Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 171 (2017) 264–292
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original article

Method of fundamental solutions for mixed and crack type problems in the classical theory of elasticity

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Received 3 February 2017; received in revised form 20 April 2017; accepted 24 April 2017
Available online 20 May 2017

Abstract

We analyse some new aspects concerning application of the fundamental solution method to the basic three-dimensional boundary value problems, mixed transmission problems, and also interior and interfacial crack type problems for steady state oscillation equations of the elasticity theory. First we present existence and uniqueness theorems of weak solutions and derive the corresponding norm estimates in appropriate function spaces. Afterwards, by means of the columns of Kupradze's fundamental solution matrix special systems of vector functions are constructed explicitly. The linear independence and completeness of these systems are proved in appropriate Sobolev–Slobodetskii and Besov function spaces. It is shown that the problem of construction of approximate solutions to the basic and mixed boundary value problems and to the interior and interfacial crack problems can be reduced to the problems of approximation of the given boundary vector functions by elements of the linear spans of the corresponding complete systems constructed by the fundamental solution vectors. By this approach the approximate solutions of the boundary value and transmission problems are represented in the form of linear combinations of the columns of the fundamental solution matrix with appropriately chosen poles distributed outside the domain under consideration. The unknown coefficients of the linear combinations are defined by the approximation conditions of the corresponding boundary and transmission data.

Keywords: Method of fundamental solutions; Theory of elasticity; Elastic vibrations; Mixed boundary value problem; Mixed transmission problem; Crack problem; Approximate solutions

1. Introduction

The Method of Fundamental Solutions (MFS) for partial differential equations was first proposed by V. Kupradze in the 1960s (see the pioneering works in this direction by V. Kupradze and M. Alexidze, [1,2], [KuAl]). The main

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Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2017.04.004

idea of the MFS is to distribute the singularity poles $\{y^{(k)}\}_{k=1}^{\infty}$ of the fundamental solution $\Gamma(x-y)$ of a differential operator outside the domain under consideration, construct the set of functions $\{\Gamma(x-y^{(k)})\}_{k=1}^{\infty}$, prove its density properties in appropriate function spaces, and then approximate the sought-for solution by a linear combination of the fundamental solutions, $\sum_{k=1}^{N} C_k\, \Gamma(x-y^{(k)})$ with unknown coefficients C_k, which are to be determined by satisfying the corresponding boundary conditions.

Starting from the 1970s, the MFS gradually became a useful technique and is used to solve a large variety boundary value problems (BVP) arising in the mathematical models of physics, engineering, and biomedicine (see [1–18], and the references therein). However, it should be mentioned that until now it has not been worked out how to apply the MFS to crack type problems in solid mechanics, since the different approaches related to MFS described in the scientific literature are not applicable to crack type problems. To work out this problematic topic and to extend the MFS to crack type boundary-value problems are among the main goals of the present investigation. We will reformulate crack type problems in the form of mixed type transmission problems introducing an artificial interface boundary containing the crack faces and then substantiate mathematically the MFS on the basis of the results obtained for mixed transmission problems.

For the basic and mixed exterior boundary value problems, as well as for the crack and mixed transmission problems of steady state elastic oscillations, here we develop the approach which is applicable for all values of the oscillation frequency parameter.

We have to mention here that the main shortage of the MFS is its poor conditioning which should be alleviated, e.g., by preconditioning of the corresponding system matrix or by iterative refinement or by some other artificial approaches available for special particular cases (see, e.g. [19]).

However, the MFS features remarkable and unusual ease of implementation due to the following reasons (see, e.g. [3,17,19]): *"Uniform character of the trial functions, complete absence of singular integral evaluations, it does not require an elaborate discretization of the boundary, simplicity of finding values of approximate solution at inner points of the domain of interest, the derivatives of the MFS approximation can also be evaluated directly, extreme abundance of the set of trial functions that results in a high adaptivity of the method, MFS can be applied even in the case of domains with irregular boundaries (e.g., for domains with Lipschitz boundaries)"*. More detailed overview of the results related to the fundamental solution method can be found in [17] and the references therein.

In this paper we prove linear independence and density property of the appropriately chosen systems of vector functions constructed by the corresponding fundamental solutions (Kupradze's matrix of fundamental solutions). These systems are associated with particular type of problems and actually they reduce the solving procedure of boundary value problems to the approximation problems of the boundary data in the appropriate non-orthogonal complete systems of vector functions.

The paper is organized as follows. In Section 2, we introduce the notions of regular, semi-regular and weak solutions and formulate classical and weak settings of boundary value and transmission problems for steady state oscillation equations of the elasticity theory. We formulate also the corresponding uniqueness theorems for the problems under consideration in the class of vector functions satisfying the Sommerfeld–Kupradze radiation conditions at infinity. In Section 3, existence and uniqueness theorems are proved for weak solutions and the corresponding estimates are obtained in appropriate function spaces. Section 4 is devoted to the fundamental solution method for basic and mixed boundary value problems, as well as for the basic and mixed transmission problems containing crack type problems as special particular cases. Special systems of vector functions are constructed explicitly by means of the columns of Kupradze's fundamental solution matrix and their linear independence and completeness are proved in appropriate Sobolev–Slobodetskii and Besov function spaces. The problem of construction of approximate solutions to the boundary value and transmission problems are reduced to the approximation problems of the given boundary vector functions by linear combinations of the elements of the corresponding nonorthogonal, linearly independent, complete vector systems. In Appendix A, we collect some auxiliary material needed in the main text of the paper concerning properties of layer potentials and the corresponding boundary operators. In Appendix B, we present alternative integral representations of radiating solutions in unbounded regions. Finally, in Appendix C, we recall some results from the theory of strongly elliptic pseudodifferential equations on manifolds with boundary in Bessel potential and Besov spaces which are the main tools for proving existence theorems for mixed boundary, boundary-transmission, and crack type problems by the potential methods.

The approach developed in this paper can be successfully applied to boundary value problems of mathematical physics for homogeneous and piecewise homogeneous bounded and unbounded composite media containing interior or interfacial cuts where the screen or crack type conditions are prescribed. In particular, it can be applied to the interior and exterior problems of statics of the theory of elasticity, as well as to the interior BVP of steady state oscillations for bounded domains. As it is well-known, the interior problems of steady state oscillations have discrete (countable) sets of resonant frequencies for arbitrary bounded domain and the corresponding nonhomogeneous BVPs are not solvable for arbitrary data (see e.g., [12, Ch. 7], [20]). However, the approach described in the paper can be applied also to the interior problems if the oscillation parameter does not belong to the set of resonant frequencies, i.e., if the corresponding homogeneous boundary value and transmission problems of steady state oscillations possess only the trivial solutions.

2. Basic equations and operators, statement of problems, and uniqueness theorems

The basic equation of elastic vibrations in the case of isotropic solids reads as [12]

$$A(\partial,\omega)u(x) \equiv \mu\,\Delta u(x) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u(x) + \varrho\,\omega^2\,u(x) = 0, \tag{2.1}$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator, $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_k := \partial/\partial x_k$, ϱ is the constant density of the homogeneous elastic solid under consideration, $\omega \in \mathbb{R}$ is the oscillation frequency parameter, $u = (u_1, u_2, u_3)^\top$ is the displacement vector (the amplitude), and $A(\partial,\omega)$ is the matrix differential operator,

$$A(\partial,\omega) = \big[\mu\,\delta_{kj}\,\Delta + (\lambda+\mu)\,\partial_k\partial_j + \varrho\,\omega^2\,\delta_{kj}\big]_{3\times 3},$$

δ_{kj} is the Kronecker symbol, λ and μ are the Lamé constants satisfying the inequalities $\mu > 0$, $\quad 2\lambda + 3\mu > 0$. When $\omega = 0$, Eq. (2.1) coincides with the Lamé equilibrium equations of statics and generates the operator $A(\partial) := A(\partial, 0)$. The principal homogeneous symbol matrix $\mathcal{A}(\xi) := [\mu\delta_{kj}|\xi|^2 + (\lambda+\mu)\xi_k\xi_j]_{3\times 3}$ of the operators $-A(\partial,\omega)$ and $-A(\partial)$ is positive definite, $\mathcal{A}(\xi)\eta\cdot\eta \geq \delta_0\,|\xi|^2\,|\eta|^2$, $\quad \forall\xi \in \mathbb{R}^3$, $\quad \forall\eta \in \mathbb{C}^3$, where δ_0 is a positive constant, $a\cdot b$ denotes the scalar product of complex valued vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$: $a\cdot b = \sum_{k=1}^3 a_k\,\overline{b_k}$; $\mathbb{R}^3$ and $\mathbb{C}^3$ stand for the set of real and complex 3-tuples respectively.

Let Ω^+ be a bounded 3-dimensional domain in $\mathbb{R}^3$ with a boundary $S = \partial\Omega^+$, $\overline{\Omega^+} = \Omega^+ \cup S$, and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. Throughout the paper, for simplicity, we assume that S is an infinitely smooth surface if not otherwise stated.

By $C^k(\overline{\Omega^\pm})$ we denote the subspace of functions from $C^k(\Omega^\pm)$ whose derivatives up to the order k are continuously extendable to S from $\Omega^\pm$.

The symbols $\{\cdot\}_S^+$ and $\{\cdot\}_S^-$ denote one-sided limits (traces) on S from Ω^+ and Ω^-, respectively. We often drop the subscript S if it does not lead to misunderstanding.

By L_p, $L_{p,loc}$, $L_{p,comp}$, W_p^r, $W_{p,loc}^r$, $W_{p,comp}^r$, H_p^s, and $B_{p,q}^s$ (with $r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) we denote the well-known Lebesgue, Sobolev–Slobodetskii, Bessel potential, and Besov function spaces, respectively (see, e.g., [21,22]). Recall that $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, for any $r \geq 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k. In our analysis we essentially employ also the following function spaces:

$$\begin{aligned}
&\widetilde{H}_p^s(M) := \big\{f :\ f \in H_p^s(M_0),\ \mathrm{supp}\,f \subset \overline{M}\big\},\\
&\widetilde{B}_{p,q}^s(M) := \big\{f :\ f \in B_{p,q}^s(M_0),\ \mathrm{supp}\,f \subset \overline{M}\big\},\\
&H_p^s(M) := \big\{r_{_M} f :\ f \in H_p^s(M_0)\big\},\\
&B_{p,q}^s(M) := \big\{r_{_M} f :\ f \in B_{p,q}^s(M_0)\big\},
\end{aligned}$$

where M_0 is a closed manifold without boundary and M is an open proper submanifold of M_0 with nonempty smooth boundary $\partial M \neq \varnothing$; $r_{_M}$ is the restriction operator onto M.

Remark 2.1. Let a function f be defined on an open proper submanifold M of a closed manifold M_0 without boundary. Let $f \in B_{p,q}^s(M)$ and $\widetilde{f}$ be an extension of f by zero to $M_0 \setminus M$. If the extension preserves the space, i.e., if $\widetilde{f} \in \widetilde{B}_{p,q}^s(M)$, then we write $f \in \widetilde{B}_{p,q}^s(M)$ instead of $f \in r_{_M}\widetilde{B}_{p,q}^s(M)$ when it does not lead to misunderstanding.

Now let us introduce some definitions (cf. [23]).

Definition 2.2. We say that w is a regular function in $\Omega^\pm$ if $w \in C^1(\overline{\Omega^\pm}) \cap C^2(\Omega^\pm)$.

Definition 2.3. Let us consider the following smooth dissection of the boundary surface $S = \overline{S}_D \cup \overline{S}_N$, $S_D \cap S_N = \varnothing$, $\ell = \overline{S}_D \cap \overline{S}_N \in C^\infty$, and let $\widetilde{\Omega}^\pm_\ell := \overline{\Omega^\pm} \setminus \ell$.

We say that w is a semi-regular function in $\widetilde{\Omega}^\pm_\ell$ and write $w \in \mathbf{C}(\widetilde{\Omega}^\pm_\ell;\delta)$ if

(i) w is continuous in $\overline{\Omega^\pm}$;
(ii) the first order derivatives of w are continuous in $\widetilde{\Omega}^\pm_\ell$ and there is a constant $\delta \in [0,1)$, such that at the collision curve ℓ the following estimates hold

$$|\partial_k w(x)| \leqslant C\,[\mathrm{dist}(x,\ell)]^{-\delta}, \quad x \in \widetilde{\Omega}^\pm_\ell, \quad C = const, \quad k = 1,2,3,$$

where $\mathrm{dist}(x,\ell)$ is the distance from the reference point x to the collision curve ℓ;
(iii) the second order derivatives of w are continuous in $\Omega^\pm$ and integrable over Ω^+ and over any subdomain of Ω^- of finite diameter.

Evidently, $\mathbf{C}(\widetilde{\Omega}^\pm_\ell;\delta) \subset \big[C(\overline{\Omega^\pm}) \cap C^1(\widetilde{\Omega}^\pm_\ell) \cap C^2(\Omega^\pm)\big]$.

Definition 2.4. Let an elastic solid occupying the domain $\Omega^\pm$ contain an interior crack. We identify the crack surface as a two-dimensional, two-sided smooth manifold $\Sigma \subset \Omega^\pm$ with the crack edge $\ell_c := \partial\Sigma$. We assume that Σ is a proper submanifold of a closed surface S_0 surrounding a bounded domain $\overline{\Omega}_0$ which is a proper subdomain of $\Omega^\pm$. We choose the direction of the unit normal vector to the fictitious surface S_0 such that it is outward with respect to the domain Ω_0. This agreement defines uniquely the direction of the normal vector to the crack surface Σ. The symbols $\{\,\cdot\,\}^+_\Sigma$ and $\{\,\cdot\,\}^-_\Sigma$ denote the one-sided limits on Σ from Ω_0 and $\Omega^\pm \setminus \overline{\Omega}_0$, respectively.

Further, let $\Omega^\pm_\Sigma := \Omega^\pm \setminus \overline{\Sigma}$ and $\widetilde{\Omega}^\pm_\Sigma := \overline{\Omega^\pm} \setminus \overline{\Sigma}$ with $\overline{\Sigma} = \Sigma \cup \ell_c$.

We say that w is a semi-regular function in $\widetilde{\Omega}^\pm_\Sigma$ and write $w \in \mathbf{C}(\widetilde{\Omega}^\pm_\Sigma;\delta)$ if

(i) w is continuous in $\widetilde{\Omega}^\pm_\Sigma$ and one-sided continuously extendable to $\overline{\Sigma}$ from Ω_0 and from $\Omega^+ \setminus \overline{\Omega}_0$, i.e., w is continuous in the regions $\widetilde{\Omega}^\pm_\Sigma$, $\overline{\Omega^\pm} \setminus \Omega_0$, and $\overline{\Omega}_0$;
(ii) the first order derivatives of w are continuous in $\widetilde{\Omega}^\pm_\Sigma$ and one-sided continuously extendable to Σ from Ω_0 and from $\Omega^\pm \setminus \overline{\Omega}_0$, and there is a constant $\delta \in [0,1)$, such that at the crack edge $\ell_c = \partial\Sigma$ the following estimates hold

$$|\partial_k w(x)| \leqslant C\,[\mathrm{dist}(x,\ell_c)]^{-\delta}, \quad x \in \widetilde{\Omega}^\pm_\Sigma, \quad C = const, \quad k = 1,2,3;$$

(iii) the second order derivatives of w are continuous in $\Omega^\pm_\Sigma$ and integrable over Ω^+_Σ and over any subdomain of Ω^-_Σ of finite diameter.

Evidently, formally we can write $\mathbf{C}(\widetilde{\Omega}^\pm_\Sigma;\delta) \subset \big[C(\overline{\Omega}_0) \cap C(\overline{\Omega^\pm} \setminus \Omega_0) \cap C^1(\widetilde{\Omega}^\pm_\Sigma) \cap C^2(\Omega^\pm_\Sigma)\big]$, which is to be understood in the following sense: if $w \in \mathbf{C}(\widetilde{\Omega}^\pm_\Sigma;\delta)$, then $r_{\overline{\Omega}_0}\, w \in C(\overline{\Omega}_0)$, $r_{\overline{\Omega^\pm} \setminus \Omega_0}\, w \in C(\overline{\Omega^\pm} \setminus \Omega_0)$, $w \in C^1(\widetilde{\Omega}^\pm_\Sigma)$, $w \in C^2(\Omega^\pm_\Sigma)$.

Definition 2.5. We say that a vector $u = (u_1,u_2,u_3)^\top$ in the exterior domain Ω^- satisfies the Sommerfeld–Kupradze type radiation conditions at infinity if u is representable in Ω^- as a sum of two metaharmonic vectors, the so called longitudinal $u^{(1)} \equiv u^{(p)}$ and transverse parts $u^{(2)} \equiv u^{(s)}$ (see, e.g., [12]),

$$u = u^{(1)} + u^{(2)} \text{ with } \Delta u^{(m)} + k_m^2\, u^{(m)} = 0, \quad m = 1,2,$$
$$k_1 \equiv k_p = \omega\sqrt{\frac{\varrho}{\lambda + 2\mu}}, \quad k_2 \equiv k_s = \omega\sqrt{\frac{\varrho}{\mu}},$$

satisfying for sufficiently large $r = |x|$ the radiating conditions

$$\frac{\partial u^{(m)}(x)}{\partial r} - i\,k_m\, u^{(m)}(x) = o\,(r^{-1}), \quad m = 1,2.$$

Denote the Sommerfeld–Kupradze class of radiating vector functions by $Z(\Omega^-)$.

Assume that the domains $\overline{\Omega^\pm}$ are occupied by an isotropic homogeneous elastic material.

Denote by $e_{kl} = e_{kl}(u)$ and $\sigma_{kl} = \sigma_{kl}(u)$ the strain and stress tensors respectively associated with the displacement vector u. Then the components of the stress vector $T(\partial,n)u$ acting upon a surface element with normal vector n read as [12]

$$\{T(\partial,n)u\}_k = \sigma_{kl}\,n_l, \ \ \sigma_{kl} = [\lambda\,\delta_{kl}\,\mathrm{div}\,u + 2\,\mu\,e_{kl}], \ \ e_{kl} = 2^{-1}\,(\partial_k u_l + \partial_l u_k).$$

Here $T(\partial,n)$ is the boundary stress operator,

$$T(\partial,n):=[T_{kl}(\partial,n)]_{3\times 3},\quad T_{kl}(\partial,n)=\lambda\, n_k\,\partial_{x_l}+\mu\, n_l\,\partial_{x_k}+\mu\,\delta_{kl}\,\partial_n, \tag{2.2}$$

where $\partial_n=\partial/\partial n$ stands for the normal derivative.

Now we formulate the basic exterior BVPs of steady state elastic oscillations.

The Dirichlet problem $(\mathrm{D})^-_\omega$: Find a regular complex-valued solution vector $u\in[C^1(\overline{\Omega^-})]^3\cap[C^2(\Omega^-)]^3\cap Z(\Omega^-)$ to the steady state oscillation equation (2.1) in Ω^- satisfying the Dirichlet type boundary condition

$$\{u(x)\}^-=f(x),\quad x\in S, \tag{2.3}$$

where $f\in[C^1(S)]^3$ is a given smooth vector function on S.

The Neumann problem $(\mathrm{N})^-_\omega$: Find a regular complex-valued solution vector $u\in[C^1(\overline{\Omega^-})]^3\cap[C^2(\Omega^-)]^3\cap Z(\Omega^-)$ to the steady state oscillation equation (2.1) in Ω^- satisfying the Neumann type boundary condition

$$\{T(\partial,n)u(x)\}^-=F(x),\quad x\in S, \tag{2.4}$$

where $F\in[C(S)]^3$ is a given vector function on S.

Mixed type problem $(\mathrm{M})^-_\omega$: Find a semi-regular complex-valued solution vector $u\in[\mathbf{C}(\widetilde{\Omega}^-_\ell;\delta)]^3\cap Z(\Omega^-)$ to the steady state oscillation equation (2.1) in Ω^- satisfying the mixed type boundary conditions:

$$\{u(x)\}^-=f^*(x),\quad x\in S_D, \tag{2.5}$$

$$\big\{T(\partial,n)u(x)\big\}^-=F^*(x),\quad x\in S_N, \tag{2.6}$$

where $f^*\in[C^1(S_D)]^3$ and $F^*\in[C(S_N)]^3$ are given vector functions.

Crack type problem $(\mathrm{C})^-_\omega$: Find a semi-regular complex-valued solution vector $u\in[\mathbf{C}(\widetilde{\Omega}^-_\Sigma;\delta)]^3\cap Z(\Omega^-_\Sigma)$ to the steady state oscillation equation (2.1) in Ω^-_Σ satisfying either the Dirichlet or Neumann type boundary condition on S and the following crack type conditions on Σ:

$$\big\{T(\partial,n)u(x)\big\}^+=F^{(+)}(x),\quad x\in\Sigma, \tag{2.7}$$

$$\big\{T(\partial,n)u(x)\big\}^-=F^{(-)}(x),\quad x\in\Sigma, \tag{2.8}$$

where $F^{(\pm)}\in[C(\Sigma)]^3$ are given vector functions.

Note that, if the mixed type boundary conditions are prescribed on the boundary surface S, then in addition we have to require that a solution is semi-regular in a neighbourhood of the collision curve ℓ.

Basic crack type problem $(\mathrm{BC})^-_\omega$: Find a semi-regular complex-valued solution vector $u\in[\mathbf{C}(\mathbb{R}^3_\Sigma;\delta)]^3\cap Z(\mathbb{R}^3_\Sigma)$ to the steady state oscillation equation (2.1) in $\mathbb{R}^3_\Sigma:=\mathbb{R}^3\setminus\overline{\Sigma}$ satisfying the crack type conditions on Σ:

$$\big\{T(\partial,n)u(x)\big\}^+=F^{(+)}(x),\quad x\in\Sigma, \tag{2.9}$$

$$\big\{T(\partial,n)u(x)\big\}^-=F^{(-)}(x),\quad x\in\Sigma, \tag{2.10}$$

where $F^{(\pm)}\in[C(\Sigma)]^3$ are given vector functions.

Now let us assume that the domains $\Omega^{(1)}=\Omega^+$ and $\Omega^{(2)}=\Omega^-$ are occupied by isotropic elastic materials with Lamé constants $\lambda^{(\kappa)}$, $\mu^{(\kappa)}$, and the density $\varrho^{(\kappa)}$, $\kappa=1,2$. In this case S is the interface of the composite elastic solid where various type transmission conditions are to be prescribed.

Basic transmission problem $(\mathrm{BT})_\omega$: Find regular complex-valued solution vectors $u^{(1)}\in[C^1(\overline{\Omega^+})]^3\cap[C^2(\Omega^+)]^3$ and $u^{(2)}\in[C^1(\overline{\Omega^-})]^3\cap[C^2(\Omega^-)]^3\cap Z(\Omega^-)$ to the steady state oscillation equations

$$\begin{aligned}A^{(\kappa)}(\partial,\omega)u^{(\kappa)}(x)&\equiv\mu^{(\kappa)}\,\Delta u(x)+(\lambda^{(\kappa)}+\mu^{(\kappa)})\,\mathrm{grad\,div}\,u(x)+\varrho^{(\kappa)}\,\omega^2\,u^{(\kappa)}(x)\\&=0,\quad x\in\Omega^{(\kappa)},\quad \kappa=1,2,\end{aligned} \tag{2.11}$$

satisfying the rigid transmission conditions

$$\{u^{(1)}(x)\}^+-\{u^{(2)}(x)\}^-=f(x),\quad x\in S, \tag{2.12}$$

$$\{T^{(1)}(\partial,n)u^{(1)}(x)\}^+-\{T^{(2)}(\partial,n)u^{(2)}(x)\}^-=F(x),\quad x\in S, \tag{2.13}$$

where $f\in[C^1(S)]^3$ and $F\in[C(S)]^3$ are given vector functions on S and

$$T^{(\kappa)}(\partial,n):=[T^{(\kappa)}_{kl}(\partial,n)]_{3\times 3},\;\; T^{(\kappa)}_{kl}(\partial,n)=\lambda^{(\kappa)}n_k\partial_{x_l}+\mu^{(\kappa)}n_l\,\partial_{x_k}+\mu^{(\kappa)}\delta_{kl}\partial_n. \tag{2.14}$$

If the interface S contains a crack along a subsurface $S_C \subset S$, then we have the following dissection $S = \overline{S_C} \cup \overline{S_T}$, where $S_T = S \setminus \overline{S_C}$ is the rigid transmission part of the interface, and $S_C \cap S_T = \varnothing$.

Basic mixed transmission problem (MT)$_\omega$: Find semi-regular complex-valued solution vectors $u^{(1)} \in [\mathbf{C}(\widetilde{\Omega}^+_\ell;\delta)]^3$ and $u^{(2)} \in [\mathbf{C}(\widetilde{\Omega}^-_\ell;\delta)]^3 \cap Z(\Omega^-)$, $\ell = \overline{S_C} \cap \overline{S_T}$, to the steady state oscillation equations (2.11) satisfying the rigid transmission conditions on S_T,

$$\{u^{(1)}(x)\}^+ - \{u^{(2)}(x)\}^- = f^{(T)}(x), \quad x \in S_T, \tag{2.15}$$

$$\{T^{(1)}(\partial,n)u^{(1)}(x)\}^+ - \{T^{(2)}(\partial,n)u^{(2)}(x)\}^- = F^{(T)}(x), \quad x \in S_T, \tag{2.16}$$

and the crack conditions on S_C,

$$\{T^{(1)}(\partial,n)u^{(1)}(x)\}^+ = F_C^{(+)}(x), \quad x \in S_C, \tag{2.17}$$

$$\{T^{(2)}(\partial,n)u^{(2)}(x)\}^- = F_C^{(-)}(x), \quad x \in S_C, \tag{2.18}$$

where $f^{(T)} \in [C^1(S_T)]^3$, $F^{(T)} \in [C(S_T)]^3$, and $F_C^{(\pm)} \in [C(S_C)]^3$ are given vector functions.

Weak setting of the problems. In the case of *weak formulation* of the above boundary value and boundary-transmission problems we look for *weak solution vectors* in the spaces $[W^1_p(\Omega^+)]^3$ and $[W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$, $1 < p < +\infty$, respectively. In this case the differential equations (2.1) and (2.11) are understood in the distributional sense, the Dirichlet type conditions (2.3), (2.5), (2.12), and (2.15) are understood in the usual trace sense, while the Neumann type conditions (2.4), (2.6)–(2.10), (2.13), (2.16)–(2.18) are understood in the generalized functional trace sense, defined with the help of Green's identity (cf. [12,24]):

$$\langle \{Tu\}^\pm\,,\ \{\overline{v}\}^\pm\rangle_S = \pm\int_{\Omega^\pm}[\,E(u,\overline{v}) - \varrho\,\omega^2\,u\cdot v\,]dx, \tag{2.19}$$

where $u \in [W^1_p(\Omega^+)]^3$, $v \in [W^1_{p'}(\Omega^+)]^3$, or $u \in [W^1_{p,loc}(\Omega^-)]^3$, $v \in [W^1_{p',comp}(\Omega^-)]^3$ with $1/p + 1/p' = 1$, $1 < p < +\infty$, the over-bar denotes complex conjugation, the symbol $\langle\cdot,\ \cdot\rangle_S$ denotes bilinear duality brackets between the mutually adjoint spaces $[B^{-\frac{1}{p}}_{p,p}(S)]^3$ and $[B^{\frac{1}{p}}_{p',p'}(S)]^3$,

$$E(u,v) = \frac{3\,\lambda + 2\,\mu}{3}\,\mathrm{div}\,u\,\mathrm{div}\,v + \frac{\mu}{2}\sum_{k\neq l}(\partial_l u_k + \partial_k u_l)\,(\partial_l v_k + \partial_k v_l) + \frac{\mu}{3}\sum_{k,l=1}^{3}(\partial_k u_k - \partial_l u_l)\,(\partial_k v_k - \partial_l v_l)\,.$$

Note that by relations (2.19) the generalized traces $\{Tu\}^\pm \in [B^{-\frac{1}{p}}_{p,p}(S)]^3$ are well defined for weak solutions of the homogeneous differential equation of steady state oscillations, $u \in [W^1_p(\Omega^+)]^3$ and $u \in [W^1_{p,loc}(\Omega^-)]^3$.

In the case of weak setting, the boundary data belong to the natural Besov spaces:

$$\begin{aligned}
&f \in [B^{1-\frac{1}{p}}_{p,p}(S)]^3,\ \ f^* \in [B^{1-\frac{1}{p}}_{p,p}(S_D)]^3,\ \ f^{(T)} \in [B^{1-\frac{1}{p}}_{p,p}(S_T)]^3,\ \ F \in [B^{-\frac{1}{p}}_{p,p}(S)]^3,\\
&F^* \in [B^{-\frac{1}{p}}_{p,p}(S_N)]^3,\ \ F^{(\pm)} \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^3,\ \ F^{(+)} - F^{(-)} \in [\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)]^3,\\
&F^{(T)} \in [B^{-\frac{1}{p}}_{p,p}(S_T)]^3,\ \ F_C^{(\pm)} \in [B^{-\frac{1}{p}}_{p,p}(S_C)]^3.
\end{aligned}$$

With the help of the Rellich–Vekua lemma the following uniqueness theorem can be proved (for details see [12,20,25–28].

Theorem 2.6. *Let the manifolds $S = \partial\Omega^\pm$, S_D, S_N, S_T, S_C, and Σ be Lipschitz. Then the BVPs $(D)^-_\omega$, $(N)^-_\omega$, $(M)^-_\omega$, $(C)^-_\omega$, $(BT)_\omega$, and $(MT)_\omega$ possess at most one weak solution for $p = 2$ and for all values of the frequency parameter ω.*

3. Existence theorems

Here we employ the notation introduced in Appendices A–C and formulate basic existence results for weak solutions and prove representability of solutions by the layer potentials.

We apply a special representation of solutions by the layer potentials and reduce the above formulated BVPs of elastic oscillations to the corresponding uniquely solvable integral (pseudodifferential) equations for arbitrary value of the oscillation parameter ω. Similar approach for the Helmholtz equation has been developed in the Refs. [29–31].

Throughout the paper, $B(R)$ denotes the ball centred at the origin and radius R such that $\Omega^+ \subset B(R)$ and $\varkappa$ is a complex number

$$\varkappa = \varkappa_1 + i\,\varkappa_2 \in \mathbb{C}, \quad \varkappa_1, \varkappa_2 \in \mathbb{R}, \quad \varkappa_2 \neq 0.$$

Theorems 2.6, A.1, B.1 and B.2 directly lead to the following existence results for the exterior Dirichlet and Neumann type problems.

Theorem 3.1. *The Dirichlet problem* $(D)^-_\omega$ *with arbitrary boundary vector function* $f \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^3$ *is uniquely solvable in the space* $[W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$, $p > 1$, *and the solution is representable as a linear combination of the double and single layer potentials*

$$u(x) = W(g)(x) + \varkappa\, V(g)(x), \quad x \in \Omega^-,$$

where the density vector function $g \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^3$ *is defined by the uniquely solvable elliptic pseudodifferential equation*

$$\mathcal{N}\, g \equiv [\,-2^{-1}\, I_3 + \widetilde{\mathcal{K}} + \varkappa\, \mathcal{H}\,]\, g = f \ \ on \ \ S.$$

Moreover, the following estimate holds

$$\|u\|_{[W^1_p(\Omega^- \cap B(R))]^3} \leqslant C_D(R)\, \|f\|_{[B^{1-\frac{1}{p}}_{p,p}(S)]^3} \tag{3.20}$$

where $C_D(R)$ *is a constant independent of* f.

Theorem 3.2. *The Neumann problem* $(N)^-_\omega$ *with arbitrary boundary vector function* $F \in \big[B^{-\frac{1}{p}}_{p,p}(S)\big]^3$ *is uniquely solvable in the space* $[W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$, $p > 1$, *and the solution is representable as a linear combination of the double and single layer potentials*

$$u(x) = W(g)(x) + \varkappa\, V(g)(x), \quad x \in \Omega^-,$$

where the density vector $g \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^3$ *is defined by the uniquely solvable elliptic pseudodifferential equation*

$$\mathcal{M}\, g \equiv \big[\mathcal{L} + \varkappa\,(2^{-1}\, I_3 + \mathcal{K}\,)\big]\, g = F \ \ on \ \ S.$$

Moreover, the following estimate holds

$$\|u\|_{[W^1_p(\Omega^- \cap B(R))]^3} \leqslant C_N(R)\, \|F\|_{[B^{-\frac{1}{p}}_{p,p}(S)]^3}$$

where $C_N(R)$ *is a constant independent of* F.

For the mixed problem we have the following assertion.

Theorem 3.3. *Let* $4/3 < p < 4$. *The mixed problem* $(M)^-_\omega$ *with arbitrary boundary data*

$$f^* \in \big[B^{1-\frac{1}{p}}_{p,p}(S_D)\big]^3, \qquad F^* \in \big[B^{-\frac{1}{p}}_{p,p}(S_N)\big]^3,$$

is uniquely solvable in the space $[W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$, *and the solution is representable as a linear combination of the double and single layer potentials*

$$u(x) = W\big(\mathcal{N}^{-1}(f_e + \widetilde{g})\big)(x) + \varkappa\, V\big(\mathcal{N}^{-1}(f_e + \widetilde{g})\big)(x), \quad x \in \Omega^-, \tag{3.21}$$

where $f_e \in \big[B^{1-\frac{1}{p}}_{p,p}(S)\big]^3$ *is some fixed extension of the vector function* f^* *from* S_D *onto the whole of* S, *while* $\widetilde{g} \in \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)\big]^3$ *is defined by the uniquely solvable elliptic pseudodifferential equation*

$$r_{S_N} \mathcal{M}\, \mathcal{N}^{-1} g = F_0 \ \ on \ \ S_N,$$

where

$$F_0 = F^* - r_{S_N} \mathcal{M}\, \mathcal{N}^{-1} f_e \in \big[B^{-\frac{1}{p}}_{p,p}(S_N)\big]^3, \quad \|f_e\|_{[B^{1-\frac{1}{p}}_{p,p}(S)]^3} \leqslant 2\|f^*\|_{[B^{1-\frac{1}{p}}_{p,p}(S_D)]^3}.$$

Moreover, the following estimate holds

$$\|u\|_{[W^1_p(\Omega^-\cap B(R))]^3} \leqslant C_M(R)\Big[\|f^*\|_{[B^{1-\frac{1}{p}}_{p,p}(S_D)]^3} + \|F^*\|_{[B^{-\frac{1}{p}}_{p,p}(S_N)]^3}\Big]$$

where $C_M(R)$ is a constant independent of f^* and F^*.

Proof. Invertibility of the operator

$$r_{S_N}\mathcal{M}\mathcal{N}^{-1} \, : \, [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N)]^3 \to [B^{-\frac{1}{p}}_{p,p}(S_N)]^3 \tag{3.22}$$

for $4/3 < p < 4$ follows from Theorems B.2 and C.2, since the principal homogeneous symbol matrix of the operator $-\mathcal{M}\mathcal{N}^{-1}$,

$$\mathfrak{S}\big(-\mathcal{M}\mathcal{N}^{-1}; x, \xi\big) = -\mathbb{L}(x,\xi)\,[\widetilde{\mathbb{K}}_-(x,\xi)]^{-1}, \quad x \in S, \quad \xi = (\xi_1, \xi_2) \neq 0,$$

is positive definite due to (A.19) in Remark A.4 and the null-space of the operator (3.22) is trivial.

Existence and uniqueness of a solution to the mixed problem and estimate (3.22) follow then from (3.21) and Theorems 2.6, B.1 and A.1. □

For the weak solution of the basic crack type problem the following existence result holds.

Theorem 3.4. *Let $4/3 < p < 4$ and $F^{(\pm)} \in \big[B^{-\frac{1}{p}}_{p,p}(\Sigma)\big]^3$ with $F^{(+)} - F^{(-)} \in \big[\widetilde{B}^{-\frac{1}{p}}_{p,p}(\Sigma)\big]^3$. Then the basic crack type problem $(BC)_\omega$ is uniquely solvable in the space $[W^1_{p,loc}(\mathbb{R}^3_\Sigma)]^3 \cap Z(\mathbb{R}^3_\Sigma)$ and the solution is representable as a linear combination of the single and double layer potentials*

$$u(x) = W\big(g\big)(x) - V\big(F^{(+)} - F^{(-)}\big)(x), \quad x \in \mathbb{R}^3_\Sigma, \tag{3.23}$$

where $g \in \big[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)\big]^3$ is defined by the uniquely solvable elliptic pseudodifferential equation

$$r_{S_N}\mathcal{L}\, g = F_0 \;\; on \;\; \Sigma, \tag{3.24}$$

where

$$F_0 = \frac{1}{2}\big(F^{(+)} + F^{(-)}\big) + r_\Sigma \mathcal{K}\left(F^{(+)} - F^{(-)}\right) \in [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^3.$$

Moreover, the following estimate holds

$$\|u\|_{[W^1_p(\mathbb{R}^3_\Sigma\cap B(R))]^3} \leqslant C_M(R)\,\big[\|F^{(+)} + F^{(-)}\|_{[B^{-\frac{1}{p}}_{p,p}(\Sigma)]^3} + \|F^{(+)} - F^{(-)}\|_{[B^{-\frac{1}{p}}_{p,p}(\Sigma)]^3}\big] \tag{3.25}$$

where $C_M(R)$ is a constant independent of $F^{(\pm)}$.

Proof. Let us rewrite the boundary conditions (2.9)–(2.10) of the crack problem $(BC)_\omega$ in the following equivalent form

$$\big\{T(\partial, n)u\big\}^+ - \big\{T(\partial, n)u\big\}^- = F^{(+)} - F^{(-)} \;\; \text{on} \;\; \Sigma, \tag{3.26}$$

$$\big\{T(\partial, n)u\big\}^+ + \big\{T(\partial, n)u\big\}^- = F^{(+)} + F^{(-)} \;\; \text{on} \;\; \Sigma. \tag{3.27}$$

The vector function (3.23) satisfies condition (3.26) automatically, while condition (3.27) leads to Eq. (3.24). Existence and uniqueness of a solution to the basic crack type problem and estimate (3.25) follow then from (3.23) and Theorems 2.6, A.1, A.3, B.2 and C.2. Indeed, the principal homogeneous symbol matrix $\mathbb{L}(x,\xi) := \mathfrak{S}(\mathcal{L}; x, \xi)$ of the operator $\mathcal{L}$ is positive definite (see Remark A.4) and the null-space of the operator

$$r_{S_N}\mathcal{L} \, : \, [\widetilde{B}^{1-\frac{1}{p}}_{p,p}(\Sigma)]^3 \to [B^{-\frac{1}{p}}_{p,p}(\Sigma)]^3 \tag{3.28}$$

is trivial implying the invertibility of the operator (3.28). Thus Eq. (3.24) is uniquely solvable and the estimate (3.25) holds. □

In the case of transmission problems, we use the same notation for potentials and the corresponding integral operators as above but equipped with superscript $^{(\kappa)}$ which indicates that the layer potentials $V^{(\kappa)}$, $W^{(\kappa)}$ and the corresponding integral operators $\mathcal{H}^{(\kappa)}, \mathcal{K}^{(\kappa)}, \widetilde{\mathcal{K}}^{(\kappa)}, \mathcal{L}^{(\kappa)}, \mathcal{N}^{(\kappa)}$, and $\mathcal{M}^{(\kappa)}$ are constructed with the help of the fundamental solution $\Gamma^{(\kappa)}(x-y,\omega)$ associated with the operator $A^{(\kappa)}(\partial,\omega)$ and the stress operator is defined by (2.14).

Theorem 3.5. *The basic transmission problem (BT)$_\omega$ with arbitrary boundary vector functions $f \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3$ and $F \in \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^3$ is uniquely solvable in the class of vector functions $[W_p^1(\Omega^+)]^3 \times \big([W_{p,loc}^1(\Omega^-)]^3 \cap Z(\Omega^-)\big)$, $p>1$, and the solution pair $\big(u^{(1)}, u^{(2)}\big)$ is representable by the layer potentials:*

$$u^{(1)}(x) = V^{(1)}(h)(x), \quad x \in \Omega^+ = \Omega^{(1)}, \tag{3.29}$$

$$u^{(2)}(x) = W^{(2)}(g)(x) + \varkappa\, V^{(2)}(g)(x), \quad x \in \Omega^- = \Omega^{(2)}, \tag{3.30}$$

where the density vectors $h \in \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^3$ and $g \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3$ are defined by the uniquely solvable elliptic system of pseudodifferential equations

$$\mathcal{H}^{(1)}\, h - \mathcal{N}^{(2)}\, g = f \quad on \quad S, \tag{3.31}$$

$$[\,-2^{-1}\, I_3 + \mathcal{K}^{(1)}\,]\, h - \mathcal{M}^{(2)}\, g = F \quad on \quad S. \tag{3.32}$$

Moreover, the following estimates hold

$$\|u^{(1)}\|_{[W_p^1(\Omega^+)]^3} \leqslant C_{BT}^{(1)} \left(\|f\|_{[B_{p,p}^{1-\frac{1}{p}}(S)]^3} + \|F\|_{[B_{p,p}^{-\frac{1}{p}}(S)]^3} \right), \tag{3.33}$$

$$\|u^{(2)}\|_{[W_p^1(\Omega^-\cap B(R))]^3} \leqslant C_{BT}^{(2)}(R) \left(\|f\|_{[B_{p,p}^{1-\frac{1}{p}}(S)]^3} + \|F\|_{[B_{p,p}^{-\frac{1}{p}}(S)]^3} \right), \tag{3.34}$$

where $C_{BT}^{(1)}$ and $C_{BT}^{(2)}(R)$ are constants independent of f and F.

Proof. The representations (3.29)–(3.30) lead to the system of pseudodifferential equations (3.31)–(3.32). Due to the invertibility property of the operators $\mathcal{N}^{(2)}$ and $\mathcal{M}^{(2)}$ (see Appendix B, Theorem B.1), we derive

$$g = \big(\mathcal{N}^{(2)}\big)^{-1}\mathcal{H}^{(1)}h - \big(\mathcal{N}^{(2)}\big)^{-1} f, \tag{3.35}$$

$$\mathcal{T}\, h = \big(\mathcal{N}^{(2)}\big)^{-1} f - \big(\mathcal{M}^{(2)}\big)^{-1} F, \tag{3.36}$$

where $\mathcal{T} = [\mathcal{T}_{kj}]_{3\times 3}$ is the pseudodifferential operator of order -1 defined by the relation

$$\mathcal{T} := \big(\mathcal{N}^{(2)}\big)^{-1}\mathcal{H}^{(1)} - \big(\mathcal{M}^{(2)}\big)^{-1}\big(-2^{-1}\, I_3 + \mathcal{K}^{(1)}\big).$$

Rewrite system (3.35)–(3.36) in matrix form

$$\mathcal{Q}\, \Phi = \Psi, \tag{3.37}$$

where $\Phi = (g,h)^\top$, $\Psi := \Big(-\big(\mathcal{N}^{(2)}\big)^{-1} f, \big(\mathcal{N}^{(2)}\big)^{-1} f - \big(\mathcal{M}^{(2)}\big)^{-1} F\Big)^\top$, and

$$\mathcal{Q} := \begin{bmatrix} I_3 & -\big(\mathcal{N}^{(2)}\big)^{-1}\mathcal{H}^{(1)} \\ [0]_{3\times 3} & \mathcal{T} \end{bmatrix}_{6\times 6}.$$

Note that $\Psi = 0$ if and only if $f = F = 0$. Therefore the homogeneous equation (3.37) corresponds to the homogeneous basic transmission problem (BT)$_\omega$.

The principal homogeneous symbol matrix of the operator $\mathcal{T}$ reads as follows (see Appendix A, Remark A.4)

$$\mathfrak{S}\big(\mathcal{T};x,\xi\big) = \big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\, \mathbb{H}^{(1)} - \big(\mathbb{L}^{(2)}\big)^{-1}\, \mathbb{K}_-^{(1)} = \big(\mathbb{L}^{(2)}\big)^{-1}\Big[\mathbb{K}_-^{(1)}\,\big(\mathbb{H}^{(1)}\big)^{-1} - \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\Big]\,\big(-\mathbb{H}^{(1)}\big), \tag{3.38}$$

where the matrices $\mathbb{K}_-^{(1)}$, $\widetilde{\mathbb{K}}_-^{(2)}$, $\mathbb{H}^{(1)}$, and $\mathbb{L}^{(2)}$ are defined in (A.15). From the results stated in Remark A.4 it follows that the matrices $\big(\mathbb{L}^{(2)}\big)^{-1}$, $\mathbb{K}_-^{(1)}\,\big(\mathbb{H}^{(1)}\big)^{-1} - \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}$, and $\big(-\mathbb{H}^{(1)}\big)$ are positive definite nonsingular matrices for all $x \in S$ and $\xi \in \mathbb{R}^2 \setminus \{0\}$. Consequently $\mathfrak{S}\big(\mathcal{T};x,\xi\big)$ is an elliptic symbol. Moreover, $\mathcal{T}$ is a composition of three

Fredholm operators with zero index and due to Atkinson's theorems (see, e.g., [32, Ch. 1, Theorem 3.3]) the index of the Fredholm operator

$$\mathcal{T} : \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^3 \to \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3$$

equals to zero. Therefore the operator

$$\mathcal{Q} : \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3 \times \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^3 \to \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3 \times \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3 \tag{3.39}$$

is Fredholm with zero index as well.

From the uniqueness theorem for the basic transmission problem $(\mathrm{BT})_\omega$ for $p = 2$ and the general theory of pseudodifferential equations on smooth manifolds without boundary it follows that the null space of the operator (3.39) is trivial for $1 < p < \infty$. Thus the operator (3.39) is invertible and the system (3.31) is uniquely solvable implying the uniqueness and existence of a solution to the problem $(\mathrm{BT})_\omega$ for $1 < p < \infty$.

The estimates (3.33)–(3.34) follow then from Theorem A.1. □

Remark 3.6. From the arguments in the proof of Theorem 3.5 and relations (3.35) and (3.36) it follows that

$$h = \mathcal{T}^{-1}\big(\mathcal{N}^{(2)}\big)^{-1} f - \mathcal{T}^{-1}\big(\mathcal{M}^{(2)}\big)^{-1} F, \tag{3.40}$$

$$g = \big(\mathcal{N}^{(2)}\big)^{-1}\big[\mathcal{H}^{(1)}\mathcal{T}^{-1}\big(\mathcal{N}^{(2)}\big)^{-1} - I\big] f - \big(\mathcal{N}^{(2)}\big)^{-1}\mathcal{H}^{(1)}\mathcal{T}^{-1}\big(\mathcal{M}^{(2)}\big)^{-1} F. \tag{3.41}$$

Now we analyse the basic mixed transmission problem $(\mathrm{MT})_\omega$. To this end, rewrite the mixed transmission conditions (2.15)–(2.18) in the formulation of problem $(\mathrm{MT})_\omega$ in the following equivalent form

$$\{u^{(1)}\}^+ - \{u^{(2)}\}^- = f^{(T)} \quad \text{on} \quad S_T, \tag{3.42}$$

$$\{T^{(1)}(\partial, n)u^{(1)}\}^+ - \{T^{(2)}(\partial, n)u^{(2)}\}^- = F_0 \quad \text{on} \quad S, \tag{3.43}$$

$$\{T^{(1)}(\partial, n)u^{(1)}\}^+ + \{T^{(2)}(\partial, n)u^{(2)}\}^- = F_C^{(+)} + F_C^{(-)} \quad \text{on} \quad S_C, \tag{3.44}$$

where

$$F_0 = \begin{cases} F^{(T)} & \text{on} \quad S_T\,, \\ F_C^{(+)} - F_C^{(-)} & \text{on} \quad S_C\,, \end{cases} \tag{3.45}$$

and we assume that the following necessary compatibility condition is fulfilled

$$F_0 \in \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^3. \tag{3.46}$$

Denote by f^* some fixed extension of the vector function $f^{(T)}$ from S_T onto the whole of S preserving the space, $f^* \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3$. Evidently, an arbitrary extension has then the form $f = f^* + \widetilde{g}$, where $\widetilde{g} \in \big[\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_C)\big]^3$.

Motivated by the existence result for the basic transmission problem described in Theorem 3.5, let us look for a solution to the basic mixed transmission problem (3.42)–(3.44) again in the form (3.29)–(3.30),

$$u^{(1)}(x) = V^{(1)}(h)(x), \quad x \in \Omega^+ = \Omega^{(1)}, \tag{3.47}$$

$$u^{(2)}(x) = W^{(2)}(g)(x) + \varkappa\, V^{(2)}(g)(x), \quad x \in \Omega^- = \Omega^{(2)}, \tag{3.48}$$

where (see (3.40)–(3.41))

$$h = \mathcal{T}^{-1}\big(\mathcal{N}^{(2)}\big)^{-1}(f^* + \widetilde{g}) - \mathcal{T}^{-1}\big(\mathcal{M}^{(2)}\big)^{-1} F_0, \tag{3.49}$$

$$g = \big(\mathcal{N}^{(2)}\big)^{-1}\big[\mathcal{H}^{(1)}\mathcal{T}^{-1}\big(\mathcal{N}^{(2)}\big)^{-1} - I\big](f^* + \widetilde{g}) - \big(\mathcal{N}^{(2)}\big)^{-1}\mathcal{H}^{(1)}\mathcal{T}^{-1}\big(\mathcal{M}^{(2)}\big)^{-1} F_0, \tag{3.50}$$

F_0 is defined in (3.45), f^* is the above introduced fixed extension, and $\widetilde{g}$ is an unknown vector function.

Due to Theorem 3.5, we find that

$$\{u^{(1)}\}^+ - \{u^{(2)}\}^- = f^* + \widetilde{g} \quad \text{on} \quad S,$$

$$\{T^{(1)}(\partial, n)u^{(1)}\}^+ - \{T^{(2)}(\partial, n)u^{(2)}\}^- = F_0 \quad \text{on} \quad S,$$

implying that the transmission conditions (3.42)–(3.43) are satisfied. The remaining condition (3.44) leads to the following pseudodifferential equation for the unknown vector function $\widetilde{g}$ on S_C,

$$\big(-2^{-1}I_3 + \mathcal{K}^{(1)}\big)h + \mathcal{M}^{(2)} g = F_C^{(+)} + F_C^{(-)} \quad \text{on} \quad S_C,$$

which can be rewritten as

$$r_{S_C}\,\mathcal{P}\,\widetilde{g} = \Psi \quad \text{on} \quad S_C, \tag{3.51}$$

where

$$\mathcal{P} := \big(-2^{-1}I_3 + \mathcal{K}^{(1)}\big)\mathcal{T}^{-1}\big(\mathcal{N}^{(2)}\big)^{-1} + \mathcal{M}^{(2)}\big(\mathcal{N}^{(2)}\big)^{-1}\big[\mathcal{H}^{(1)}\mathcal{T}^{-1}\big(\mathcal{N}^{(2)}\big)^{-1} - I_3\big], \tag{3.52}$$

$$\begin{aligned}\Psi := {} & F_C^{(+)} + F_C^{(-)} - r_{S_C}\big\{\big(-2^{-1}I_3 + \mathcal{K}^{(1)}\big)\mathcal{T}^{-1}\big[\big(\mathcal{N}^{(2)}\big)^{-1} f^* - \big(\mathcal{M}^{(2)}\big)^{-1} F_0\big]\big\} \\ & - r_{S_C}\big\{\mathcal{M}^{(2)}\big(\mathcal{N}^{(2)}\big)^{-1}\big([\mathcal{H}^{(1)}\mathcal{T}^{-1}\big(\mathcal{N}^{(2)}\big)^{-1} - I_3]f^* - \mathcal{H}^{(1)}\mathcal{T}^{-1}\big(\mathcal{M}^{(2)}\big)^{-1}F_0\big)\big\}.\end{aligned} \tag{3.53}$$

Due to mapping properties of the operators involved in (3.52) and (3.53) we have (see Appendices A and B)

$$\Psi \in [B_{p,p}^{-\frac{1}{p}}(S_C)]^3 \tag{3.54}$$

and the operator

$$r_{S_C}\mathcal{P} \;:\; [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_C)]^3 \longrightarrow [B_{p,p}^{-\frac{1}{p}}(S_C)]^3 \tag{3.55}$$

is continuous.

In view of the relations derived in Appendix A, Remark A.4, and the equality (3.38), for the principal homogeneous symbol matrix $\mathfrak{S}\big(\mathcal{P}; x, \xi\big)$ of the operator $\mathcal{P}$ we have:

$$\begin{aligned}\mathfrak{S}\big(\mathcal{P}\big) &= \mathbb{K}_-^{(1)}\big(-\mathbb{H}^{(1)}\big)^{-1}\big[\mathbb{K}_-^{(1)}\big(\mathbb{H}^{(1)}\big)^{-1} - \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big]^{-1}\mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1} \\ &\quad + \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big\{-\big[\mathbb{K}_-^{(1)}\big(\mathbb{H}^{(1)}\big)^{-1} - \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big]^{-1}\mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1} - I_3\big\} \\ &= -\big[\mathbb{K}_-^{(1)}\big(\mathbb{H}^{(1)}\big)^{-1} + \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big]\big[\mathbb{K}_-^{(1)}\big(\mathbb{H}^{(1)}\big)^{-1} - \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big]^{-1}\mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1} - \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1} \\ &= \big\{\big[\mathbb{K}_-^{(1)}\big(\mathbb{H}^{(1)}\big)^{-1} + \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big]\big[\mathbb{K}_-^{(1)}\big(\mathbb{H}^{(1)}\big)^{-1} - \mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big]^{-1} + I_3\big\}\big[-\mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}\big].\end{aligned}$$

Note that, due to the relations presented in Remark A.4, the matrices $\mathbf{A} := \mathbb{K}_-^{(1)}\big(\mathbb{H}^{(1)}\big)^{-1}$ and $\mathbf{B} := -\mathbb{L}^{(2)}\big(\widetilde{\mathbb{K}}_-^{(2)}\big)^{-1}$ are positive definite and consequently they are self-adjoint. The symbol $\mathfrak{S}\big(\mathcal{P}\big)$ can be rewritten as

$$\begin{aligned}\mathfrak{S}\big(\mathcal{P}\big) &= \big\{\big(\mathbf{A} - \mathbf{B}\big)\big(\mathbf{A} + \mathbf{B}\big)^{-1} + I_3\big\}\,\mathbf{B} = \big\{\big(\mathbf{A} - \mathbf{B}\big)\big(\mathbf{A} + \mathbf{B}\big)^{-1} + \big(\mathbf{A} + \mathbf{B}\big)\big(\mathbf{A} + \mathbf{B}\big)^{-1}\big\}\,\mathbf{B} \\ &= 2\,\mathbf{A}\,\big(\mathbf{A} + \mathbf{B}\big)^{-1}\mathbf{B} = 2\,\big(\mathbf{A}^{-1} + \mathbf{B}^{-1}\big)^{-1},\end{aligned}$$

implying that the symbol $\mathfrak{S}\big(\mathcal{P}\big)$ is positive definite. Therefore by Remark C.1 and Theorem C.2, operator (3.55) is invertible if (see (C.1) with $s = 1 - \frac{1}{p}$, $\nu = 1$, and $\delta_j = 0$, $j = 1, 2, 3$)

$$\frac{3}{4} < p < 4.$$

The above results lead to the following existence theorem.

Theorem 3.7. *Let $\frac{3}{4} < p < 4$,*

$$f^{(T)} \in [B_{p,p}^{1-\frac{1}{p}}(S_T)]^3, \quad F^{(T)} \in [B_{p,p}^{-\frac{1}{p}}(S_T)]^3, \quad F_C^{(\pm)} \in [B_{p,p}^{-\frac{1}{p}}(S_C)]^3\,,$$

and let the vector function F_0 defined in (3.45) satisfy the inclusion (3.46).

Then the basic mixed transmission problem $(MT)_\omega$ is uniquely solvable in the class of vector functions $[W_p^1(\Omega^+)]^3 \times \big([W_{p,loc}^1(\Omega^-)]^3 \cap Z(\Omega^-)\big)$, and the solution pair $\big(u^{(1)}, u^{(2)}\big)$ is representable by the layer potentials (3.47)–(3.48) with densities given by (3.49)–(3.50), where the unknown vector function $\widetilde{g}$ is defined by the uniquely solvable pseudodifferential equation (3.51).

Moreover, the following estimates hold

$$\|u^{(1)}\|_{[W_p^1(\Omega^+)]^3} \leqslant C_{MT}^{(1)}\Big(\|f^{(T)}\|_{[B_{p,p}^{1-\frac{1}{p}}(S)]^3} + \|F_0\|_{[B_{p,p}^{-\frac{1}{p}}(S)]^3} + \|F_C^{(+)} + F_C^{(-)}\|_{[B_{p,p}^{-\frac{1}{p}}(S_C)]^3}\Big),$$

$$\|u^{(2)}\|_{[W_p^1(\Omega^- \cap B(R))]^3} \leqslant C_{MT}^{(2)}(R)\Big(\|f^{(T)}\|_{[B_{p,p}^{1-\frac{1}{p}}(S)]^3} + \|F_0\|_{[B_{p,p}^{-\frac{1}{p}}(S)]^3} + \|F_C^{(+)} + F_C^{(-)}\|_{[B_{p,p}^{-\frac{1}{p}}(S_C)]^3}\Big),$$

where $C_{MT}^{(1)}$ and $C_{MT}^{(2)}(R)$ are constants independent of $f^{(T)}$, F_0, and $F_C^{(+)} + F_C^{(-)}$.

Remark 3.8. The above formulated existence theorems with $p = 2$ remain valid also for Lipschitz domains, i.e., when the surfaces S, S_D, S_N, S_T, S_C, Σ, and their boundaries belong to Lipschitz continuous classes.

Remark 3.9. Applying the same arguments as in [23] for mixed and crack type problems, it can be shown that for sufficiently smooth data weak solutions to the Problems $(\mathrm{D})^-_\omega$, $(\mathrm{N})^-_\omega$, $(\mathrm{BT})^-_\omega$ actually are regular vector functions (see also [12]), while the weak solutions to the Problems $(\mathrm{M})^-_\omega$, $(\mathrm{BC})^-_\omega$, $(\mathrm{C})^-_\omega$, and $(\mathrm{MT})^-_\omega$ actually are semi-regular vector functions in the corresponding domains (cf. [25,27,33–36]). Therefore all the boundary, transmission, and crack type conditions can be understood in the classical pointwise sense.

Remark 3.10. Note that the crack problem can be considered as a particular case of the mixed transmission problem. Indeed, if we assume that in the formulation of the problem $(\mathrm{MT})_\omega$ both domains Ω^+ and Ω^- are occupied by the same type materials, i.e., all material constants in both domains are the same, and on S_T the homogeneous transmission conditions are prescribed (i.e. $f^{(T)} = 0$ and $F^{(T)} = 0$ on S_T in (2.15) and (2.16)), then the corresponding differential operators are the same in both domains and the transmission part S_T of the interface S becomes a formal interface since the continuity of the displacement and stress vectors across the surface S_T implies that in fact the differential equation is satisfied also at the points of the surface S_T and the corresponding solution actually is an analytic function in $\mathbb{R}^3 \setminus \overline{S_C}$. Evidently we arrive at the basic crack problem with $\Sigma = S_C$.

4. Method of fundamental solutions

Here we develop the Fundamental Solution Method for the above formulated boundary value and transmission problems for the elastic oscillation system for arbitrary values of the frequency parameter ω.

4.1. Auxiliary lemmata

Let Ω_0^+ be an arbitrary simply connected subdomain of Ω^+ such that $\overline{\Omega_0^+} \subset \Omega^+$ and denote $S_0^+ = \partial\Omega_0^+$.

Further, let Ω_0^- be an arbitrary simply connected bounded subdomain of Ω^- such that $\overline{\Omega_0^-} \subset \Omega^-$ and denote $S_0^- = \partial\Omega_0^-$.

We assume that S_0^+ and S_0^- are simply connected surfaces.

Let $\{z^{(k)}\}_{k=1}^\infty$ be an everywhere dense countable set of points in Ω_0^+ and $\{y^{(k)}\}_{k=1}^\infty$ be an everywhere dense countable set of points in Ω_0^-.

Denote by $\Gamma^{(j)}(x,\omega)$ the jth column of Kupradze's fundamental matrix $\Gamma(x,\omega)$ (see (A.1) in Appendix A).

Consider the systems of functions which can be employed for constructing approximate solutions to the Dirichlet problem,

$$\Phi_D^{(-)} := \big\{\varphi^{(l)}(x)\big\}_{l=1}^\infty, \quad x \in \overline{\Omega^-}, \qquad \Phi_D^{(+)} := \big\{\psi^{(l)}(x)\big\}_{l=1}^\infty, \quad x \in \overline{\Omega^+},$$

where

$$\varphi^{(l)}(x) := \begin{cases} \Gamma^{(1)}(x - z^{(k)},\omega) & \text{for} \quad l = 3(k-1)+1, \\ \Gamma^{(2)}(x - z^{(k)},\omega) & \text{for} \quad l = 3(k-1)+2, \quad k = 1,2,3,\dots, \quad z^{(k)} \in \Omega_0^+, \\ \Gamma^{(3)}(x - z^{(k)},\omega) & \text{for} \quad l = 3k, \end{cases} \tag{4.56}$$

$$\psi^{(l)}(x) := \begin{cases} \Gamma^{(1)}(x - y^{(k)},\omega) & \text{for} \quad l = 3(k-1)+1, \\ \Gamma^{(2)}(x - y^{(k)},\omega) & \text{for} \quad l = 3(k-1)+2, \quad k = 1,2,3,\dots, \quad y^{(k)} \in \Omega_0^-. \\ \Gamma^{(3)}(x - y^{(k)},\omega) & \text{for} \quad l = 3k, \end{cases} \tag{4.57}$$

Note that due to definition (4.56) to each point $z^{(k)}$ there corresponds the triplet of vector functions

$$\varphi^{(3(k-1)+1)}, \quad \varphi^{(3(k-1)+2)}, \quad \varphi^{(3k)}, \qquad k = 1,2,\dots. \tag{4.58}$$

Similarly, in view of (4.57), to each point $y^{(k)}$ there corresponds the triplet of vector functions

$$\psi^{(3(k-1)+1)}, \quad \psi^{(3(k-1)+2)}, \quad \psi^{(3k)}, \qquad k = 1,2,\dots. \tag{4.59}$$

Evidently, $\varphi^{(l)}$ are radiating, complex valued analytic vector functions in $\mathbb{R}^3 \setminus \overline{\Omega_0^+}$, while $\psi^{(l)}$ are radiating, complex valued analytic vector functions in $\mathbb{R}^3 \setminus \overline{\Omega_0^-}$. Moreover, $\varphi^{(l)}$ and $\psi^{(l)}$ solve the homogeneous equation (2.1) in the corresponding domains.

Now we prove several lemmas which play a crucial role in our further analysis.

Lemma 4.1. *The system $\Phi_D^{(-)}$ is linearly independent on S.*

Proof. We have to prove that any finite subsystem of $\Phi_D^{(-)}$ is linearly independent on S. Let m be an arbitrary natural number and for some complex valued constants C_l the following equality holds

$$u^{(m)}(x) = \sum_{l=1}^{m} C_l\, \varphi^{(l)}(x) = 0, \quad x \in S, \quad m \in \mathbb{N}. \tag{4.60}$$

Denote by $z^{(k)}$, $k = 1, 2, \ldots, m_0$, the points involved in the expression (4.60). Without loss of generality we can assume that for each $z^{(k)}$, $k = 1, 2, \ldots, m_0$, the expression (4.60) contains all three vector functions associated with the point $z^{(k)}$ (see (4.58)). If necessary, we can add the corresponding terms with zero coefficients. Therefore, in what follows we assume that m is multiple of 3, $m = 3m_0$.

Evidently, $u^{(m)}$ is a radiating analytic vector function in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}$ which solves the homogeneous differential equation

$$A(\partial, \omega)u^{(m)}(x) = 0, \quad x \in \mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}. \tag{4.61}$$

Then in view of (4.60) and (4.61), we see that $u^{(m)}$ solves the homogeneous exterior Dirichlet problem $(\mathrm{D})_\omega^-$ and due to the existence and uniqueness Theorem 3.1 we conclude that $u^{(m)} = 0$ in Ω^-. By the analyticity then we get

$$u^{(m)}(x) = 0 \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}. \tag{4.62}$$

Let $B(z^{(j)}, \varepsilon)$ be a ball centred at the point $z^{(j)}$ and radius ε such that $z^{(k)} \notin B(z^{(j)}, \varepsilon)$ for $k \leqslant m_0$ and $k \neq j$. Denote $\Sigma(z^{(j)}, \varepsilon) = \partial B(z^{(j)}, \varepsilon)$, $j = 1, 2, \ldots, m_0$.

On the one hand, in view of (4.62) we have

$$\int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))u^{(m)}(x)\, dS = 0, \quad j = 1, \ldots, m_0. \tag{4.63}$$

On the other hand, there holds the equality (see [23, Appendix D], [12, Ch.5])

$$\lim_{\varepsilon\to 0} \int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))\Gamma(x - z^{(j)})\, dS = I_3,$$

where $\Gamma(x - z^{(j)})$ is Kelvin's matrix and I_3 is the 3×3 unit matrix. Therefore, in view of (A.2) in Appendix A, for $q = 1, 2, 3$, and $j = 1, 2, \ldots, m_0$, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))\Gamma^{(q)}(x - z^{(j)}, \omega)\, dS \\ &\quad = \lim_{\varepsilon\to 0} \int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))\Gamma^{(q)}(x - z^{(j)})\, dS = (\delta_{1q}, \delta_{2q}, \delta_{3q})^\top. \end{aligned} \tag{4.64}$$

Keeping in mind (4.64) and passing to the limit in (4.63) as $\varepsilon \to 0$, we find

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))u^{(m)}(x)\, dS &= C_{3(j-1)+1} \lim_{\varepsilon\to 0} \int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))\Gamma^{(1)}(x - z^{(j)})\, dS \\ &\quad + C_{3(j-1)+2} \lim_{\varepsilon\to 0} \int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))\Gamma^{(2)}(x - z^{(j)})\, dS \\ &\quad + C_{3(j-1)+3} \lim_{\varepsilon\to 0} \int_{\Sigma(z^{(j)},\varepsilon)} T(\partial_x, n(x))\Gamma^{(3)}(x - z^{(j)})\, dS \\ &= (C_{3(j-1)+1}, C_{3(j-1)+2}, C_{3(j-1)+3})^\top = 0, \end{aligned}$$

for $j = 1, 2, \ldots, m_0$, which implies that $C_l = 0$ for $l = 1, 2, \ldots, 3m_0$. This completes the proof. □

Lemma 4.2. *The system $\Phi_D^{(-)}$ is complete in $\big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3$ for $p \in (1, +\infty)$.*

Proof. We have to show that the linear span of the system $\Phi_D^{(-)}$ is dense in $\big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3$. To this end we will apply the following fact from the functional analysis which is a direct consequence of the Hahn–Banach theorem (see, e.g., [37, Ch. 1, Section 5]). Let $\mathbf{B}$ be a Banach space and $\mathbf{B}^*$ be its adjoint space. A subset $\mathbf{X} \subset \mathbf{B}$ is dense in $\mathbf{B}$ if and only if the relation

$$\langle f, x\rangle = 0 \quad \text{for all} \quad x \in \mathbf{X}$$

with $f \in \mathbf{B}^*$ implies that f is the zero functional.

Thus to prove the density of the linear span of the system $\Phi_D^{(-)}$ in $[B_{p,p}^{1-\frac{1}{p}}(S)]^3$ it suffices to show that if a vector function χ belongs to the adjoint space, $\chi \in [B_{p',p'}^{-1+\frac{1}{p}}(S)]^3$ with $1/p + 1/p' = 1$, and

$$\langle \chi, \varphi^{(l)}\rangle_S = 0, \quad l = 1, 2, \dots, \tag{4.65}$$

then $\chi = 0$. As above, here the symbol $\langle\cdot, \cdot\rangle_S$ denotes duality brackets between the mutually adjoint spaces $[B_{p,p}^{1-\frac{1}{p}}(S)]^3$ and $[B_{p',p'}^{-1+\frac{1}{p}}(S)]^3$.

Condition (4.65) can be rewritten as

$$\big\langle \chi, \Gamma^{(j)}(\cdot - z^{(k)}, \omega)\big\rangle_S = 0, \quad j = 1, 2, 3, \quad k = 1, 2, \dots.$$

Due to the density of the set $\{z^{(k)}\}_{k=1}^{\infty}$ in Ω_0^+, we get

$$\big\langle \chi, \Gamma^{(j)}(\cdot - z, \omega)\big\rangle_S = 0, \quad z \in \Omega_0^+, \quad j = 1, 2, 3.$$

This implies that

$$V(\chi)(z) = \big\langle \chi, \Gamma(\cdot - z, \omega)\big\rangle_S = 0, \quad z \in \Omega_0^+, \tag{4.66}$$

where $V(\chi)$ is a single layer potential with the integration surface S and with the density χ (see (A.3)). Since the single layer potential is analytic in $\Omega^{\pm}$ we conclude from (4.66)

$$V(\chi)(z) = 0, \quad z \in \Omega^+. \tag{4.67}$$

Moreover, by Theorem A.1 we have

$$V(\chi) \in \big[W_{p'}^1(\Omega^+)\big]^3, \quad V(\chi) \in \big[W_{p',loc}^1(\Omega^-)\big]^3 \cap Z(\Omega^-), \quad p' > 1.$$

Further, by Theorem A.2, formula (A.11), and relation (4.67) it follows that the single layer potential $V(\chi)$ solves the homogeneous exterior Dirichlet problem in Ω^- and in accordance with Theorem 3.1 vanishes in Ω^-. Therefore due to Theorem A.2

$$\big\{T(\partial, n)V(\chi)\big\}^- - \big\{T(\partial, n)V(\chi)\big\}^+ = \chi = 0 \quad \text{on} \quad S,$$

which completes the proof. □

Now, let us introduce the following systems of functions on S which can be employed for constructing approximate solutions to the Neumann problem,

$$\Phi_N^{(-)} := \big\{ T(\partial, n(x))\varphi^{(l)}(x) \big\}_{l=1}^{\infty}, \quad \Phi_N^{(+)} := \big\{ T(\partial, n(x))\psi^{(l)}(x) \big\}_{l=1}^{\infty}, \quad x \in S,$$

where $T(\partial, n)$ is the boundary stress operator (2.2), $\varphi^{(l)}$ and $\psi^{(l)}$ are defined in (4.56) and (4.57) respectively.

Lemma 4.3. *The system $\Phi_N^{(-)}$ is linearly independent on S.*

Proof. We have to prove that any finite subsequence of the system $\Phi_N^{(-)}$ is linearly independent. Let $m \in \mathbb{N}$ be an arbitrary natural number and

$$\sum_{l=1}^{m} C_l\, T(\partial, n(x))\varphi^{(l)}(x) = 0, \quad x \in S, \tag{4.68}$$

where C_l are complex valued constants.

As in the proof of Lemma 4.1, we denote by $z^{(k)}$, $k = 1, 2, \dots, m_0$, the points involved in the expression (4.68) and without loss of generality we assume that for each $z^{(k)}$, $k = 1, 2, \dots, m_0$, the expression (4.68) contains all three vector functions associated with the point $z^{(k)}$ (see (4.58)) implying that $m = 3m_0$.

Now, let us set

$$u^{(m)}(x) := \sum_{l=1}^{m} C_l \, \varphi^{(l)}(x), \quad x \notin \overline{\Omega_0^+}. \tag{4.69}$$

Evidently, $u^{(m)}$ is a radiating analytic vector function in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}$ which solves the homogeneous differential equation

$$A(\partial, \omega) u^{(m)}(x) = 0, \quad x \in \mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}. \tag{4.70}$$

In view of (4.69), (4.68), and (4.70) it then follows that $u^{(m)}$ solves the homogeneous exterior Neumann problem $(\mathrm{N})_\omega^-$ and due to uniqueness Theorem 2.6 we conclude that $u^{(m)} = 0$ in Ω^- and consequently, by the analyticity property, $u^{(m)} = 0$ in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}$. By the same arguments as in the proof of Lemma 4.1 we show that all the constants C_l, $l = 1, 2, \dots, m$, equal to zero, which completes the proof. □

Lemma 4.4. *The system $\Phi_N^{(-)}$ is complete in $\big[B_{p,p}^{-\frac{1}{p}}(S)\big]^3$ for $p \in (1, +\infty)$.*

Proof. As in Lemma 4.2, to prove the density of the linear span of the system $\Phi_N^{(-)}$ in $[B_{p,p}^{-\frac{1}{p}}(S)]^3$ it suffices to show that if a vector function χ belongs to the adjoint space, $\chi \in [B_{p',p'}^{\frac{1}{p}}(S)]^3$ with $1/p + 1/p' = 1$, and

$$\langle \chi \,,\, T(\partial, n)\varphi^{(l)} \rangle_S = 0, \quad l = 1, 2, \dots, \tag{4.71}$$

then $\chi = 0$. Here the symbol $\langle \cdot, \cdot \rangle_S$ again denotes duality brackets between the mutually adjoint spaces $[B_{p,p}^{-\frac{1}{p}}(S)]^3$ and $[B_{p',p'}^{\frac{1}{p}}(S)]^3$.

Condition (4.71) can be rewritten as

$$\langle \chi \,,\, T(\partial, n)\Gamma^{(j)}(\cdot - z^{(k)}, \omega) \rangle_S = 0, \quad j = 1, 2, 3, \ \ k = 1, 2, \dots$$

which is equivalent to the relation

$$W(\chi)(z^{(k)}) = 0, \quad k = 1, 2, \dots, \tag{4.72}$$

where $W(\chi)$ is the double layer potential with the integration surface S and with the density χ (see (A.4)). Due to the density property of the set $\{z^{(k)}\}_{k=1}^{\infty}$ in Ω_0^+, from (4.72) we deduce

$$W(\chi)(z) = 0, \quad z \in \Omega_0^+. \tag{4.73}$$

By analyticity property of the double layer potential in domains $\Omega^\pm$ we conclude

$$W(\chi)(z) = 0, \quad z \in \Omega^+.$$

Note that by Theorem A.1 we have

$$W(\chi) \in \big[W_{p'}^1(\Omega^+)\big]^3, \quad W(\chi) \in \big[W_{p',loc}^1(\Omega^-)\big]^3 \cap Z(\Omega^-), \quad p' > 1.$$

Further, by Theorem A.2, formula (A.12), and relation (4.73) it follows that the double layer potential $W(\chi)$ solves the homogeneous exterior Neumann problem in Ω^- and in accordance with Theorem 3.2 vanishes in Ω^-. Therefore due to Theorem A.2

$$\big\{W(\chi)\big\}^+ - \big\{W(\chi)\big\}^- = \chi = 0 \quad \text{on} \quad S,$$

which completes the proof. □

Further, let us introduce the system which can be employed for constructing approximate solutions to the mixed Dirichlet–Neumann problem,

$$\Phi_M^{(-)} := \big\{ \nu^{(l)}(x) \big\}_{l=1}^{\infty}, \quad x \in S, \tag{4.74}$$

where

$$\nu^{(l)}(x) := \begin{cases} \varphi^{(l)}(x) & \text{for} \quad x \in S_D\,, \\ T(\partial, n(x))\varphi^{(l)}(x) & \text{for} \quad x \in S_N\,, \end{cases} \tag{4.75}$$

where $\varphi^{(l)}$ is given in (4.56), S_D and S_N are the Dirichlet and Neumann parts in the mixed boundary value problem $(\mathrm{M})^-_\omega$.

It is evident that the vector $\nu^{(l)}$ can be considered as the following pair of restrictions

$$\widetilde{\nu}^{(l)} = \big(r_{S_D}\, \nu^{(l)}, r_{S_N}\, \nu^{(l)}\big) \equiv \big(r_{S_D}\, \varphi^{(l)}, r_{S_N}\, T(\partial, n(x))\varphi^{(l)}\big)\,.$$

Similarly, the system $\Phi^{(-)}_M$ defined in (4.74) can be identified with the system

$$\widetilde{\Phi}^{(-)}_M := \{\widetilde{\nu}^{(l)}\}_{l=1}^{\infty}.$$

Lemma 4.5. *The system $\Phi^{(-)}_M$ is linearly independent on S.*

Proof. Let $m \in \mathbb{N}$ be a natural number and

$$\sum_{l=1}^{m} C_l\, \nu^{(l)}(x) = 0, \quad x \in S, \tag{4.76}$$

where C_l are complex valued constants.

As in the proof of Lemma 4.1, we denote by $z^{(k)}$, $k = 1, 2, \dots, m_0$, the points involved in the expression (4.76) and without loss of generality we assume again that for each $z^{(k)}$, $k = 1, 2, \dots, m_0$, the expression (4.76) contains all three vector functions associated with the point $z^{(k)}$ (see (4.58)) implying that m is multiple of 3, $m = 3m_0$.

Now, let us construct the vector

$$u^{(m)}(x) := \sum_{l=1}^{m} C_l\, \varphi^{(l)}(x), \quad x \notin \overline{\Omega^+_0}. \tag{4.77}$$

Evidently, $u^{(m)}$ is a radiating analytic vector function in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}$ which solves the homogeneous differential equation (4.70) in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}$. In view of (4.77), (4.76), and (4.75) it then follows that $u^{(m)}$ solves the exterior homogeneous mixed problem $(\mathrm{M})^-_\omega$ and due to the existence and uniqueness Theorem 3.3 we conclude that $u^{(m)} = 0$ in Ω^- and consequently, by the analyticity property, $u^{(m)} = 0$ in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{m_0}$. By the same arguments as in the proof of Lemma 4.1 we show that all the constants C_l, $l = 1, 2, \dots, m$, equal to zero, which completes the proof. □

Lemma 4.6. *The system $\widetilde{\Phi}^{(-)}_M$ is complete in $\big[B^{1-\frac{1}{p}}_{p,p}(S_D)\big]^3 \times \big[B^{-\frac{1}{p}}_{p,p}(S_N)\big]^3$ for $4/3 < p < 4$.*

Proof. As in Lemma 4.2, to prove the density of the linear span of the system $\widetilde{\Phi}^{(-)}_M$ in $\big[B^{1-\frac{1}{p}}_{p,p}(S_D)\big]^3 \times \big[B^{-\frac{1}{p}}_{p,p}(S_N)\big]^3$ it suffices to show that if a pair of vector functions $\widetilde{\chi} = (\chi_D, \chi_N)$ belongs to the adjoint space, $\widetilde{\chi} = (\chi_D, \chi_N) \in \big[\widetilde{B}^{-1+\frac{1}{p}}_{p',p'}(S_D)\big]^3 \times \big[\widetilde{B}^{\frac{1}{p}}_{p',p'}(S_N)\big]^3$ with $1/p + 1/p' = 1$, and

$$\begin{aligned} \langle \widetilde{\chi}\,,\, \widetilde{\nu}^{(l)} \rangle_S &:= \langle \chi_D\,,\, \nu^{(l)} \rangle_{S_D} + \langle \chi_N\,,\, \nu^{(l)} \rangle_{S_N} \\ &= \langle \chi_D\,,\, \varphi^{(l)} \rangle_{S_D} + \langle T(\partial, n(x))\varphi^{(l)}\,,\, \chi_N \rangle_{S_N} = 0, \quad l = 1, 2, \dots, \end{aligned} \tag{4.78}$$

then $\widetilde{\chi} = 0$.

Here the symbols $\langle\cdot,\,\cdot\rangle_{S_D}$ and $\langle\cdot,\,\cdot\rangle_{S_N}$ denote duality brackets between the mutually adjoint pairs of Besov spaces $[\widetilde{B}^{-1+\frac{1}{p}}_{p,p}(S_D)]^3$ and $[B^{1-\frac{1}{p}}_{p',p'}(S_D)]^3$, and $[B^{-\frac{1}{p}}_{p,p}(S_N)]^3$ and $[\widetilde{B}^{\frac{1}{p}}_{p',p'}(S_N)]^3$, respectively.

Condition (4.78) can be rewritten as

$$\langle \chi_D\,,\, \Gamma^{(j)}(\cdot - z^{(k)}) \rangle_{S_D} + \langle T(\partial, n)\Gamma^{(j)}(\cdot - z^{(k)})\,,\, \chi_N \rangle_{S_N} = 0, \quad j = 1, 2, 3, \quad k = 1, 2, \dots$$

which is equivalent to the relation

$$V(\chi_D)(z^{(k)}) + W(\chi_N)(z^{(k)}) = 0, \quad k = 1, 2, \dots, \tag{4.79}$$

where $V(\chi_D)$ and $W(\chi_N)$ are the single and double layer potentials on S with the densities

$$\chi_D \in \big[\widetilde{B}^{-1+\frac{1}{p}}_{p',\,p'}(S_D)\big]^3, \qquad \chi_N \in \big[\widetilde{B}^{\frac{1}{p}}_{p',\,p'}(S_N)\big]^3. \tag{4.80}$$

Due to the density property of the set $\{z^{(k)}\}_{k=1}^{\infty}$ in Ω_0^+, from (4.79) we get

$$U(z) := V(\chi_D)(z) + W(\chi_N)(z) = 0, \quad z \in \Omega_0^+.$$

By analyticity property of the layer potentials in domains $\Omega^\pm$ we conclude

$$U(z) = V(\chi_D)(z) + W(\chi_N)(z) = 0, \quad z \in \Omega^+. \tag{4.81}$$

Note that if $4/3 < p < 4$, then $4/3 < p' < 4$, and by Theorem A.1 we have

$$U \in \big[W^1_{p'}(\Omega^+)\big]^3, \quad U \in \big[W^1_{p',\,loc}(\Omega^-)\big]^3 \cap Z(\Omega^-), \;\; 4/3 < p' < 4. \tag{4.82}$$

Further, by Theorem A.2 and relations (4.81) and (4.80) we find that

$$\big\{U\big\}^+ - \big\{U\big\}^- = \chi_N = 0 \text{ on } S_D, \quad \big\{T(\partial,n)U\big\}^+ - \big\{T(\partial,n)U\big\}^- = -\chi_D = 0 \text{ on } S_N.$$

Whence it follows that U belongs to the class (4.82) and solves the homogeneous exterior mixed problem $(\mathrm{M})^-_\omega$. In accordance with the existence and uniqueness Theorem 3.3 U vanishes in Ω^- implying $\chi_D = \chi_N = 0$ on S. □

Next, we introduce the system of vector functions which can be employed for constructing approximate solutions to the basic transmission problem $(\mathrm{BT})_\omega$.

By $\varphi^{(\kappa,l)}(x)$ and $\psi^{(\kappa,l)}(x)$ we denote the vector functions defined by formulas (4.56) and (4.57) respectively constructed by the columns $\Gamma^{(j,\kappa)}$, $\kappa = 1, 2$, $j = 1, 2, 3$, of the fundamental matrix $\Gamma^{(\kappa)}$ associated with the operator $A^{(\kappa)}(\partial, \omega)$. Here $\kappa = 1$ corresponds to the bounded domain $\Omega^+ = \Omega^{(1)}$, while $\kappa = 2$ corresponds to the exterior unbounded domain $\Omega^- = \Omega^{(2)}$. The set of points $\{z^{(k)}\}_{k=1}^{\infty} \subset \overline{\Omega_0^+} \subset \Omega^{(1)}$ and $\{y^{(k)}\}_{k=1}^{\infty} \subset \overline{\Omega_0^-} \subset \Omega^{(2)}$ are the same as above.

Let

$$\Phi_{BT} := \big\{\, \Psi^{(l,1)}(x)\,,\, \Phi^{(l,2)}(x)\big\}_{l=1}^{\infty}, \quad x \in S,$$

where $\Psi^{(l,1)}(x)$ and $\Phi^{(l,2)}(x)$ are six vectors defined on S by the relations

$$\Psi^{(l,1)}(x) = \big(\psi^{(l,1)}(x),\; T^{(1)}(\partial, n(x))\psi^{(l,1)}(x)\big)^\top, \quad x \in S, \tag{4.83}$$

$$\Phi^{(l,2)}(x) = \big(-\varphi^{(l,2)}(x),\; -T^{(2)}(\partial, n(x))\varphi^{(l,2)}(x)\big)^\top, \quad x \in S. \tag{4.84}$$

Lemma 4.7. *The system Φ_{BT} is linearly independent on S.*

Proof. Let $m_1, m_2 \in \mathbb{N}$ be arbitrary natural numbers and

$$\sum_{l=1}^{m_1} C_{l,1}\, \Psi^{(l,1)}(x) + \sum_{l=1}^{m_2} C_{l,2}\, \Phi^{(l,2)}(x) = 0, \quad x \in S, \tag{4.85}$$

where $C_{l,\kappa}$, $\kappa = 1, 2$, are complex valued constants.

Denote by $z^{(k)}$, $k = 1, 2, \ldots, p_2$, and $y^{(k)}$, $k = 1, 2, \ldots, p_1$, the points involved in the expression (4.85) and without loss of generality assume that for each $z^{(k)}$, $k = 1, 2, \ldots, p_2$, and for each $y^{(k)}$, $k = 1, 2, \ldots, p_1$, the expression (4.85) contains all three vector functions corresponding to the points $z^{(k)}$ and $y^{(k)}$ (see (4.58) and (4.59)) implying that m_2 and m_1 are multiples of 3, $m_2 = 3p_2$ and $m_1 = 3p_1$.

Now, let us construct the vectors

$$u^{(m_1,1)}(x) := \sum_{l=1}^{m_1} C_{l,1}\, \psi^{(l,1)}(x), \quad x \notin \overline{\Omega_0^-}, \tag{4.86}$$

$$u^{(m_2,2)}(x) := \sum_{l=1}^{m_2} C_{l,2}\, \varphi^{(l,2)}(x), \quad x \notin \overline{\Omega_0^+}. \tag{4.87}$$

Evidently, $u^{(m_1,1)}$ and $u^{(m_2,2)}$ are radiating analytic vector function in $\mathbb{R}^3 \setminus \{y^{(k)}\}_{k=1}^{p_1}$ and $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{p_2}$ respectively and solve the homogeneous differential equations

$$A^{(1)}(\partial, \omega)u^{(m_1,1)}(x) = 0, \quad x \notin \overline{\Omega_0^-}, \tag{4.88}$$

$$A^{(2)}(\partial, \omega)u^{(m_2,2)}(x) = 0, \quad x \notin \overline{\Omega_0^+}. \tag{4.89}$$

In view of relations (4.83)–(4.89) it then follows that the pair $\big(u^{(m_1,1)}, u^{(m_2,2)}\big)$ solves the homogeneous basic transmission problem $(\mathrm{BT})_\omega$ and due to the existence and uniqueness Theorem 3.5 we conclude that $u^{(m_1,1)} = 0$ in Ω^+ and $u^{(m_2,2)} = 0$ in Ω^-. Consequently, by the analyticity property, $u^{(m_1,1)} = 0$ in $\mathbb{R}^3 \setminus \{y^{(k)}\}_{k=1}^{p_1}$ and $u^{(m_2,2)} = 0$ in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{p_2}$. Now, by the same arguments as in the proof of Lemma 4.1 we derive that $C_{l,1} = 0, l = 1, 2, \ldots, m_1$, and $C_{l,2} = 0, l = 1, 2, \ldots, m_2$, which completes the proof. □

Lemma 4.8. *The system Φ_{BT} is complete in $\big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3 \times \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^3$ for $p > 1$.*

Proof. To prove the density property of the linear span of the system Φ_{BT} in $\big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3 \times \big[B_{p,p}^{-\frac{1}{p}}(S)\big]^3$ it suffices to show that if a pair of vector functions $\chi = (g, h)$ belongs to the adjoint space, $\chi = (g, h) \in \big[B_{p',p'}^{-1+\frac{1}{p}}(S)\big]^3 \times \big[B_{p',p'}^{\frac{1}{p}}(S)\big]^3$ with $1/p + 1/p' = 1$, and

$$\langle \chi\,,\ \Psi^{(l,1)}\rangle_S := \langle g\,,\ \psi^{(l,1)}\rangle_S + \langle h\,,\ T^{(1)}(\partial, n)\psi^{(l,1)}\rangle_S = 0, \quad l = 1, 2, \ldots, \tag{4.90}$$

$$\langle \chi\,,\ \Phi^{(l,2)}\rangle_S := \langle g\,,\ \varphi^{(l,2)}\rangle_S + \langle h\,,\ T^{(2)}(\partial, n)\varphi^{(l,2)}\rangle_S = 0, \quad l = 1, 2, \ldots, \tag{4.91}$$

then $\chi = (g, h) = 0$.

Condition (4.90) and (4.91) can be rewritten as

$$\langle g\,,\ \Gamma^{(j,1)}(\cdot - y^{(k)})\rangle_S + \langle T^{(1)}(\partial, n)\Gamma^{(j,1)}(\cdot - y^{(k)})\,,\ h\rangle_S = 0, \quad j = 1, 2, 3, \quad k = 1, 2, \ldots$$

$$\langle g\,,\ \Gamma^{(j,2)}(\cdot - z^{(k)})\rangle_S + \langle T^{(2)}(\partial, n)\Gamma^{(j,2)}(\cdot - z^{(k)})\,,\ h\rangle_S = 0, \quad j = 1, 2, 3, \quad k = 1, 2, \ldots$$

which is equivalent to the relation

$$V^{(1)}(g)(y^{(k)}) + W^{(1)}(h)(y^{(k)}) = 0, \quad k = 1, 2, \ldots, \quad y^{(k)} \in \Omega_0^-, \tag{4.92}$$

$$V^{(2)}(g)(z^{(k)}) + W^{(2)}(h)(z^{(k)}) = 0, \quad k = 1, 2, \ldots, \quad z^{(k)} \in \Omega_0^+, \tag{4.93}$$

where $V^{(\kappa)}(g)$ and $W^{(\kappa)}(h)$ are the single and double layer potentials with the integration surface S constructed by the fundamental matrix $\Gamma^{(\kappa)}$ with the densities

$$g \in \big[B_{p',p'}^{-1+\frac{1}{p}}(S)\big]^3, \qquad h \in \big[B_{p',p'}^{\frac{1}{p}}(S)\big]^3, \qquad p' > 1.$$

Due to the density property of the sets $\{y^{(k)}\}_{k=1}^{\infty}$ in Ω_0^- and $\{z^{(k)}\}_{k=1}^{\infty}$ in Ω_0^+, from (4.92) and (4.93) we get

$$V^{(1)}(g)(z) + W^{(1)}(h)(z) = 0, \quad z \in \Omega_0^-, \tag{4.94}$$

$$V^{(2)}(g)(z) + W^{(2)}(h)(z) = 0, \quad z \in \Omega_0^+. \tag{4.95}$$

Due to analyticity of the layer potentials in domains $\Omega^{\pm}$ we conclude

$$U^{(1)}(z) := V^{(1)}(g)(z) + W^{(1)}(h)(z) = 0, \quad z \in \Omega^-, \tag{4.96}$$

$$U^{(2)}(z) := -V^{(2)}(g)(z) - W^{(2)}(h)(z) = 0, \quad z \in \Omega^+. \tag{4.97}$$

Note that by Theorem A.1 we have

$$U^{(1)} \in \big[W_{p'}^1(\Omega^+)\big]^3, \quad U^{(2)} \in \big[W_{p',loc}^1(\Omega^-)\big]^3 \cap Z(\Omega^-), \quad p' > 1.$$

Further, by Theorem A.2 and with the help of relations (4.94)–(4.97) we find that

$$\big\{U^{(1)}\big\}^+ - \big\{U^{(2)}\big\}^- = 0 \text{ on } S,$$

$$\big\{T^{(1)}(\partial, n)U^{(1)}\big\}^+ - \big\{T^{(2)}(\partial, n)U^{(2)}\big\}^- = 0 \text{ on } S.$$

Whence it follows that ($U^{(1)}$ and $U^{(2)}$) belong to the appropriate classes of vector functions and solve the homogeneous basic transmission problem $(\mathrm{BT})_\omega^-$. In accordance with the existence and uniqueness Theorem 3.5 then $U^{(1)}$ vanishes in Ω^+ and $U^{(2)}$ vanishes in Ω^- which along with (4.96) and (4.97) imply $g = h = 0$ on S. $\square$

Finally, we introduce the system of vector functions which can be employed for constructing approximate solutions to the mixed transmission problem $(\mathrm{MT})_\omega$ which as a particular case covers the crack type problem.

Let us define the three vectors

$$\Lambda^{(l,1)}(x) := \begin{cases} \psi^{(l,1)}(x) & \text{for} \quad x \in S_T, \\ T^{(1)}(\partial, n(x))\psi^{(l,1)}(x) & \text{for} \quad x \in S_C, \end{cases} \qquad l = 1, 2, \dots \tag{4.98}$$

$$\Lambda^{(l,2)}(x) := \begin{cases} -\varphi^{(l,2)}(x) & \text{for} \quad x \in S_T, \\ T^{(2)}(\partial, n(x))\varphi^{(l,2)}(x) & \text{for} \quad x \in S_C, \end{cases} \qquad l = 1, 2, \dots \tag{4.99}$$

where $\psi^{(l,1)}$ and $\varphi^{(l,2)}$ are the same as in the above introduced system Φ_{BT} for the basic transmission problem.

Further, we define the six vectors

$$\Theta^{(l,1)}(x) = \big(\Lambda^{(l,1)}(x),\ T^{(1)}(\partial, n(x))\psi^{(l,1)}(x)\big)^\top, \qquad x \in S, \tag{4.100}$$

$$\Theta^{(l,2)}(x) = \big(\Lambda^{(l,2)}(x),\ -T^{(2)}(\partial, n(x))\varphi^{(l,2)}(x)\big)^\top, \quad x \in S, \tag{4.101}$$

and set

$$\Phi_{MT} := \big\{\, \Theta^{(l,1)}(x)\ ,\ \Theta^{(l,2)}(x)\big\}_{l=1}^{\infty}, \quad x \in S. \tag{4.102}$$

Lemma 4.9. *The system Φ_{MT} is linearly independent on S.*

Proof. Let $m_1, m_2 \in \mathbb{N}$ be arbitrary natural numbers and let

$$\sum_{l=1}^{m_1} C_{l,1}\, \Theta^{(l,1)}(x) + \sum_{l=1}^{m_2} C_{l,2}\, \Theta^{(l,2)}(x) = 0, \quad x \in S, \tag{4.103}$$

with some complex constants $C_{l,\kappa}$, $\kappa = 1, 2,$.

Denote by $z^{(k)}$, $k = 1, 2, \dots, p_2$, and $y^{(k)}$, $k = 1, 2, \dots, p_1$, the points involved in the expression (4.103) and without loss of generality assume again that for each $z^{(k)}$, $k = 1, 2, \dots, p_2$, and for each $y^{(k)}$, $k = 1, 2, \dots, p_1$, the expression (4.85) contains all three vector functions corresponding to the points $z^{(k)}$ and $y^{(k)}$ implying that m_2 and m_1 are multiples of 3, $m_2 = 3p_2$ and $m_1 = 3p_1$.

Now, let us construct the vectors

$$u^{(m_1,1)}(x) := \sum_{l=1}^{m_1} C_{l,1}\, \psi^{(l,1)}(x), \quad x \notin \overline{\Omega_0^-}, \tag{4.104}$$

$$u^{(m_2,2)}(x) := \sum_{l=1}^{m_2} C_{l,2}\, \varphi^{(l,2)}(x), \quad x \notin \overline{\Omega_0^+}. \tag{4.105}$$

Evidently, $u^{(m_1,1)}$ and $u^{(m_2,2)}$ are radiating analytic vector function in $\mathbb{R}^3 \setminus \{y^{(k)}\}_{k=1}^{p_1}$ and $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{p_2}$ respectively and solve the homogeneous differential equations

$$A^{(1)}(\partial, \omega)u^{(m_1,1)}(x) = 0, \quad x \notin \overline{\Omega_0^-}, \tag{4.106}$$

$$A^{(2)}(\partial, \omega)u^{(m_2,2)}(x) = 0, \quad x \notin \overline{\Omega_0^+}. \tag{4.107}$$

In view of relations (4.98)–(4.107) it then follows that the pair $\big(u^{(m_1,1)}, u^{(m_2,2)}\big)$ solves the homogeneous mixed transmission problem $(\mathrm{MT})_\omega$ with equivalently transformed conditions (3.42)–(3.44) and due to the existence and uniqueness Theorem 3.7 we conclude that $u^{(m_1,1)} = 0$ in Ω^+ and $u^{(m_2,2)} = 0$ in Ω^-. Consequently, by the analyticity property, $u^{(m_1,1)} = 0$ in $\mathbb{R}^3 \setminus \{y^{(k)}\}_{k=1}^{p_1}$ and $u^{(m_2,2)} = 0$ in $\mathbb{R}^3 \setminus \{z^{(k)}\}_{k=1}^{p_2}$. Now, by the same arguments as in the proof of Lemma 4.1 we derive that $C_{l,1} = 0$, $l = 1, 2, \dots, m_1$, and $C_{l,2} = 0$, $l = 1, 2, \dots, m_2$, which completes the proof. $\square$

As in the case of the system $\Phi_M^{(-)}$, here we can identify the system Φ_{MT} with the system $\widetilde{\Phi}_{MT}$ defined as

$$\widetilde{\Phi}_{MT} := \{\widetilde{\Theta}^{(l,1)}, \widetilde{\Theta}^{(l,2)}\}_{l=1}^{\infty},$$

where

$$\widetilde{\Theta}^{(l,1)}(x) = \big(r_{S_T}\,\psi^{(l,1)},\ r_{S_C}\,T^{(1)}(\partial,n(x))\psi^{(l,1)},\ r_S T^{(1)}(\partial,n(x))\psi^{(l,1)}\big)^\top,$$
$$\widetilde{\Theta}^{(l,2)}(x) = \big(r_{S_T}\,\varphi^{(l,2)},\ r_{S_C}\,T^{(2)}(\partial,n(x))\varphi^{(l,2)},\ r_S T^{(2)}(\partial,n(x))\varphi^{(l,2)}\big)^\top.$$

Lemma 4.10. *The system $\widetilde{\Phi}_{MT}$ is complete in the space*

$$\big[B^{1-\frac{1}{p}}_{p,p}(S_T)\big]^3 \times \big[B^{-\frac{1}{p}}_{p,p}(S_C)\big]^3 \times \big[B^{-\frac{1}{p}}_{p,p}(S)\big]^3 \tag{4.108}$$

for $4/3 < p < 4$.

Proof. To prove the density property of the linear span of the system $\widetilde{\Phi}_{MT}$ in the space (4.108) it suffices to show that if a pair of vector functions $\widetilde{\chi} = (f, g, h)$ belongs to the adjoint space,

$$\widetilde{\chi} = (f,g,h) \in \Big[\big[\widetilde{B}^{-1+\frac{1}{p}}_{p',p'}(S_T)\big]^3 \times \big[\widetilde{B}^{\frac{1}{p}}_{p',p'}(S_C)\big]^3\Big] \times \big[B^{\frac{1}{p}}_{p',p'}(S)\big]^3$$

with $1/p + 1/p' = 1$, and for all $l = 1, 2, \dots$,

$$\langle\widetilde{\chi}\,,\ \widetilde{\Theta}^{(l,1)}\rangle_S := \langle f\,,\ \psi^{(l,1)}\rangle_{S_T} + \langle g\,,\ T^{(1)}(\partial,n)\psi^{(l,1)}\rangle_{S_C} + \langle h\,,\ T^{(1)}(\partial,n)\psi^{(l,1)}\rangle_S = 0, \tag{4.109}$$
$$\langle\widetilde{\chi}\,,\ \widetilde{\Theta}^{(l,2)}\rangle_S := -\langle f\,,\ \varphi^{(l,2)}\rangle_{S_T} + \langle g\,,\ T^{(2)}(\partial,n)\varphi^{(l,2)}\rangle_{S_C} - \langle h\,,\ T^{(2)}(\partial,n)\varphi^{(l,2)}\rangle_S = 0, \tag{4.110}$$

then $\widetilde{\chi} = (f, g, h) = 0$.

Condition (4.109) and (4.110) can be rewritten as

$$\langle f\,,\ \Gamma^{(j,1)}(\cdot - y^{(k)})\rangle_{S_T} + \langle g\,,\ T^{(1)}(\partial,n)\Gamma^{(j,1)}(\cdot - y^{(k)})\rangle_{S_C} + \langle T^{(1)}(\partial,n)\Gamma^{(j,1)}(\cdot - y^{(k)})\,,\ h\rangle_S = 0,$$
$$-\langle f\,,\ \Gamma^{(j,2)}(\cdot - z^{(k)})\rangle_{S_T} + \langle g\,,\ T^{(2)}(\partial,n)\Gamma^{(j,2)}(\cdot - z^{(k)})\rangle_{S_C} - \langle T^{(2)}(\partial,n)\Gamma^{(j,2)}(\cdot - z^{(k)})\,,\ h\rangle_S = 0,$$
$$j = 1,2,3, \quad k = 1,2,\dots$$

which is equivalent to the relations

$$V^{(1)}(f)(y^{(k)}) + W^{(1)}(g)(y^{(k)}) + W^{(1)}(h)(y^{(k)}) = 0, \qquad k = 1,2,\dots,\quad y^{(k)} \in \Omega_0^-,$$
$$-V^{(2)}(f)(z^{(k)}) + W^{(2)}(g)(z^{(k)}) - W^{(2)}(h)(z^{(k)}) = 0, \quad k = 1,2,\dots,\quad z^{(k)} \in \Omega_0^+,$$

where $V^{(\kappa)}(f)$, $V^{(\kappa)}(g)$, and $W^{(\kappa)}(h)$ are the single and double layer potentials constructed by the fundamental matrix $\Gamma^{(\kappa)}$ with the integration surface S and the densities

$$f \in \big[\widetilde{B}^{-1+\frac{1}{p}}_{p',p'}(S_T)\big]^3, \quad g \in \big[\widetilde{B}^{\frac{1}{p}}_{p',p'}(S_C)\big]^3, \quad h \in \big[B^{\frac{1}{p}}_{p',p'}(S)\big]^3, \quad 4/3 < p' < 4. \tag{4.111}$$

Due to the density property of the sets $\{y^{(k)}\}_{k=1}^{\infty}$ in Ω_0^- and $\{z^{(k)}\}_{k=1}^{\infty}$ in Ω_0^+, from (4.92) and (4.93) we get

$$V^{(1)}(f)(z) + V^{(1)}(g)(z) + W^{(1)}(h)(z) = 0, \quad z \in \Omega_0^-,$$
$$-V^{(2)}(f)(z) + W^{(2)}(g)(z) - W^{(2)}(h)(z) = 0, \quad z \in \Omega_0^+.$$

Due to analyticity of the layer potentials in domains $\Omega^{\pm}$ we conclude

$$U^{(1)}(z) := V^{(1)}(f)(z) + W^{(1)}(g)(z) + W^{(1)}(h)(z) = 0, \quad z \in \Omega^-, \tag{4.112}$$
$$U^{(2)}(z) := -V^{(2)}(f)(z) + W^{(2)}(g)(z) - W^{(2)}(h)(z) = 0, \quad z \in \Omega^+. \tag{4.113}$$

Note that by Theorem A.1 we have

$$U^{(1)} \in \big[W^1_{p'}(\Omega^+)\big]^3, \quad U^{(2)} \in \big[W^1_{p',loc}(\Omega^-)\big]^3 \cap Z(\Omega^-), \quad 4/3 < p' < 4. \tag{4.114}$$

Further, by Theorem A.2 and relations (4.112) and (4.113) we find that

$$\big\{U^{(1)}\big\}^+ = g + h \text{ on } S, \tag{4.115}$$
$$\big\{T^{(1)}(\partial,n)U^{(1)}\big\}^+ = -f \text{ on } S, \tag{4.116}$$
$$\big\{U^{(2)}\big\}^- = -g + h \text{ on } S, \tag{4.117}$$
$$\big\{T^{(2)}(\partial,n)U^{(2)}\big\}^- = -f \text{ on } S. \tag{4.118}$$

Whence in view of (4.111) we find

$$\{U^{(1)}\}^+ - \{U^{(2)}\}^- = 0 \text{ on } S_T, \tag{4.119}$$

$$\{T^{(1)}(\partial, n)U^{(1)}\}^+ + \{T^{(2)}(\partial, n)U^{(2)}\}^- = 0 \text{ on } S_C, \tag{4.120}$$

$$\{T^{(1)}(\partial, n)U^{(1)}\}^+ - \{T^{(2)}(\partial, n)U^{(2)}\}^- = 0 \text{ on } S. \tag{4.121}$$

From (4.119)–(4.121) and (4.112)–(4.114) it follows that the pair $(U^{(1)}, U^{(2)})$ belongs to the appropriate class of functions and solves the homogeneous mixed transmission problem $(\mathrm{MT})^-_\omega$. In accordance with the existence and uniqueness Theorem 3.7 then $U^{(1)}$ vanishes in Ω^+ and $U^{(2)}$ vanishes in Ω^- which along with (4.115)–(4.118) imply $f = g = h = 0$ on S. □

4.2. *Construction of approximate solutions*

In this subsection we describe how to construct approximate solutions of the above considered boundary value problems. In what follows $\varphi^{(l)}$, $\psi^{(l)}$, $\varphi^{(l,\kappa)}$, $\psi^{(l,\kappa)}$, $\kappa = 1, 2$, are the vector functions introduced in the previous subsection.

4.2.1. *The Dirichlet problem*

Let us look for an approximate solution of the exterior Dirichlet problem $(\mathrm{D})^-_\omega$, (see (2.1), (2.3)) in the form

$$u^{(m)}(x) = \sum_{l=1}^{m} a_l\, \varphi^{(l)}(x), \quad x \in \Omega^-, \quad m \in \mathbb{N}, \tag{4.122}$$

where a_l are sought-for complex valued constants. These constants are to be chosen in such a way that the norm $\|u - u^{(m)}\|_{[W^1_p(\Omega^- \cap B(R))]^3}$ of the difference of the exact solution u and the approximate solution $u^{(m)}$ should be small.

Note that for all m the vector function $u^{(m)}$ solves the homogeneous differential equation (2.1) and is analytic and radiating in $\mathbb{R}^3 \setminus \overline{\Omega^+_0}$.

Due to Theorem 3.1 and estimate (3.20), if the trace of $u^{(m)}$ on the boundary S approximates the boundary function f with a sufficiently good accuracy in the space $[B^{1-1/p}_{p,p}(S)]^3$, then the norm $\|u - u^{(m)}\|_{[W^1_p(\Omega^- \cap B(R))]^3}$ for fixed R will also be sufficiently small and $u^{(m)}$ can be considered as a good approximation of the exact solution u in the region $\Omega^- \cap B(R)$).

Lemmas 4.1 and 4.2 show that a good approximation on S of a boundary vector function $f \in [B^{1-1/p}_{p,p}(S)]^3$ is possible within an arbitrary accuracy by the linear combinations of type (4.122):

$$\sum_{l=1}^{m} a_l\, \varphi^{(l)}(x) \approx f(x), \quad x \in S. \tag{4.123}$$

Thus, construction of an approximate solution of the Dirichlet BVP is reduced to the approximation problem for the boundary vector function into the linearly independent complete system of vector functions $\Phi^{(-)}_D$ explicitly constructed by the columns of the fundamental solution matrix.

This approximation can be practically carried out by choosing finite sets of functions from the system $\Phi^{(-)}_D$ appropriately and then applying some well-known methods, e.g., Galerkin, collocation, least square, adaptive cross approximation etc. However, this is a very serious problem which needs a special investigation from the point of view of numerical analysis (cf. [16–19,38,39]).

Similar approach with word for word arguments can be applied to all BVP considered in Section 3. Therefore, below we will write down schematically only the expressions of approximate solutions in the corresponding domains and formulate the desired boundary approximation problems (counterparts of (4.122) and (4.123)).

4.2.2. *The Neumann problem*

Approximate solution of the exterior Neumann problem (see (2.1), (2.4))

$$u^{(m)}(x) = \sum_{l=1}^{m} a_l\, \varphi^{(l)}(x), \quad x \in \Omega^-, \quad m \in \mathbb{N},$$

where a_l are sought-for complex valued constants.

Desired boundary approximation of the vector function $F \in [B^{-1/p}_{p,p}(S)]^3$ in the system $\Phi^{(-)}_N$ (see Theorem 3.2 and Lemmas 4.3 and 4.4):

$$\sum_{l=1}^{m} a_l\, T(\partial, n(x))\varphi^{(l)} \approx F \quad \text{on} \quad S.$$

4.2.3. The mixed problem

Approximate solution of the exterior mixed problem (see (2.1), (2.5), (2.6))

$$u^{(m)}(x) = \sum_{l=1}^{m} a_l\, \varphi^{(l)}(x), \qquad x \in \Omega^-, \qquad m \in \mathbb{N},$$

where a_l are sought-for complex valued constants.

Desired boundary approximation on S_D and S_N of the vector functions $f^* \in [B^{1-1/p}_{p,p}(S_D)]^3$ and $F^* \in [B^{-1/p}_{p,p}(S_N)]^3$ in the system $\widetilde{\Phi}^{(-)}_M$ (see Theorem 3.3 and Lemmas 4.5 and 4.6):

$$\sum_{l=1}^{m} a_l\, \varphi^{(l)} \approx f^* \quad \text{on} \quad S_D,$$
$$\sum_{l=1}^{m} a_l\, T(\partial, n)\varphi^{(l)} \approx F^* \quad \text{on} \quad S_N.$$

4.2.4. The basic transmission problem

Approximate solution of the basic transmission problem (see (2.1), (2.12)–(2.13))

$$u^{(m_1,1)}(x) := \sum_{l=1}^{m_1} a_{l,1}\, \psi^{(l,1)}(x), \qquad x \in \Omega^+, \qquad m_1 \in \mathbb{N},$$
$$u^{(m_2,2)}(x) := \sum_{l=1}^{m_2} a_{l,2}\, \varphi^{(l,2)}(x), \qquad x \in \Omega^-, \qquad m_2 \in \mathbb{N},$$

where $a_{l,1}$ and $a_{l,2}$ are sought-for complex valued constants.

Desired boundary approximation on S_D and S_N of the vector functions $f \in [B^{1-1/p}_{p,p}(S)]^3$ and $F \in [B^{-1/p}_{p,p}(S)]^3$ in the system Φ_{BT} (see Theorem 3.5 and Lemmas 4.7 and 4.8):

$$\sum_{l=1}^{m_1} a_{l,1}\, \psi^{(l,1)} - \sum_{l=1}^{m_2} a_{l,2}\, \varphi^{(l,2)} \approx f \quad \text{on} \quad S,$$
$$\sum_{l=1}^{m_1} a_{l,1}\, T^{(1)}(\partial, n)\psi^{(l,1)} - \sum_{l=1}^{m_2} a_{l,2}\, T^{(2)}(\partial, n)\varphi^{(l,2)} \approx F \quad \text{on} \quad S.$$

4.2.5. The mixed transmission problem

Approximate solution of the mixed transmission problem (see (2.1), (2.15)–(2.18), and (3.42)–(3.45))

$$u^{(m_1,1)}(x) := \sum_{l=1}^{m_1} a_{l,1}\, \psi^{(l,1)}(x), \qquad x \in \Omega^+, \qquad m_1 \in \mathbb{N},$$
$$u^{(m_2,2)}(x) := \sum_{l=1}^{m_2} a_{l,2}\, \varphi^{(l,2)}(x), \qquad x \in \Omega^-, \qquad m_2 \in \mathbb{N},$$

where $a_{l,1}$ and $a_{l,2}$ are sought-for complex valued constants.

Desired boundary approximation on S_T, S_C, and S of the vector functions

$$f^{(T)} \in [B^{1-1/p}_{p,p}(S_T)]^3, \qquad F^{(+)} + F^{(-)} \in [B^{-1/p}_{p,p}(S_C)]^3, \qquad F_0 \in [B^{-1/p}_{p,p}(S)]^3$$

in the system $\widetilde{\Phi}_{MT}$ (see Theorem 3.7 and Lemmas 4.9 and 4.10):

$$\sum_{l=1}^{m_1} a_{l,1}\,\psi^{(l,1)} - \sum_{l=1}^{m_2} a_{l,2}\,\varphi^{(l,2)} \approx f^{(T)} \quad \text{on} \quad S_T,$$
$$\sum_{l=1}^{m_1} a_{l,1}\,T^{(1)}(\partial,n)\psi^{(l,1)} + \sum_{l=1}^{m_2} a_{l,2}\,T^{(2)}(\partial,n)\varphi^{(l,2)} \approx F^{(+)} + F^{(-)} \quad \text{on} \quad S_C,$$
$$\sum_{l=1}^{m_1} a_{l,1}\,T^{(1)}(\partial,n)\psi^{(l,1)} - \sum_{l=1}^{m_2} a_{l,2}\,T^{(2)}(\partial,n)\varphi^{(l,2)} \approx F_0 \quad \text{on} \quad S.$$

Due to Remark 3.10 the crack problem is a particular case of a special mixed transmission problem and its approximate solution can be constructed in accordance with the approach described in the present subsection.

Acknowledgements

This research was supported by the Shota Rustaveli National Science Foundation (SRNSF) Grant No. FR/286/5-101/13.

Appendix A. Layer potentials and their properties

Here we collect some auxiliary material needed in the main text of the paper concerning properties of layer potentials and the corresponding boundary operators.

Denote by $\Gamma(x,\omega)$ and $\Gamma(x)$ respectively Kupradze's and Kelvin's matrices of fundamental solutions of the differential operator of elastic oscillations $A(\partial,\omega)$ and its principal homogeneous part $A(\partial)$ (Lamé's operator)

$$A(\partial,\omega)\Gamma(x,\omega) = I_3\,\delta(x), \qquad A(\partial)\Gamma(x) = I_3\,\delta(x),$$

where $\delta(x)$ is the Dirac delta function. These matrices read as (see [12, Ch. 2])

$$\Gamma(x,\omega) = [\Gamma_{kj}(x,\omega)]_{3\times3}, \qquad \Gamma_{kj}(x,\omega) = \sum_{l=1}^{2}(\delta_{kj}\alpha_l + \beta_l\,\partial_k\,\partial_j)\,\frac{e^{ik_l|x|}}{|x|}, \tag{A.1}$$
$$\alpha_l = -\frac{\delta_{2l}}{4\pi\,\mu}, \quad \beta_l = \frac{(-1)^{l+1}}{4\pi\,\varrho\,\omega^2}, \quad k_1 \equiv k_p = \omega\sqrt{\frac{\varrho}{\lambda+2\mu}}, \quad k_2 \equiv k_s = \omega\sqrt{\frac{\varrho}{\mu}},$$
$$\Gamma(x) = [\Gamma_{kj}(x)]_{3\times3}, \qquad \Gamma_{kj}(x) = \frac{\delta_{kj}\,\lambda'}{|x|} + \frac{\mu'\,x_k\,x_j}{|x|^3},$$
$$\lambda' = -\frac{\lambda+3\mu}{8\pi\,\mu\,(\lambda+2\mu)}, \quad \mu' = -\frac{\lambda+\mu}{8\pi\,\mu\,(\lambda+2\mu)}.$$

The following relations hold true

$$\begin{aligned}
&\Gamma(x,\omega) = \Gamma(-x,\omega) = [\Gamma(x,\omega)]^\top, && \Gamma(x) = \Gamma(-x) = [\Gamma(x)]^\top,\\
&|\Gamma_{pq}(x,\omega)| \le c(\lambda,\mu)\,|x|^{-1}, && |\Gamma(_{pq}x,\omega) - \Gamma_{pq}(x)| \le |\omega|\,c(\lambda,\mu),\\
&|\partial_j\Gamma_{pq}(x,\omega) - \partial_j\Gamma_{pq}(x)| \le |\omega|^2 c(\lambda,\mu), && |\partial_j\partial_l\Gamma_{pq}(x,\omega) - \partial_j\partial_l\Gamma_{pq}(x)| \le c(\lambda,\mu,\omega)|x|^{-1},
\end{aligned} \tag{A.2}$$

where $c(\lambda,\mu)$ and $c(\lambda,\mu,\omega)$ are positive constants. These relations show that the Kelvin matrix of statics $\Gamma(x)$ is the principal singular homogeneous part of Kupradze's matrix $\Gamma(x,\omega)$. It is evident that the entries of $\Gamma(x,\omega)$ and $\Gamma(x)$ are analytic functions of the real variable $x \in \mathbb{R}^3 \setminus \{0\}$ and, moreover, the columns of $\Gamma(x,\omega)$ satisfy the Sommerfeld–Kupradze radiation conditions at infinity.

Introduce the single and double layer potentials of elastic oscillations

$$V(g)(x) := \int_S \Gamma(x-y,\omega)\,g(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{A.3}$$
$$W(h)(x) := \int_S [T(\partial_y, n(y))\Gamma(x-y,\omega)]^\top\,h(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{A.4}$$

where $g = (g_1, g_2, g_3)^\top$ and $h = (h_1, h_2, h_3)^\top$ are densities of the potentials.

By standard arguments and with the help of Green's second formula one can derive the following integral representation formula for a regular solution u to the homogeneous equation $A(\partial,\omega)u = 0$ in the domain Ω^+,

$$W(\{u\}^+)(x) - V(\{Tu\}^+)(x) = \begin{cases} u(x) & \text{in} \quad \Omega^+, \\ 0 & \text{in} \quad \Omega^-. \end{cases} \tag{A.5}$$

Similarly, for a radiating regular solution of the homogeneous equation $A(\partial,\omega)u = 0$ in the domain Ω^- we have an analogous representation formula (see [12,27])

$$-W(\{u\}^-)(x) + V(\{Tu\}^-)(x) = \begin{cases} 0 & \text{in} \quad \Omega^+, \\ u(x) & \text{in} \quad \Omega^-. \end{cases} \tag{A.6}$$

These representation formulae can be extended to the classes $[W^1_p(\Omega^+)]^3$ and $[W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$, and to Lipschitz domains. From these formulae it is evident that any solution to the homogeneous equation is actually an analytic vector function of the real variable $x \in \Omega^\pm$. Further, if u solves the homogeneous equation $A(\partial,\omega)u = 0$ in Ω^+ and Ω^-, and $r_{\Omega^+} u \in [W^1_p(\Omega^+)]^3$, $r_{\Omega^-} u \in [W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$ then by adding formulae (A.5) and (A.6) we get

$$\begin{aligned} &u(x) = W\big([u]_S\big)(x) - V\big([Tu]_S\big)(x) \text{ in } \Omega^+ \cup \Omega^- \\ &\quad \text{with} \quad [u]_S := \{u\}^+_S - \{u\}^-_S, \qquad [Tu]_S := \{Tu\}^+_S - \{Tu\}^-_S, \end{aligned}$$

which shows that if on some open part $S_1 \subset S$ of the common boundary S of the adjacent domains Ω^+ and Ω^- the jumps of the Cauchy data equal to zero, i.e., $r_{S_1}[\{u\}^+ - \{u\}^-] = 0$ and $r_{S_1}[\{Tu\}^+ - \{Tu\}^-] = 0$, then the vector-function $\widetilde{u}$ defined by the equality

$$\widetilde{u} := \begin{cases} u(x) & \text{for} \quad x \in \Omega^+, \\ u(x) & \text{for} \quad x \in \Omega^-, \\ \{u(x)\}^+ & \text{for} \quad x \in S_1, \end{cases}$$

is an analytic vector function in the connected domain $\mathbb{R}^3 \setminus \overline{S_2}$ with $S_2 = S \setminus \overline{S_1}$.

Further we introduce the boundary operators generated by the single and double layer potentials,

$$(\mathcal{H}\,g)(x) := \int_S \Gamma(x-y,\omega)\, g(y)\, dS_y\,, \qquad x \in S, \tag{A.7}$$

$$(\mathcal{K}\,g)(x) := \int_S \big[\, T(\partial_x, n(x))\Gamma(x-y,\omega)\,\big]\, g(y)\, dS_y, \qquad x \in S, \tag{A.8}$$

$$(\widetilde{\mathcal{K}}\,h)(x) := \int_S \big[\, T(\partial_y, n(y))\,\Gamma(x-y,\omega)\,\big]^\top\, h(y)\, dS_y, \qquad x \in S, \tag{A.9}$$

$$(\mathcal{L}h)(x) := \big\{\, T(\partial_x, n(x)) W(h)(x)\,\big\}^\pm, \qquad x \in S. \tag{A.10}$$

The boundary operators $\mathcal{H}$ and $\mathcal{L}$ are pseudodifferential operators of order -1 and 1, respectively, while the operators $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ are mutually adjoint singular integral operators–pseudodifferential operators of order 0 (for details see [12,40–45]).

We will employ the same notation equipped with subscript "0" for the elastostatic potentials constructed by the Kelvin matrix $\Gamma(x-y)$ and the corresponding boundary operators.

Now we describe the basic mapping and jump properties of the above introduced layer potentials. They can be found in [24–26,35,40,41,43–55].

Theorem A.1. *Let S be C^∞-smooth and $1 < p < \infty$, $1 \le t \le \infty$, and $s \in \mathbb{R}$. The operators*

$$\begin{aligned} V \quad &: \quad \big[B^s_{p,p}(S)\big]^3 \longrightarrow \big[H_p^{s+1+\frac{1}{p}}(\Omega^+)\big]^3 \qquad \Big[\big[B^s_{p,p}(S)\big]^3 \longrightarrow \big[H_{p,loc}^{s+1+\frac{1}{p}}(\Omega^-)\big]^3 \cap Z(\Omega^-)\Big], \\ &: \quad \big[B^s_{p,t}(S)\big]^3 \longrightarrow \big[B_{p,t}^{s+1+\frac{1}{p}}(\Omega^+)\big]^3 \qquad \Big[\big[B^s_{p,t}(S)\big]^3 \longrightarrow \big[B_{p,t,loc}^{s+1+\frac{1}{p}}(\Omega^-)\big]^3 \cap Z(\Omega^-)\Big], \\ W \quad &: \quad \big[B^s_{p,p}(S)\big]^3 \longrightarrow \big[H_p^{s+\frac{1}{p}}(\Omega^+)\big]^3 \qquad \Big[\big[B^s_{p,p}(S)\big]^3 \longrightarrow \big[H_{p,loc}^{s+\frac{1}{p}}(\Omega^-)\big]^3 \cap Z(\Omega^-)\Big], \\ &: \quad \big[B^s_{p,t}(S)\big]^3 \longrightarrow \big[B_{p,t}^{s+\frac{1}{p}}(\Omega^+)\big]^3 \qquad \Big[\big[B^s_{p,t}(S)\big]^3 \longrightarrow \big[B_{p,t,loc}^{s+\frac{1}{p}}(\Omega^-)\big]^3 \cap Z(\Omega^-)\Big], \end{aligned}$$

are continuous.

If S is Lipschitz, then the operators

$$V \; : \; \big[H_2^{-\frac{1}{2}}(S)\big]^3 \longrightarrow \big[H_2^1(\Omega^+)\big]^3 \qquad \Big[\big[H_2^{-\frac{1}{2}}(S)\big]^3 \longrightarrow \big[H^1_{2,\,loc}(\Omega^-)\big]^3 \cap Z(\Omega^-)\Big],$$
$$W \; : \; \big[H_2^{\frac{1}{2}}(S)\big]^3 \longrightarrow \big[H_2^1(\Omega^+)\big]^3 \qquad \Big[\big[H_2^{\frac{1}{2}}(S)\big]^3 \longrightarrow \big[H^1_{2,\,loc}(\Omega^-)\big]^3 \cap Z(\Omega^-)\Big],$$

are continuous.

Theorem A.2. *Let S be C^∞-smooth and $1 < p < \infty$, $1 \le t \le \infty$, and $g \in \big[B_{p,t}^{-\frac{1}{p}}(S)\big]^3$, $h \in \big[B_{p,t}^{1-\frac{1}{p}}(S)\big]^3$. Then*

$$\big\{V(g)\big\}^+ = \big\{V(g)\big\}^- = \mathcal{H}\, g \;\; on \;\; S, \tag{A.11}$$
$$\big\{T(\partial,n)V(g)\big\}^\pm = \big[\mp 2^{-1} I_3 + \mathcal{K}\big]\, g \;\; on \;\; S,$$
$$\big\{W(h)\big\}^\pm = \big[\pm 2^{-1} I_3 + \widetilde{\mathcal{K}}\big]\, h \;\; on \;\; S,$$
$$\big\{\,T(\partial,n)W(h)\,\big\}^+ = \big\{\,T(\partial,n)W(h)\,\big\}^- = \mathcal{L}\, h \;\; on \;\; S. \tag{A.12}$$

The same relations hold for a Lipschitz boundary S and $p = t = 2$.

Theorem A.3. (i) *Let S be C^∞-smooth and $1 < p < \infty$, $1 \le t \le \infty$, $s \in \mathbb{R}$. The operators*

$$\begin{aligned}
\mathcal{H} \; &: \; \big[H_p^s(S)\big]^3 \longrightarrow \big[H_p^{s+1}(S)\big]^3 \qquad &&\Big[\big[B_{p,t}^s(S)\big]^3 \longrightarrow \big[B_{p,t}^{s+1}(S)\big]^3\Big],\\
\pm 2^{-1} I_3 + \mathcal{K},\; \pm 2^{-1} I_3 + \widetilde{\mathcal{K}} \; &: \; \big[H_p^s(S)\big]^3 \longrightarrow \big[H_p^s(S)\big]^3 \qquad &&\Big[\big[B_{p,t}^s(S)\big]^3 \longrightarrow \big[B_{p,t}^s(S)\big]^3\Big],\\
\mathcal{L} \; &: \; \big[H_p^{s+1}(S)\big]^3 \longrightarrow \big[H_p^s(S)\big]^3 \qquad &&\Big[\big[B_{p,t}^{s+1}(S)\big]^3 \longrightarrow \big[B_{p,t}^s(S)\big]^3\Big],
\end{aligned}$$

are continuous Fredholm operators with zero index. The principal homogeneous symbol matrices of these operators are non-degenerate. Moreover, the principal homogeneous symbol matrices of the operators $-\mathcal{H}$ and $\mathcal{L}$ are positive definite.

(ii) *If S is Lipschitz, then the operators*

$$\begin{aligned}
\mathcal{H} \; &: \; \big[H_2^{-\frac{1}{2}}(S)\big]^3 \longrightarrow \big[H_2^{\frac{1}{2}}(S)\big]^3,\\
\pm 2^{-1} I_3 + \mathcal{K} \; &: \; \big[H_2^{-\frac{1}{2}}(S)\big]^3 \longrightarrow \big[H_2^{-\frac{1}{2}}(S)\big]^3,\\
\pm 2^{-1} I_3 + \widetilde{\mathcal{K}} \; &: \; \big[H_2^{\frac{1}{2}}(S)\big]^3 \longrightarrow \big[H_2^{\frac{1}{2}}(S)\big]^3,\\
\mathcal{L} \; &: \; \big[H_2^{\frac{1}{2}}(S)\big]^3 \longrightarrow \big[H_2^{-\frac{1}{2}}(S)\big]^3,
\end{aligned}$$

are continuous Fredholm operators with zero index, and moreover, there exist positive constants C_k, $k = 1,2,3,4$, such that

$$\begin{aligned}
&\langle h\,,\,-\mathcal{H}h\rangle_S \ge C_1\,\|h\|^2_{[H_2^{-\frac{1}{2}}(S)]^3} - C_2\,\|\mathcal{T}h\|^2_{[H_2^{-\frac{1}{2}}(S)]^3} \quad for\ all \;\; h \in [H_2^{-\frac{1}{2}}(S)]^3,\\
&\langle \mathcal{L}g\,,\,g\rangle_S \ge C_3\,\|g\|^2_{[H_2^{\frac{1}{2}}(S)]^3} - C_4\,\|g\|^2_{[H_2^0(S)]^3} \quad for\ all \;\; g \in [H_2^{\frac{1}{2}}(S)]^3,
\end{aligned} \tag{A.13}$$

where the symbol $\langle\cdot,\cdot\rangle_S$ denotes the duality brackets between the mutually adjoint spaces $[H_2^{-\frac{1}{2}}(S)]^3$ and $[H_2^{\frac{1}{2}}(S)]^3$, and $\mathcal{T} : [H_2^{-\frac{1}{2}}(S)]^3 \longrightarrow [H_2^{-\frac{1}{2}}(S)]^3$ is a compact operator.

(iii) *The following operator equalities hold in appropriate function spaces:*

$$\widetilde{\mathcal{K}}\,\mathcal{H} = \mathcal{H}\,\mathcal{K}, \quad \mathcal{L}\,\widetilde{\mathcal{K}} = \mathcal{K}\,\mathcal{L}, \quad \mathcal{L}\,\mathcal{H} = -4^{-1} I_3 + \mathcal{K}^2, \quad \mathcal{H}\,\mathcal{L} = -4^{-1} I_3 + \widetilde{\mathcal{K}}^2. \tag{A.14}$$

Remark A.4. In the static case, i.e., for the operators constructed by the Kelvin fundamental matrix $\Gamma(x-y)$, the operators $\mathcal{H}_0$, $2^{-1} I_3 + \widetilde{\mathcal{K}}_0$ and $2^{-1} I_3 + \mathcal{K}_0$ in items (i) and (ii) of Theorem A.3 are invertible. Moreover, $\widetilde{\mathcal{K}}_0$ and $\mathcal{K}_0$ are mutually adjoint singular integral operators and the inequality (A.13) holds with $C_2 = 0$ [36].

In view of the relations (A.2), it is evident that the operators of elastostatics $\mathcal{H}_0$, $\pm 2^{-1} I_3 + \mathcal{K}_0$, $\pm 2^{-1} I_3 + \widetilde{\mathcal{K}}_0$, $\mathcal{L}_0$, and elasto-oscillations $\mathcal{H}$, $\pm 2^{-1} I_3 + \mathcal{K}$, $\pm 2^{-1} I_3 + \widetilde{\mathcal{K}}$, $\mathcal{L}$ have the same principal homogeneous symbol matrices respectively:

$$\begin{aligned}
&\mathbb{H}(x,\xi) := \mathfrak{S}(\mathcal{H}_0; x,\xi) = \mathfrak{S}(\mathcal{H}; x,\xi),\\
&\mathbb{K}_{\pm}(x,\xi) := \mathfrak{S}(\pm 2^{-1} I_3 + \mathcal{K}_0)(x,\xi) = \mathfrak{S}(\pm 2^{-1} I_3 + \mathcal{K})(x,\xi),\\
&\mathbb{L}(x,\xi) := \mathfrak{S}(\mathcal{L}_0; x,\xi) = \mathfrak{S}(\mathcal{L}; x,\xi),\\
&\widetilde{\mathbb{K}}_{\pm}(x,\xi) := \mathfrak{S}(\pm 2^{-1} I_3 + \widetilde{\mathcal{K}}_0)(x,\xi) = \mathfrak{S}(\pm 2^{-1} I_3 + \widetilde{\mathcal{K}})(x,\xi),\\
&x \in S, \quad \xi \in \mathbb{R}^2 \setminus \{0\}.
\end{aligned} \tag{A.15}$$

The matrices $-\mathbb{H}(x,\xi)$ and $\mathbb{L}(x,\xi)$ are positive definite matrices with entries being real valued even functions in ξ, while the matrices $\mathbb{K}_{\pm}(x,\xi)$ and $\widetilde{\mathbb{K}}_{\pm}(x,\xi)$ are non-degenerate and mutually adjoint, i.e., $\widetilde{\mathbb{K}}_{\pm}(x,\xi) = [\,\overline{\mathbb{K}_{\pm}(x,\xi)}\,]^{\top}$. The following matrices related to the so called Steklov–Poincaré operators of statics $[-2^{-1} I_3 + \mathcal{K}_0]\,\mathcal{H}_0^{-1}$ and $-[2^{-1} I_3 + \mathcal{K}_0]\,\mathcal{H}_0^{-1}$ corresponding to the interior and exterior domains, respectively,

$$\mathfrak{S}\big([-2^{-1} I_3 + \mathcal{K}_0]\,\mathcal{H}_0^{-1}; x,\xi\big) = \mathbb{K}_{-}(x,\xi)\,[\mathbb{H}(x,\xi)]^{-1}, \tag{A.16}$$

$$-\,\mathfrak{S}\big([2^{-1} I_3 + \mathcal{K}_0]\,\mathcal{H}_0^{-1}; x,\xi\big) = -\,\mathbb{K}_{+}(x,\xi)\,[\mathbb{H}(x,\xi)]^{-1}, \tag{A.17}$$

are positive definite as well. Moreover, from (A.14) it follows that

$$\begin{aligned}
&\widetilde{\mathbb{K}}_{\pm}(x,\xi)\,\mathbb{H}(x,\xi) = \mathbb{H}(x,\xi)\,\mathbb{K}_{\pm}(x,\xi), \quad \mathbb{L}(x,\xi)\,\widetilde{\mathbb{K}}_{\pm}(x,\xi) = \mathbb{K}_{\pm}(x,\xi)\,\mathbb{L}(x,\xi),\\
&\mathbb{L}(x,\xi)\,\mathbb{H}(x,\xi) = \mathbb{K}_{+}(x,\xi)\,\mathbb{K}_{-}(x,\xi), \quad \mathbb{H}(x,\xi)\mathbb{L}(x,\xi) = \widetilde{\mathbb{K}}_{+}(x,\xi)\,\widetilde{\mathbb{K}}_{-}(x,\xi).
\end{aligned} \tag{A.18}$$

Note that the matrices $\widetilde{\mathbb{K}}_{-}(x,\xi)$ and $\widetilde{\mathbb{K}}_{+}(x,\xi)$, as well as the matrices $\mathbb{K}_{-}(x,\xi)$ and $\mathbb{K}_{+}(x,\xi)$ commute each other. Therefore from (A.18) we derive

$$\begin{aligned}
\pm\,\mathbb{L}(x,\xi)\,[\widetilde{\mathbb{K}}_{\pm}(x,\xi)]^{-1} &= \pm\,[\mathbb{H}(x,\xi)]^{-1}\,\widetilde{\mathbb{K}}_{+}(x,\xi)\,\widetilde{\mathbb{K}}_{-}(x,\xi)\,[\widetilde{\mathbb{K}}_{\pm}(x,\xi)]^{-1}\\
&= \pm\,[\mathbb{H}(x,\xi)]^{-1}\,\widetilde{\mathbb{K}}_{\mp}(x,\xi) = \pm\,\mathbb{K}_{\mp}(x,\xi)\,[\mathbb{H}(x,\xi)]^{-1},
\end{aligned} \tag{A.19}$$

implying that the matrices $\pm\,\mathbb{L}(x,\xi)\,[\widetilde{\mathbb{K}}_{\pm}(x,\xi)]^{-1}$ are positive definite in view of the positive definiteness of the matrices (A.16) and (A.17).

Moreover, it can be shown that the entries of the matrices $\mathbb{H}(x,\xi)$ and $\mathbb{L}(x,\xi)$ are real valued functions, while $\mathbb{K}_{\pm}(x,\xi) = \pm 2^{-1} I_3 + i\,\mathbf{K}(x,\xi)$ and $\widetilde{\mathbb{K}}_{\pm}(x,\xi) = \pm 2^{-1} I_3 + i\,\widetilde{\mathbf{K}}(x,\xi)$, where the entries of the matrices $\mathbf{K}$ and $\widetilde{\mathbf{K}}$ are real valued odd functions in ξ (see Appendix C in [23]).

Appendix B. Alternative representations of radiating solutions

Let Ω^+, Ω^-, and S be the same as in Appendix A, and consider the following linear combination of the single and double layer potentials

$$u(x) = W(g)(x) + \varkappa\, V(g)(x), \quad x \in \Omega^-,$$

where $\varkappa = \varkappa_1 + i\,\varkappa_2 \in \mathbb{C}$ with $\varkappa_1, \varkappa_2 \in \mathbb{R}$ and $\varkappa_2 \neq 0$, and $g \in (g_1, g_2, g_3)^{\top} \in [B_{p,p}^{1-\frac{1}{p}}(S)]^3$ is a density vector.

Evidently, $u \in [W^1_{p,\,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$ in view of Theorem A.1, while by Theorem A.2 we have

$$\{u\}^- = [\,-2^{-1}\, I_3 + \widetilde{\mathcal{K}} + \varkappa\,\mathcal{H}\,]\,g \equiv \mathcal{N}\,g, \qquad \{Tu\}^- = \mathcal{L}\,g + \varkappa\,[\,2^{-1}\, I_3 + \mathcal{K}\,]\,g \equiv \mathcal{M}\,g,$$

where $\mathcal{H}$, $\mathcal{K}$, $\widetilde{\mathcal{K}}$ and $\mathcal{L}$ are given by equalities (A.7)–(A.10) respectively.

In Refs. [25] and [27] the following assertions are proved.

Theorem B.1. *Let $S \in C^\infty$, $s \in \mathbb{R}$, $1 < p < +\infty$, and $1 \le t \le +\infty$. Then the operators*

$$\mathcal{N} : [H_p^{s+1}(S)]^3 \longrightarrow [H_p^{s+1}(S)]^3 \quad \big[\,[B_{p,t}^{s+1}(S)]^3 \longrightarrow [B_{p,t}^{s+1}(S)]^3\,\big]\,, \tag{B.1}$$

$$\mathcal{M} : [H_p^{s+1}(S)]^3 \longrightarrow [H_p^{s}(S)]^3 \quad \big[\,[B_{p,t}^{s+1}(S)]^3 \longrightarrow [B_{p,t}^{s}(S)]^3\,\big]\,, \tag{B.2}$$

are invertible.

For a Lipschitz manifold S these operators are also invertible for $p = t = 2$ and $s = -1/2$ and, moreover, there are positive constants C_1 and C_2 such that the following inequality holds true

$$Re\,\big\langle -\mathcal{M}[\mathcal{N}]^{-1} g,\, \overline{g}\big\rangle_S \geqslant C_1\, \|g\|^2_{[H_2^{\frac{1}{2}}(S)]^3} - C_2\, \|g\|^2_{[H_2^0(S)]^3} \ \textit{for all } g \in [H_2^{\frac{1}{2}}(S)]^3.$$

Theorem B.2. *If $u \in [W^1_{p,loc}(\Omega^-)]^3 \cap Z(\Omega^-)$, $1 < p < +\infty$, solves the homogeneous equation $A(\partial,\omega)u(x) = 0$ in Ω^-, then u can be represented uniquely in the following two equivalent to each other forms*

$$u(x) = W(\mathcal{N}^{-1}g)(x) + \varkappa\, V(\mathcal{N}^{-1}g)(x), \quad x \in \Omega^-,$$
$$u(x) = W(\mathcal{M}^{-1}h)(x) + \varkappa\, V(\mathcal{M}^{-1}h)(x), \quad x \in \Omega^-,$$

where $\mathcal{N}^{-1}$ and $\mathcal{M}^{-1}$ are the operators inverse to $\mathcal{N}$ and $\mathcal{M}$ respectively defined in (B.1) and (B.2), *while the densities g and h, are related to the vector u by the equalities*

$$g = \{u\}^-_S \in [B^{1-\frac{1}{p}}_{p,p}(S)]^3\,, \qquad h = \{Tu\}^-_S \in [B^{-\frac{1}{p}}_{p,p}(S)]^3\,.$$

In the case of a Lipschitz surface S, the same assertion holds true with $p = 2$.

Appendix C. Fredholm properties of strongly elliptic pseudodifferential operators on manifolds with boundary

Here we recall some results from the theory of strongly elliptic pseudodifferential equations on manifolds with boundary in Bessel potential and Besov spaces which are the main tools for proving existence theorems for mixed boundary, boundary-transmission, and crack problems by the potential methods. They can be found in [56–58].

Let $\overline{M} \in C^\infty$ be a compact, n-dimensional, nonintersecting manifold with boundary $\partial M \in C^\infty$ and let $\mathcal{A}$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{M}$. Denote by $\mathfrak{S}(\mathcal{A}; x, \xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system ($x \in \overline{M},\ \xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \dots, \lambda_N(x)$ be the eigenvalues of the matrix

$$[\,\mathfrak{S}(\mathcal{A}; x, 0, \dots, 0, +1)\,]^{-1}[\,\mathfrak{S}(\mathcal{A}; x, 0, \dots, 0, -1)\,], \quad x \in \partial\overline{M},$$

and introduce the notation $\delta_j(x) = \mathrm{Re}\,\big[(2\pi\, i)^{-1} \ln \lambda_j(x)\big]$, $j = 1, \dots, N$, where $\ln\zeta$ denotes the branch of the logarithm function analytic in the complex plane cut along $(-\infty,\ 0]$. Due to the strong ellipticity of $\mathcal{A}$ we have the strong inequality $-1/2 < \delta_j(x) < 1/2$ for $x \in \overline{M}$, $j = 1, 2, \dots, N$. The numbers $\delta_j(x)$ do not depend on a particular choice of the local coordinate system at a fixed pint $x \in \partial M$.

Remark C.1. Note that if $\mathfrak{S}(\mathcal{A}; x, \xi)$ is a positive definite matrix for every $x \in \overline{M}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, we have $\delta_j(x) = 0$ for $j = 1, \dots, N$, since all the eigenvalues $\lambda_j(x)$ ($j = 1, \dots, N$) are positive numbers for any $x \in \overline{M}$. The same holds if $\mathfrak{S}(\mathcal{A}; x, \xi)$ is representable in the form

$$\mathfrak{S}(\mathcal{A}; x, \xi) = Q^{(1)}(x, \xi)\, Q(x, \xi)\, Q^{(2)}(x, \xi),$$

where $Q(x, \xi) = \|Q_{kj}(x, \xi)\|_{N\times N}$ and $Q^{(m)}(x, \xi) = \|Q^{(m)}_{kj}(x, \xi)\|_{N\times N}$, $m = 1, 2$, are positive definite matrices and, in addition, the entries $Q^{(m)}_{kj}(x, \xi)$ are even functions in ξ.

The Fredholm properties of strongly elliptic pseudo-differential operators on manifolds with boundary are characterized by the following theorem.

Theorem C.2. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le t \le \infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $\nu \in \mathbb{R}$, that is, there is a positive constant C_0 such that Re $\mathfrak{S}(\mathcal{A}; x, \xi)\,\eta \cdot \eta \ge C_0\,|\eta|^2$ for $x \in \overline{M}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\eta \in \mathbb{C}^N$.*

Then the operators

$$\mathcal{A} : \big[\widetilde{H}^s_p(M)\big]^N \longrightarrow \big[H^{s-\nu}_p(M)\big]^N \quad \Big[\big[\widetilde{B}^s_{p,t}(M)\big]^N \longrightarrow \big[B^{s-\nu}_{p,t}(M)\big]^N\Big], \tag{C.1}$$

are Fredholm with zero index if

$$\frac{1}{p} - 1 + \sup_{x\in\partial M,\, 1\le j\le N} \delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{x\in\partial M,\, 1\le j\le N} \delta_j(x). \tag{C.2}$$

Moreover, the null-spaces and indices of the operators (C.1) *are the same (for all values of the parameter $t \in [1, +\infty]$) provided p and s satisfy inequality* (C.2).

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