatorname{Ind} r_{\mathcal{C}^c}\mathfrak{B}(D) = 0$). Then to prove that the operator $r_{\mathcal{C}^c}\mathfrak{B}(D)$ in (4.17) is invertible, it suffices to show that the kernel is trivial (i.e., that $\operatorname{Ker} r_{\mathcal{C}^c}\mathfrak{B}(D) = \{0\}$). The latter follows immediately for $\omega = i\beta$ from the positive definiteness (4.19).

By introducing into the Green's formula (2.5) the values

$$\boldsymbol{U}^\pm = \mathbf{V}(\mathbf{V}_{-1})^{-1}\mathbf{\Phi}^\pm, \quad \mathbf{\Phi}^\pm = \mathcal{P}_{\tau,\pm}^{-1}\mathbf{\Psi}, \mathbf{\Psi}\in\widetilde{\mathbb{H}}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c)$$

and summing them up, we get

$$\begin{aligned}\big(\mathbf{\Psi},\mathfrak{B}(D)\mathbf{\Psi}\big)_{\mathcal{S}} &= \boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U}^+,\boldsymbol{U}^+)_{\Omega^+} + \boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U}^-,\boldsymbol{U}^-)_{\Omega^-}\\ &\quad - \omega^2\big(\varepsilon\boldsymbol{U}^+,\boldsymbol{U}^+\big)_{\Omega^+} - \omega^2\big(\varepsilon\boldsymbol{U}^-,\boldsymbol{U}^-\big)_{\Omega^-}.\end{aligned} \tag{4.20}$$

Since $\operatorname{Im}\omega \neq 0$, by equating in (4.20) the real and the imaginary parts to 0, we get that $\big(\mathbf{\Psi},\mathfrak{B}(D)\mathbf{\Psi}\big)_{\mathcal{S}} = 0$ implies that

$$0 = (\varepsilon\boldsymbol{U}^\pm,\boldsymbol{U}^\pm)_{\Omega^\pm} \geq c\big\|\boldsymbol{U}^\pm \mid \mathbb{L}_2(\Omega^\pm)\big\|^2 \Longrightarrow \boldsymbol{U}^\pm \equiv 0 \quad \text{in } \Omega^\pm.$$

Thus $\gamma_{\mathcal{S}}^\pm\boldsymbol{U}^\pm = \mathbf{\Phi}^\pm \equiv 0$ on $\mathcal{S}$ and, therefore, $\mathcal{P}_\pm\mathbf{\Phi}^\pm = \mathbf{\Psi} \equiv 0$ on $\mathcal{S}$. □

Corollary 4.6. *The operator $r_{\mathcal{C}^c}\mathfrak{B}(D)$ is invertible in the following space setting*

$$r_{\mathcal{C}^c}\mathfrak{B}(D) : \widetilde{\mathbb{H}}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c) \to \mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c).$$

Theorem 4.7. *Let $\ell g^+ \in H^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C})$ be a fixed extension of the function g^+ up to the entire closed surface $\mathcal{S}$, while $\ell_0(g^+ - g^-) \in H^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{S})$ be the extension by zero of the function $g^+ - g^-$.*

The Nemann BVP (2.7), (2.8) has a unique solution $\boldsymbol{U} \in \mathbb{H}^1_{\boldsymbol{\nu}\times,0}(\mathbb{R}^3_{\mathcal{C}})$ of the form

$$\boldsymbol{U} = \begin{cases} -\mathbf{V}(\mathbf{V}_{-\mathbf{1}})^{-1}[\mathcal{P}^{-1}_{\tau,+}(\ell g^+)\boldsymbol{\nu} + \mathcal{P}^{-1}_{\tau,+}\boldsymbol{\Psi}] & \text{in } \Omega^+, \\ \mathbf{V}(\mathbf{V}_{-\mathbf{1}})^{-1}[\mathcal{P}^{-1}_{\tau,-}((\ell g^+)\boldsymbol{\nu} - \ell_0(g^+ - g^-)\boldsymbol{\nu}) + \mathcal{P}^{-1}_{\tau,-}\boldsymbol{\Psi}] & \text{in } \Omega^-, \end{cases}$$

where $\boldsymbol{\Psi} \in \widetilde{\mathbb{H}}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c)$ is a unique solution to the system

$$r_{\mathcal{C}^c}\mathfrak{B}(D)\boldsymbol{\Psi} = \mathbf{F} \quad \text{on } \mathcal{C}^c,$$
$$\mathbf{F} := r_{\mathcal{C}^c}(\mathcal{P}_{\tau,-})^{-1}\ell_0(g^+ - g^-)\boldsymbol{\nu} - r_{\mathcal{C}^c}\mathfrak{B}(D)(\ell g^+)\boldsymbol{\nu}, \quad \mathbf{F} \in \mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c).$$

The pseudodifferential operator of order -1

$$r_{\mathcal{C}^c}\mathfrak{B}(D) = r_{\mathcal{C}^c}[\mathcal{P}^{-1}_{\tau,+} + \mathcal{P}^{-1}_{\tau,-}] : \widetilde{\mathbb{H}}^{-1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c) \to \mathbb{H}^{1/2}_{\boldsymbol{\nu}\times,0}(\mathcal{C}^c)$$

is invertible.

Proof. The proof follows directly from Lemma 4.5 and Corollary 4.6. □

Acknowledgment. Supported by Shota Rustaveli National Science Foundation, grant number 31/39.

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