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THE SCREEN TYPE BOUNDARY VALUE PROBLEMS FOR ANISOTROPIC PSEUDO-MAXWELL'S EQUATIONS

Introduction

Let Ω denote either a bounded $\Omega^+ \subset \mathbb{R}^3$ or an unbounded $\Omega^- := \mathbb{R}^3 \backslash \overline{\Omega^+}$ domain with smooth boundary $\mathcal{S} := \partial\Omega^+$ and let $\boldsymbol{\nu}$ be the outer unit normal vector field to $\mathcal{S}$.

By $\mathcal{C}$ we denote an orientable smooth open surface in $\mathbb{R}^3$ (a screen) with the smooth boundary $\partial\mathcal{C}$. The screen has two faces $\mathcal{C}^-$ and $\mathcal{C}^+$ distinguished by the orientation of the normal vector field: $\boldsymbol{\nu}$ is pointing from $\mathcal{C}^+$ to $\mathcal{C}^-$. Moreover, we assume that $\mathcal{C}$ is a part of some smooth and simple (non self intersecting) hypersurface $\mathcal{S}$ that divides the space $\mathbb{R}^3$ into two disjoint domains Ω^+ and $\Omega^- := \mathbb{R}^3 \backslash \overline{\Omega^+}$ such that Ω^+ is bounded and $\mathcal{S} = \partial\Omega^\pm$.

Our purpose is to investigate the screen-type boundary value problem for pseudo-Maxwell's equations

$$\operatorname{curl} \mu^{-1} \operatorname{curl} \boldsymbol{U} - s\,\varepsilon \operatorname{grad} \operatorname{div} (\varepsilon \boldsymbol{U}) - \omega^2 \varepsilon \boldsymbol{U} = 0 \quad \text{in} \quad \mathbb{R}^3_{\mathcal{C}}, \tag{1}$$

where $\mathbb{R}^3_{\mathcal{C}} := \mathbb{R}^3 \setminus \mathcal{C}$ is the domain with a screen, using the potential method.

The present investigation covers the anisotropic case when the matrices in (1)

$$\varepsilon = [\varepsilon_{jk}]_{3\times 3}\,, \quad \mu = [\mu_{jk}]_{3\times 3}\,, \tag{2}$$

are real valued, constant, symmetric and positive definite, i.e.,

$$\langle \varepsilon\xi, \xi \rangle \geq c|\xi|^2\,, \quad \langle \mu\xi, \xi \rangle \geq d|\xi|^2\,, \quad \forall \xi \in \mathbb{R}^3\,,$$

for some positive constants $c > 0$, $d > 0$, where

$$\langle \eta, \xi \rangle := \sum_{j=1}^{3} \eta_j \overline{\xi}_j, \quad \eta,\ \xi \in \mathbb{C}^3.$$

s is a positive real number and the frequency parameter ω is assumed to be non-zero and complex valued, i.e., $\operatorname{Im} \omega \neq 0$.

The study of boundary value problems in electromagnetism naturally leads us to the pseudo-Maxwell's equations inherited with tangent boundary

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conditions, which are in some sense non-standard for the elliptic equations (1), cf. works of Buffa, Costabel, Christiansen, Dauge, Hazard, Lenoir, Mitrea, Niciase and others. The case with the Dirichlet type boundary condition $\boldsymbol{\nu} \times \boldsymbol{U}$ is mostly investigated by variational methods, here $\boldsymbol{\nu}$ is the unit normal to the boundary $\partial\Omega$. Our goal is investigate well posedness of the Neumann type boundary value problems for (1) as well as its unique solvability in unbounded domains with screen $\mathbb{R}^3_{\mathcal{C}}$.

For rigorous formulation of conditions for the unique solvability of the formulated boundary value problems we use the Bessel potential $\mathbb{H}^r(\Omega)$, $\mathbb{H}^r(\mathcal{S})$ spaces. We quote [3] for definitions and properties of these spaces.

The space $\widetilde{\mathbb{H}}^r(\mathcal{C})$ comprises those functions $\varphi \in \mathbb{H}^r(\mathcal{S})$ which are supported in $\overline{\mathcal{C}}$ (functions with the "vanishing traces on the boundary"). For the detailed definitions and properties of these spaces we refer, e.g., to [3]).

It is well known that the space $\mathbb{H}^{r-1/2}(\mathcal{S})$ is a trace space for $\mathbb{H}^r(\Omega)$, provided that $r > 1/2$ and the corresponding trace operator is denoted by $\gamma_{\mathcal{S}}$. For the detailed definitions and properties of these spaces we refer, e.g., to [3].

We introduce the following spaces:

$$\mathbb{H}^r_{\varepsilon\boldsymbol{\nu},0}(\mathcal{S}) := \Big\{\boldsymbol{U} \in \mathbb{H}^r(\mathcal{S}) \ : \ \langle\varepsilon\boldsymbol{\nu}, \boldsymbol{U}\rangle = 0\Big\},$$
$$\mathbb{H}^1_{\varepsilon\boldsymbol{\nu},0}(\Omega^+) = \Big\{\boldsymbol{U} \in \mathbb{H}^1(\Omega^+) \ : \ \langle\varepsilon\boldsymbol{\nu}, \gamma_{\mathcal{S}}\boldsymbol{U}\rangle = 0 \quad \text{on} \quad \mathcal{S}\Big\},$$
$$\mathbb{H}^1_{\varepsilon\boldsymbol{\nu},0}(\mathbb{R}^3_{\mathcal{C}}) = \Big\{\boldsymbol{U} \in \mathbb{H}^1(\mathbb{R}^3_{\mathcal{C}}) \ : \ \langle\varepsilon\boldsymbol{\nu}, \gamma_{\mathcal{C}^\pm}\boldsymbol{U}\rangle = 0 \quad \text{on} \quad \mathcal{C}\Big\}.$$

Theorem 0.1. *The operator in* (1)

$$\boldsymbol{A}(D)\boldsymbol{U} := \operatorname{curl}\mu^{-1}\operatorname{curl}\boldsymbol{U} - s\,\varepsilon\operatorname{grad}\operatorname{div}(\varepsilon\boldsymbol{U}) - \omega^2\varepsilon\boldsymbol{U}$$

is elliptic, has the positive definite principal symbol

$$\mathcal{A}_{\rm pr}(\xi) := \sigma_{\rm curl}(\xi)\mu^{-1}\sigma_{\rm curl}(\xi) + s\,\varepsilon\big[\xi_j\xi_k]_{3\times3}\varepsilon, \quad \xi = (\xi_1, \xi_2, \xi_3)^\top \in \mathbb{R}^3, \quad (3)$$

where

$$\sigma_{\rm curl}(\xi) := \begin{bmatrix} 0 & i\xi_3 & -i\xi_2 \\ -i\xi_3 & 0 & i\xi_1 \\ i\xi_2 & -i\xi_1 & 0 \end{bmatrix},$$

is non-vanishing $\det\mathcal{A}_{\rm pr}(\xi) \neq 0$ *for* $\xi \neq 0$ *and positive definite*

$$\langle\mathcal{A}_{\rm pr}(\xi)\eta, \eta\rangle \geq c|\xi|^2|\eta|^2 \quad c = \text{const} > 0, \quad \forall\,\xi \in \mathbb{R}^3, \quad \forall\eta \in \mathbb{C}^3. \qquad (4)$$

Moreover, the operator $\boldsymbol{A}(D)$ *is self-adjoint and the following Green's formula holds*

$$(\boldsymbol{A}(D)\boldsymbol{U}, \boldsymbol{V})_{\Omega^+} = (\mathfrak{N}(D, \boldsymbol{\nu})\boldsymbol{U}, \boldsymbol{V})_{\mathcal{S}} + \boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U}, \boldsymbol{V})_{\Omega^+} - \omega^2(\varepsilon\,\boldsymbol{U}, \boldsymbol{V})_{\Omega^+}, \quad (5)$$

for all $\boldsymbol{U}, \boldsymbol{V} \in \mathbb{H}^1(\Omega^+)$. *Here* $\mathfrak{N}(D, \boldsymbol{\nu})$ *is the Neumann's boundary operator*

$$\mathfrak{N}(D, \boldsymbol{\nu})\boldsymbol{U} := \boldsymbol{\nu} \times \mu^{-1}\operatorname{curl}\boldsymbol{U} - s\operatorname{div}(\varepsilon\boldsymbol{U})\varepsilon\boldsymbol{\nu}, \quad \boldsymbol{U} \in \mathbb{H}^1(\Omega^+) \qquad (6)$$

and $\boldsymbol{a}_{\varepsilon,\mu}$ *is the natural bilinear differential form associated with the Green formula*

$$\boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U},\boldsymbol{V})_\Omega := \big(\mu^{-1}\mathrm{curl}\,\boldsymbol{U}, \mathrm{curl}\,\boldsymbol{V}\big)_\Omega + s\,\big(\mathrm{div}(\varepsilon\boldsymbol{U}), \mathrm{div}(\varepsilon\boldsymbol{V})\big)_\Omega. \qquad (7)$$

Based on this fact we obtain that the Neumann's trace $\mathfrak{N}(D,\boldsymbol{\nu})\boldsymbol{U} \in \mathbb{H}^{-\frac{1}{2}}(\mathcal{S})$.

Let us mention the well known fact, that the Neumann boundary value problem

$$\boldsymbol{A}(D)\boldsymbol{U} = 0 \quad \text{in} \ \ \Omega^+, \quad \mathfrak{N}(D,\boldsymbol{\nu})\,\boldsymbol{U} = g \quad \text{on} \ \ \mathcal{S}, \quad g \in \mathbb{H}^{-\frac{1}{2}}(\mathcal{S}),$$

is not an elliptic boundary value problem in the sense of the Shapiro-Lopatinski condition. To overcome the problem we consider the tangent boundary conditions and look for a solution in tangent spaces. First, for any $\boldsymbol{V} \in \mathbb{H}^1_{\varepsilon\boldsymbol{\nu},0}(\Omega^+)$ we have $\pi_{\varepsilon\boldsymbol{\nu}}\boldsymbol{V} = \boldsymbol{V}$, where $\pi_{\varepsilon\boldsymbol{\nu}}\boldsymbol{U} := \boldsymbol{U} - \langle \boldsymbol{U}, \varepsilon\boldsymbol{\nu}\rangle\varepsilon\boldsymbol{\nu}$ is a projection on the hyperplane, orthogonal to the vector field $\varepsilon\boldsymbol{\nu}$. Therefore from (6) and (7) we obtain

$$\big(\mathfrak{N}(D,\boldsymbol{\nu})\boldsymbol{U},\boldsymbol{V}\big) = \big(\mathfrak{N}(D,\boldsymbol{\nu})\boldsymbol{U},\pi_{\varepsilon\boldsymbol{\nu}}\boldsymbol{V}\big) = \big(\pi_{\varepsilon\boldsymbol{\nu}}\mathfrak{N}(D,\boldsymbol{\nu})\boldsymbol{U},\pi_{\varepsilon\boldsymbol{\nu}}\boldsymbol{V}\big).$$

Thus $\pi_{\varepsilon\boldsymbol{\nu}}\mathfrak{N}(D,\boldsymbol{\nu})\boldsymbol{U}$ is well-defined as a functional on $\mathbb{H}^{\frac{1}{2}}_{\varepsilon\boldsymbol{\nu},0}(\mathcal{S})$ and belongs to $\mathbb{H}^{-\frac{1}{2}}_{\varepsilon\boldsymbol{\nu},0}(\mathcal{S})$.

An important role in the investigation goes to the following lemma, which was proved by M. Costabel in [2] for a compact domain, We have extended the result for a non-compact domains, including domains with a screen.

Lemma 0.2. *The bilinear differential form* $\boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U},\boldsymbol{U})_{\Omega^+}$ *in* (7) *is coercive, i.e., there exist positive constants* c_1 *and* c_2 *such that*

$$\mathrm{Re}\,\boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U},\boldsymbol{U})_{\Omega^\pm} \geq c_1\big\|\boldsymbol{U}\big|\mathbb{H}^1(\Omega^\pm)\big\|^2 - c_2\big\|\boldsymbol{U}\big|\mathbb{L}_2(\Omega^\pm)\big\|^2 \qquad (8)$$

on the space $\mathbb{H}^1_{\varepsilon\boldsymbol{\nu},0}(\Omega^+)$.

Moreover, the bilinear differential form $\boldsymbol{a}_{\varepsilon,\mu}(\boldsymbol{U},\boldsymbol{U})_{\Omega^-}$ *is coercive for all vector fields* $\boldsymbol{U} \in \mathbb{H}^1_{\varepsilon\boldsymbol{\nu},0}(\Omega^-)$ *provided they are solutions to pseudo-Maxwell's equation.*

1. Basic Results

Our main goal is to investigate following screen type Neumann boundary value problem (BVP) for pseudo-Maxwell's equations:

Problem. Find $\boldsymbol{U} \in \mathbb{H}^1_{\varepsilon\boldsymbol{\nu},0}(\mathbb{R}^3_{\mathcal{C}})$ such that

$$\begin{cases} \boldsymbol{A}(D)\boldsymbol{U} = \mathrm{curl}\,\mu^{-1}\mathrm{curl}\boldsymbol{U} - s\,\varepsilon\,\mathrm{grad}\,\mathrm{div}(\varepsilon\boldsymbol{U}) - \omega^2\varepsilon\boldsymbol{U} = 0 \quad \text{in} \quad \mathbb{R}^3_{\mathcal{C}}\,, \\ \gamma^\pm_{\mathcal{C}}\,(\pi_{\varepsilon\boldsymbol{\nu}}\mathfrak{N}(D,\boldsymbol{\nu})\,\boldsymbol{U}) = \boldsymbol{g}^\pm \quad \text{on} \quad \mathcal{C}\,, \end{cases} \qquad (9)$$

where s is an arbitrary positive constant and the given data $\boldsymbol{g}^\pm$ satisfy the conditions

$$\boldsymbol{g}^\pm \in \mathbb{H}^{-1/2}_{\varepsilon\boldsymbol{\nu},0}(\mathcal{C}), \quad \boldsymbol{g}^+ - \boldsymbol{g}^- \in r_{\mathcal{C}}\widetilde{\mathbb{H}}^{-1/2}_{\varepsilon\boldsymbol{\nu},0}(\mathcal{C}). \qquad (10)$$

Let us consider, respectively, the *single layer* and *double layer* potential operators

$$\mathbf{V}\boldsymbol{U}(x) := \oint_{\mathcal{S}} \mathbf{F}_{\boldsymbol{A}}(x-\tau)\boldsymbol{U}(\tau)\,dS,$$

$$\mathbf{W}\boldsymbol{U}(x) := \oint_{\mathcal{S}} [(\mathfrak{N}(D,\boldsymbol{\nu}(\tau))\mathbf{F}_{\boldsymbol{A}})(x-\tau)]^{\top}\boldsymbol{U}(\tau)\,dS, \quad x \in \Omega, \tag{11}$$

related to pseudo-Maxwell's equations in (9), where $\mathbf{F}_{\boldsymbol{A}}$ is a fundamental solution to $\boldsymbol{A}(D)$.

Lemma 1.1. *The direct value $\mathbf{V}_{-1}$ of the single layer potential in* (11) *is invertible in the following space settings*

$$\mathbf{V}_{\mathbf{-1}} \;:\; \mathbb{H}^{r}(\mathcal{S}) \to \mathbb{H}^{r+1}(\mathcal{S}) \quad \forall r \in \mathbb{R}.$$

The principal symbol of the pseudodifferential operator $\mathbf{V}_{\mathbf{-1}}$ is positive definite

$$\langle V_{-1,\mathrm{pr}}(\varkappa,\xi)\eta,\eta\rangle \geq c_0|\eta|^2|\xi|^{-1} \quad \forall \eta \in C^3\,, \quad \varkappa \in \mathcal{S}, \quad \xi \in \mathbb{R}^3,$$

for some positive constant c_0.

The foregoing Lemma 1.1 enables to look for a solution of the BVP (9)–(10) in the form

$$\boldsymbol{U}(x) = \begin{cases} \mathbf{V}(\mathbf{V}_{\mathbf{-1}})^{-1}\boldsymbol{\Phi}^{+}(x) & x \in \Omega^{+}, \\ \mathbf{V}(\mathbf{V}_{\mathbf{-1}})^{-1}\boldsymbol{\Phi}^{-}(x) & x \in \Omega^{-} \quad \text{for some} \quad \boldsymbol{\Phi}^{\pm} \in \mathbb{H}^{1/2}_{\varepsilon\boldsymbol{\nu},0}(\mathcal{S}), \end{cases}$$

where $\Omega^{\pm}$ are the domains bordered by a surface $\mathcal{S} = \partial\Omega^{+} = \partial\Omega^{-}$, which contains $\mathcal{C}$ as a subsurface $\mathcal{C} \subset \mathcal{S}$. Then $\boldsymbol{U}$ satisfies the basic differential equation from BVP (9) in the domains $\Omega^{\pm}$ and, due to the mapping properties of $\mathbf{V}$ we have $\boldsymbol{U} \in \mathbb{H}^{1}_{\varepsilon\boldsymbol{\nu},0}(\mathbb{R}^3_{\mathcal{C}})$. Further we need to fulfill the boundary conditions (cf. (6))

$$r_{\mathcal{C}}\gamma_{\mathcal{S}^{\pm}}\,(\pi_{\varepsilon\boldsymbol{\nu}}\mathfrak{N}(D,\boldsymbol{\nu})\boldsymbol{U}) = \boldsymbol{g}^{\pm} \quad \text{on} \quad \mathcal{C}.$$

Due to the Plemelji formulae we derive the following boundary pseudodifferential equations

$$r_{\mathcal{C}}\mathcal{P}_{\pm}\boldsymbol{\Phi}^{\pm} = \boldsymbol{g}^{\pm} \quad \text{on} \quad \mathcal{C},$$

where

$$\mathcal{P}_{\pm} := \pi_{\varepsilon\boldsymbol{\nu}}\Big(\frac{1}{2}I \mp (\mathbf{W_0})^{*}\Big)(\mathbf{V}_{\mathbf{-1}})^{-1}$$

are the *modified Poincaré-Steklov* pseudodifferential operators of order 1. $\mathbf{W_0}$ is the direct value of the double layer potential in (11), while $(\mathbf{W_0})^{*}$ is the adjoint operator.

Lemma 1.2. *For an open subsurface $\mathcal{C} \subset \mathcal{S}$ the operators*

$$r_{\mathcal{C}}\mathcal{P}_{\pm} \quad : \quad \widetilde{\mathbb{H}}^{1/2}_{\varepsilon\boldsymbol{\nu},0}(\mathcal{C}) \to \mathbb{H}^{-1/2}_{\varepsilon\boldsymbol{\nu},0}(\mathcal{C})$$

are coercive

$$\operatorname{Re}\left(r_{\mathcal{C}}\mathcal{P}_{\pm}\mathbf{\Phi},\mathbf{\Phi}\right)_{\mathcal{C}} \geq c_0\left\|\mathbf{\Phi}\middle|\widetilde{\mathbb{H}}^{1/2}_{\varepsilon\boldsymbol{\nu},0}(\mathcal{C})\right\|^2 - c_1\left\|\mathbf{\Phi}\middle|\mathbb{L}_{2,\varepsilon\boldsymbol{\nu},0}(\mathcal{C})\right\|^2$$

for some positive constants c_0, c_1 *and all* $\mathbf{\Phi} \in \widetilde{\mathbb{H}}^{1/2}_{\varepsilon\boldsymbol{\nu},0}(\mathcal{C})$. *Moreover, the operators have the trivial kernels* $\operatorname{Ker} r_{\mathcal{C}}\mathcal{P}_{\pm} = \{0\}$ *and are invertible.*

If the frequency is purely imaginary $\omega = i\beta \neq 0$, $\beta \in \mathbb{R}$, *the operators* $r_{\mathcal{C}}\mathcal{P}_{\pm}$ *are positive definite*

$$\left(r_{\mathcal{C}}\mathcal{P}_{\pm}\mathbf{\Phi},\mathbf{\Phi}\right)_{\mathcal{C}} \geq M_{\pm}\left\|\mathbf{\Phi}\middle|\widetilde{\mathbb{H}}^{1/2}_{\varepsilon\boldsymbol{\nu},0}(\mathcal{C})\right\|$$

for some positive constants $M_{\pm}$.

Based on the foregoing lemmata in [1] we have proved the following result.

Theorem 1.3. *Let* $0 \leq r < \frac{1}{2}$ *and the condition*

$$\boldsymbol{g}^{\pm} \in \mathbb{H}^{r-1/2}_{\varepsilon\boldsymbol{\nu},0}(\mathcal{C}), \quad \boldsymbol{g}^{+} - \boldsymbol{g}^{-} \in r_{\mathcal{C}}\widetilde{\mathbb{H}}^{r-1/2}_{\varepsilon\boldsymbol{\nu},0}(\mathcal{C}).$$

hold. Then the elliptic BVP (9) *has a unique solution* $\boldsymbol{U} \in \mathbb{H}^{r+1}_{\varepsilon\boldsymbol{\nu},0}(\mathbb{R}^3_{\mathcal{C}})$.

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