

# Localized boundary-domain singular integral equations of Dirichlet problem for self-adjoint second-order strongly elliptic PDE systems

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The paper deals with the three-dimensional Dirichlet boundary value problem (BVP) for a second-order strongly elliptic self-adjoint system of partial differential equations in the divergence form with variable coefficients and develops the integral potential method based on a localized parametrix. Using Green's representation formula and properties of the localized layer and volume potentials, we reduce the Dirichlet BVP to a system of localized boundary-domain integral equations. The equivalence between the Dirichlet BVP and the corresponding localized boundary-domain integral equation system is studied. We establish that the obtained localized boundary-domain integral operator belongs to the Boutet de Monvel algebra. With the help of the Wiener–Hopf factorization method, we investigate corresponding Fredholm properties and prove invertibility of the localized operator in appropriate Sobolev (Bessel potential) spaces. Copyright © 2016 The Authors *Mathematical Methods in the Applied Sciences* Published by John Wiley & Sons Ltd.

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## 1. Introduction

We consider the Dirichlet boundary value problem (BVP) for a second-order strongly elliptic self-adjoint system of partial differential equations in the divergence form with variable coefficients and develop the generalized integral potential method based on a localized parametrix.

The BVP treated in the paper is well investigated in the literature by the variational method and also by the classical integral potential method, when the corresponding fundamental solution is available in explicit form (e.g. [1–3]) or when at least its properties are known to be good enough (see, e.g. [4, 5] and references therein).

Our goal here is to develop a localized integral potential method for general second-order strongly elliptic self-adjoint systems of partial differential equations with variable coefficients. We show that a solution of the problem can be represented by explicit localized parametrix-based potentials and that the corresponding localized boundary-domain integral operator (LBDIO) is invertible, which is important for analysis of convergence and stability of localized boundary-domain integral equation (LBDIE)-based numerical methods for PDEs (e.g. [6–13]).

Using Green's representation formula and properties of the localized layer and volume potentials, we reduce the Dirichlet BVP to a system of LBDIEs. First, we establish the equivalence between the original BVP and the corresponding LBDIE system, which appeared to be quite non-trivial task and plays a crucial role in our analysis. Afterwards, we establish that the LBDIO of the system belongs to the Boutet de Monvel operator algebra. Employing the Vishik–Eskin theory, based on the Wiener–Hopf factorization

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method, we investigate corresponding Fredholm properties and prove invertibility of the localized operator in appropriate Sobolev (Bessel potential) spaces.

In the references [14–20], the traditional and localised boundary-domain integral equation methods have been developed for the case of scalar elliptic second-order partial differential equations with variable coefficients, and here, we extend the LBDIE method to PDE systems.

## 2. Boundary value problem and parametrix-based operators

### 2.1. Formulation of the boundary value problems and localized Green's third identity

Consider a uniformly strongly elliptic second-order self-adjoint matrix partial differential operator

$$A = A(x, \partial_x) = [A_{pq}(x, \partial_x)]_{p,q=1}^3 = \left[ \frac{\partial}{\partial x_k} \left( a_{kj}^{pq}(x) \frac{\partial}{\partial x_j} \right) \right]_{p,q=1}^3, \tag{2.1}$$

where  $\partial_x = (\partial_1, \partial_2, \partial_3)$ ,  $\partial_j = \partial_{x_j} = \partial/\partial x_j$ ,  $a_{kj}^{pq} = a_{jk}^{qp} = a_{pj}^{kq} \in C^\infty$ ,  $j, k, p, q = 1, 2, 3$ . Here and in what follows, the Einstein summation by repeated indices from 1 to 3 is assumed if not otherwise stated.

We assume that the coefficients  $a_{kj}^{pq}$  are real and the quadratic form  $a_{kj}^{pq}(x) \eta_{kp} \eta_{qj}$  is uniformly positive definite with respect to symmetric variables  $\eta_{kp} = \eta_{pk} \in \mathbb{R}$ , which implies that the principal homogeneous symbol of the operator  $A(x, \partial_x)$  with opposite sign,  $A(x, \xi) = [a_{kj}^{pq}(x) \xi_k \xi_j]_{3 \times 3}$  is uniformly positive definite, which for the real symmetric coefficients  $a_{kj}^{pq}$  means there are positive constants  $c_1$  and  $c_2$  such that

$$c_1 |\xi|^2 |\zeta|^2 \leq \bar{\zeta} \cdot A(x, \xi) \zeta \leq c_2 |\xi|^2 |\zeta|^2 \quad \forall x \in \mathbb{R}^3, \quad \forall \xi \in \mathbb{R}^3, \quad \forall \zeta \in \mathbb{C}^3. \tag{2.2}$$

Here,  $a \cdot b := a^\top b := \sum_{j=1}^3 a_j b_j$  is the bilinear product of two-column vectors  $a, b \in \mathbb{C}^3$ .

Further, let  $\Omega = \Omega^+$  be a bounded domain in  $\mathbb{R}^3$  with a simply connected boundary  $\partial\Omega = S \in C^\infty$ ,  $\bar{\Omega} = \Omega \cup S$ . Throughout the paper,  $n = (n_1, n_2, n_3)$  denotes the unit normal vector to  $S$  directed outward the domain  $\Omega$ . Set  $\Omega^- := \mathbb{R}^3 \setminus \bar{\Omega}$ .

By  $H^r(\Omega) = H_r^r(\Omega)$  and  $H^r(S) = H_r^r(S)$ ,  $r \in \mathbb{R}$ , we denote the Bessel potential spaces on a domain  $\Omega$  and on a closed manifold  $S$  without boundary, while  $\mathcal{D}(\mathbb{R}^3)$  and  $\mathcal{D}(\Omega)$  stand for  $C^\infty$  functions with compact support in  $\mathbb{R}^3$  and in  $\Omega$ , respectively, and  $S(\mathbb{R}^3)$  denotes the Schwartz space of rapidly decreasing functions in  $\mathbb{R}^3$ . Recall that  $H^0(\Omega) = L_2(\Omega)$  is a space of square integrable functions in  $\Omega$ . For a vector  $u = (u_1, u_2, u_3)^\top$ , the inclusion  $u = (u_1, u_2, u_3)^\top \in H^r$  means that each component  $u_j$  belongs to the space  $H^r$ .

Let us denote by  $\gamma^+ u$  and  $\gamma^- u$  the traces of  $u$  on  $S$  from the interior and exterior of  $\Omega^+$ , respectively.

We also need the following subspace of  $H^1(\Omega)$ , see, for example, [21],

$$H^{1,0}(\Omega; A) := \left\{ u = (u_1, u_2, u_3)^\top \in H^1(\Omega) : Au \in H^0(\Omega) \right\}. \tag{2.3}$$

The Dirichlet BVP reads as follows:

Find a vector function  $u = (u_1, u_2, u_3)^\top \in H^{1,0}(\Omega, A)$  satisfying the differential equation

$$Au = f \text{ in } \Omega \tag{2.4}$$

and the Dirichlet boundary condition

$$\gamma^+ u = \varphi_0 \text{ on } S, \tag{2.5}$$

where  $\varphi_0 = (\varphi_{01}, \varphi_{02}, \varphi_{03})^\top \in H^{1/2}(S)$  and  $f = (f_1, f_2, f_3)^\top \in H^0(\Omega)$  are given vector functions. Equation (2.4) is understood in the distributional sense, while the Dirichlet boundary condition (2.5) is understood in the usual trace sense.

The classical co-normal derivative operators,  $T^\pm$ , associated with the differential operator  $A(x, \partial_x)$ , are well defined in terms of the gradient traces on the boundary  $S$  for a sufficiently smooth vector function  $v$ , say  $v \in H^2(\Omega)$ , as follows:

$$\left[ T^\pm(x, \partial_x) v(x) \right]_p := a_{kj}^{pq}(x) n_k(x) \gamma^\pm \partial_{x_j} v_q(x), \quad x \in S, \quad p = 1, 2, 3. \tag{2.6}$$

The co-normal derivative operator defined in (2.6) can be extended by continuity to the space  $H^{1,0}(\Omega; A)$ . The extension is inspired by Green's first identity (cf. [3, 21, 22]) as follows:

$$\langle T^+ v, g \rangle_S := \int_\Omega [\gamma^{-1} g(x)] \cdot A(x, \partial_x) v(x) dx + \int_\Omega E(v(x), \gamma^{-1} g(x)) dx, \quad \forall g \in H^{1/2}(S), \quad \forall v \in H^{1,0}(\Omega; A), \tag{2.7}$$

where  $\langle \cdot, \cdot \rangle_S$  denotes the duality between the adjoint spaces  $H^{-1/2}(S)$  and  $H^{1/2}(S)$ , which extends the usual bilinear  $L_2(S)$  inner product, while  $E(v(x), u(x)) = a_{kj}^{pq}(x) [\partial_{x_j} v_q(x)] [\partial_{x_k} u_p(x)]$ . By  $\gamma^{-1}$ , we denote a (non-unique) continuous linear extension operator acting from  $H^{1/2}(S)$  into  $H^1(\mathbb{R}^3)$ . The restrictions of  $\gamma^{-1}$  on  $\Omega^+$  and  $\Omega^-$  are the right inverse operators to the corresponding trace operators  $\gamma^+$  and  $\gamma^-$ . Clearly, definition (2.7) does not depend on the extension operator.

Moreover, by [21, Lemma 3.4], and [3, Lemma 4.3], for any  $v \in H^{1,0}(\Omega; A)$  and  $u \in H^1(\Omega)$  the first Green identity holds in the form

$$\langle T^+ v, \gamma^+ u \rangle_S = \int_\Omega [u \cdot Av + E(v, u)] dx. \tag{2.8}$$

Remark 2.1

From condition (2.2), it follows that the quadratic form  $E(u(x), u(x))$  rewritten as

$$E(u(x), u(x)) = a_{kj}^{pq}(x) \varepsilon_{qj}(x) \varepsilon_{pk}(x)$$

where

$$\varepsilon_{qj}(x) = (\partial_j u_q(x) + \partial_q u_j(x)) / 2,$$

is positive definite in the symmetric variables  $\varepsilon_{qj}$ . Therefore, Green's first identity (2.8) and Korn's inequality along with the Lax–Milgram lemma imply that the Dirichlet BVP (2.4)–(2.5) is uniquely solvable in the space  $H^{1,0}(\Omega; A)$  (e.g. [1–3, 23]).

2.2. Parametrix-based operators and integral identities

As it has already been mentioned, our goal here is to develop the LBDIE method for the Dirichlet BVP (2.4)–(2.5).

Let  $F_\Delta(x) := -1/[4\pi|x|]$  denote the scalar fundamental solution of the Laplace operator,  $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ . Let us define a localized matrix parametrix for the the matrix operator  $l\Delta$  as

$$P(x) \equiv P_\chi(x) := P_\Delta(x) l = \chi(x) F_\Delta(x) l = -\frac{\chi(x)}{4\pi|x|} l \tag{2.9}$$

where  $P_\Delta(x) \equiv P_{\chi\Delta}(x) := \chi(x) F_\Delta(x)$  is a scalar function of the vector argument  $x$ ,  $l$  is the unit  $3 \times 3$  matrix and  $\chi$  is a localizing function (Appendix A)

$$\chi \in X_+^k, \quad k \geq 3, \quad \text{with } \chi(0) = 1, \tag{2.10}$$

Throughout the paper, we assume that condition (2.10) is satisfied if not otherwise stated. Note that the function  $\chi$  can have a compact support, which is useful for numerical implementations, but generally this is not necessary, and the class  $X_+^k$  include also the functions not compactly supported but sufficiently fast decreasing at infinity, see [24] and Appendix A for details.

For sufficiently smooth vector functions  $u$  and  $v$ , say  $u, v \in C^2(\overline{\Omega})$ , there holds Green's second identity

$$\int_{\Omega} [v \cdot A(x, \partial_x)u - u \cdot A(x, \partial_x)v] dx = \int_S [\gamma^+ v \cdot T^+ u - T^+ v \cdot \gamma^+ u] dS. \tag{2.11}$$

Denote by  $B(y, \varepsilon)$  a ball centred at point  $y$ , with radius  $\varepsilon > 0$ , and let  $\Sigma(y, \varepsilon) := \partial B(y, \varepsilon)$ . Let us take as  $v(x)$ , successively, the columns of the matrix  $P(x - y)$ , where  $y$  is an arbitrarily fixed interior point in  $\Omega$ , and write the identity (2.11) for the region  $\Omega_\varepsilon := \Omega \setminus B(y, \varepsilon)$  with  $\varepsilon > 0$  such that  $\overline{B(y, \varepsilon)} \subset \Omega$ . Keeping in mind that  $P^\top(x - y) = P(x - y)$  and  $[A(x, \partial_x)P(x - y)]^\top = [A(x, \partial_x)P(x - y)]$ , we arrive at the equality,

$$\begin{aligned} \int_{\Omega_\varepsilon} [P(x - y) A(x, \partial_x)u(x) - \{A(x, \partial_x)P(x - y)\} u(x)] dx = \\ \int_S [P(x - y) T^+(x, \partial_x)u(x) - \{T(x, \partial_x)P(x - y)\}^\top \gamma^+ u(x)] dS_x \\ - \int_{\Sigma(y, \varepsilon)} [P(x - y) T^+(x, \partial_x)u(x) - \{T(x, \partial_x)P(x - y)\}^\top \gamma^+ u(x)] dS_x. \end{aligned} \tag{2.12}$$

The normal vector on  $\Sigma(y, \varepsilon)$  is directed inward  $\Omega_\varepsilon$ .

Let the operator  $\mathcal{N}$  defined as

$$\mathcal{N} u(y) := \text{v.p.} \int_{\Omega} [A(x, \partial_x)P(x - y)] u(x) dx := \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} [A(x, \partial_x)P(x - y)] u(x) dx \tag{2.13}$$

be the Cauchy principal value singular integral operator, which is well defined if the limit in the right-hand side exists. The similar operator with integration over the whole space  $\mathbb{R}^3$  is denoted as

$$\mathbf{N} u(y) := \text{v.p.} \int_{\mathbb{R}^3} [A(x, \partial_x)P(x - y)] u(x) dx. \tag{2.14}$$

Note that

$$\frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x - y|} = -\frac{4\pi \delta_{kj}}{3} \delta(x - y) + \text{v.p.} \frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x - y|}, \tag{2.15}$$

where  $\delta_{kj}$  is the Kronecker delta, and  $\delta(\cdot)$  is the Dirac distribution, the left-hand side in (2.15) is also understood in the distributional sense, while the second summand in the right-hand side is a Cauchy-integrable function. Therefore, in view of (2.9) and taking into account that  $\chi(0) = 1$ , we can write the following equality in the distributional sense

$$\begin{aligned}
 [A(x, \partial_x)P(x-y)]_{pq} &= \left[ \frac{\partial}{\partial x_k} \left( a_{kj}^{pq}(x) \frac{\partial P_{rq}(x-y)}{\partial x_j} \right) \right] = \left[ \frac{\partial}{\partial x_k} \left( a_{kj}^{pq}(x) \delta_{rq} \frac{\partial P_{\Delta}(x-y)}{\partial x_j} \right) \right] \\
 &= a_{kj}^{pq}(x) \frac{\partial^2 P_{\Delta}(x-y)}{\partial x_k \partial x_j} + \frac{\partial a_{kj}^{pq}(x)}{\partial x_k} \frac{\partial P_{\Delta}(x-y)}{\partial x_j} = a_{kj}^{pq}(x) \left[ \frac{\delta_{kj}}{3} \delta(x-y) + \text{v.p.} \frac{\partial^2 P_{\Delta}(x-y)}{\partial x_k \partial x_j} \right] + \frac{\partial a_{kj}^{pq}(x)}{\partial x_k} \frac{\partial P_{\Delta}(x-y)}{\partial x_j} \\
 &= \beta_{pq}(x) \delta(x-y) + \text{v.p.} [A(x, \partial)P(x-y)]_{pq},
 \end{aligned} \tag{2.16}$$

where

$$\beta(x) = [\beta_{pq}(x)]_{p,q=1}^3, \quad \beta_{pq}(x) = \frac{1}{3} a_{kk}^{pq}(x), \tag{2.17}$$

$$\text{v.p.}[A(x, \partial_x)P(x-y)]_{pq} = \text{v.p.} \left[ -\frac{a_{kj}^{pq}(x)}{4\pi} \frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x-y|} \right] + R_{pq}(x, y) = \text{v.p.} \left[ -\frac{a_{kj}^{pq}(y)}{4\pi} \frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x-y|} \right] + R_{pq}^{(1)}(x, y), \tag{2.18}$$

$$R(x, y) = [R_{pq}(x, y)]_{p,q=1}^3, \quad R^{(1)}(x, y) = [R_{pq}^{(1)}(x, y)]_{p,q=1}^3, \tag{2.19}$$

$$\begin{aligned}
 R_{pq}(x, y) &:= -\frac{a_{kj}^{pq}(x)}{4\pi} \left\{ [\chi(x-y) - 1] \frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x-y|} + \frac{\partial^2 \chi(x-y)}{\partial x_k \partial x_j} \frac{1}{|x-y|} + \frac{\partial \chi(x-y)}{\partial x_j} \frac{\partial}{\partial x_k} \frac{1}{|x-y|} \right. \\
 &\quad \left. + \frac{\partial \chi(x-y)}{\partial x_k} \frac{\partial}{\partial x_j} \frac{1}{|x-y|} \right\} - \frac{1}{4\pi} \frac{\partial a_{kj}^{pq}(x)}{\partial x_k} \left[ \frac{\partial \chi(x-y)}{\partial x_j} \frac{1}{|x-y|} + \chi(x-y) \frac{\partial}{\partial x_j} \frac{1}{|x-y|} \right],
 \end{aligned} \tag{2.20}$$

$$R_{pq}^{(1)}(x, y) := R_{pq}(x, y) - \frac{a_{kj}^{pq}(x) - a_{kj}^{pq}(y)}{4\pi} \frac{\partial^2}{\partial x_k \partial x_j} \frac{1}{|x-y|}. \tag{2.21}$$

Clearly, the entries of the matrix functions  $R(x, y)$  and  $R^{(1)}(x, y)$  possess weak singularities of type  $\mathcal{O}(|x-y|^{-2})$  as  $x \rightarrow y$ .

Denote by  $\mathring{E}$  the extension operator by zero from  $\Omega$  onto  $\Omega^-$ . From the definitions (2.13) and (2.14), it is evident that

$$(\mathcal{N}u)(y) = (\mathbf{N}\mathring{E}u)(y) \quad \text{for } y \in \Omega, \quad u \in H^r(\Omega), \quad r \geq 0. \tag{2.22}$$

The definition of  $\mathcal{N}$  can be extended to smaller  $r$  as

$$(\mathcal{N}u)(y) := (\mathbf{N}\tilde{E}^r u)(y) \quad \text{for } y \in \Omega, \quad u \in H^r(\Omega), \quad -1/2 < r < 1/2, \tag{2.23}$$

where  $\tilde{E}^r : H^r(\Omega) \rightarrow \tilde{H}^r(\Omega)$  is the extension operator, uniquely defined for  $-1/2 < r < 1/2$ , see, for example, [22, Theorem 2.16]. For  $0 \leq r < 1/2$ ,  $\tilde{E}^r = \mathring{E}$ , and thus, the expressions (2.22) and (2.23) coincide for such  $r$ .

From decomposition (2.18), it follows that (e.g. [25], [1, Theorem 8.6.1]) if  $\chi \in X^k$  with integer  $k \geq 2$ , then

$$r_{\Omega} \mathcal{N} = r_{\Omega} \mathbf{N}\mathring{E} : H^r(\Omega) \rightarrow H^r(\Omega), \quad 0 \leq r, \tag{2.24}$$

$$r_{\Omega} \mathcal{N} = r_{\Omega} \mathbf{N}\tilde{E}^r : H^r(\Omega) \rightarrow H^r(\Omega), \quad -1/2 < r < 1/2, \tag{2.25}$$

are bounded because the principal homogeneous symbol of  $\mathbf{N}$  is rational ((4.2) in Section 4), and the operators with the kernel functions either  $R(x, y)$  or  $R_1(x, y)$  map  $H^r(\Omega)$  into  $H^{r+1}(\Omega)$  (cf. [24, Theorem 5.4]). Here and throughout the paper,  $r_{\Omega}$  denotes the restriction operator to  $\Omega$ .

Further, by direct calculations one can easily verify that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma(y, \varepsilon)} P(x-y) T(x, \partial_x) u(x) d\Sigma(y, \varepsilon) = 0, \tag{2.26}$$

$$\lim_{\varepsilon \rightarrow 0} \int_{\Sigma(y, \varepsilon)} \{T(x, \partial_x)P(x-y)\} u(x) d\Sigma(y, \varepsilon) = \left[ \frac{a_{kj}^{pq}(y)}{4\pi} \int_{\Sigma_1} \eta_k \eta_j d\Sigma_1 \right]_{3 \times 3} u(y) = \left[ \frac{a_{kj}^{pq}(y)}{4\pi} \frac{4\pi \delta_{kj}}{3} \right]_{3 \times 3} u(y) = \beta(y) u(y), \tag{2.27}$$

where  $\Sigma_1$  is a unit sphere,  $\eta = (\eta_1, \eta_2, \eta_3) \in \Sigma_1$  and  $\beta$  is defined by (2.17).

Passing to the limit in (2.12) as  $\varepsilon \rightarrow 0$  and using relations (2.13), (2.26), and (2.27), we obtain

$$\beta(y) u(y) + \mathcal{N}u(y) - V(T^+u)(y) + W(\gamma^+u)(y) = \mathcal{P}(Au)(y), \quad y \in \Omega, \tag{2.28}$$

where  $\mathcal{N}$  is a localized singular integral operator given by (2.13), while  $V$ ,  $W$  and  $\mathcal{P}$  are the localized vector single layer, double layer and Newtonian volume potentials,

$$Vg(y) := - \int_S P(x-y) g(x) dS_x, \tag{2.29}$$

$$Wg(y) := - \int_S [T(x, \partial_x)P(x-y)] g(x) dS_x, \tag{2.30}$$

$$\mathcal{P}h(y) := \int_{\Omega} P(x-y) h(x) dx. \tag{2.31}$$

Here, the densities  $g$  and  $h$  are three-dimensional vector functions. Introducing the following localized scalar Newtonian volume potential

$$\mathcal{P}_\Delta h_0(y) := \int_{\Omega} P_\Delta(x-y) h_0(x) dx \quad (2.32)$$

with  $h_0$  being a scalar density function, we evidently obtain,

$$[\mathcal{P}h(y)]_p = \mathcal{P}_\Delta h_p(y), \quad p = \overline{1,3},$$

for any vector function  $h = (h_1, h_2, h_3)^\top$ .

We will also need the localized vector Newtonian volume potential similar to (2.31) but with integration over the whole space  $\mathbb{R}^3$ ,

$$\mathbf{P}h(y) := \int_{\mathbb{R}^3} P(x-y) h(x) dx. \quad (2.33)$$

Mapping properties of potentials (2.29)–(2.33) are investigated in [15, 24] and provided in Appendix B.

We refer to relation (2.28) as Green's third identity. Because of the density of  $\mathcal{D}(\overline{\Omega})$  in  $H^{1,0}(\Omega; A)$  ([22, Theorem 3.12]) and the mapping properties of the potentials, Green's third identity (2.28) is valid also for  $u \in H^{1,0}(\Omega; A)$ . In this case, the co-normal derivative  $T^+u$  is understood in the sense of definition (2.7). In particular, (2.28) holds true for solutions of the previously formulated Dirichlet BVP (2.4)–(2.5).

On the other hand, applying the first Green identity (2.8) on  $\Omega_\varepsilon$  to  $u \in H^1(\Omega)$  and to  $P(x-y)$ , as  $v(x)$ , and taking the limit as  $\varepsilon \rightarrow 0$ , one can easily derive another, more general form of the third Green identity,

$$\beta(y) u(y) + \mathcal{N} u(y) + \mathcal{W}(\gamma^+ u)(y) = \mathcal{Q} u(y), \quad \forall y \in \Omega, \quad (2.34)$$

where for the  $p$ -th component of the vector  $\mathcal{Q} u(y)$ , we have

$$[\mathcal{Q} u(y)]_p := - \int_{\Omega} a_{kl}^{pq}(x) \frac{\partial P_\Delta(x-y)}{\partial x_k} \frac{\partial u_q(x)}{\partial x_l} dx = \partial_k \mathcal{P}_\Delta (a_{kl}^{pq} \partial_l u_q)(y), \quad \forall y \in \Omega. \quad (2.35)$$

Using the properties of localized potentials described in Appendix B (Theorems B.1 and B.4) and taking the trace of Equation (2.28) on  $S$ , we arrive at the relation for  $u \in H^{1,0}(\Omega^+; A)$ ,

$$\mathcal{N}^+ u - \mathcal{V}(T^+ u) + (\beta - \mu) \gamma^+ u + \mathcal{W}(\gamma^+ u) = \mathcal{P}^+(Au) \quad \text{on } S, \quad (2.36)$$

where the localized boundary integral operators  $\mathcal{V}$  and  $\mathcal{W}$  are generated by the localized single and double layer potentials and are defined in (B1) and (B2), the matrix  $\mu$  is defined by (B17), while

$$\mathcal{N}^+ := \gamma^+ \mathcal{N}, \quad \mathcal{P}^+ := \gamma^+ \mathcal{P}.$$

Now, we prove the following technical lemma.

**Lemma 2.2**

Let  $\chi \in X^3$ ,  $f \in H^0(\Omega)$ ,  $F \in H^{1,0}(\Omega, \Delta)$ ,  $\psi \in H^{-\frac{1}{2}}(S)$ , and  $\varphi \in H^{\frac{1}{2}}(S)$ . Moreover, let  $u \in H^1(\Omega)$  and the following equation hold

$$\beta(y) u(y) + \mathcal{N} u(y) - \mathcal{V} \psi(y) + \mathcal{W} \varphi(y) = F(y) + \mathcal{P} f(y), \quad y \in \Omega. \quad (2.37)$$

Then,  $u \in H^{1,0}(\Omega, A)$ .

**Proof**

Note that by Theorem B.1,  $\mathcal{P} f \in H^2(\Omega)$  for arbitrary  $f \in H^0(\Omega)$ , while by Theorem B.2, the inclusions  $\mathcal{V} \psi, \mathcal{W} \varphi \in H^{1,0}(\Omega, \Delta)$  hold for arbitrary  $\psi \in H^{-\frac{1}{2}}(S)$  and  $\varphi \in H^{\frac{1}{2}}(S)$ . In view of the relations (2.34)–(2.35), Equation (2.37) can be rewritten component-wise as

$$\partial_k \mathcal{P}_\Delta (a_{kl}^{pq} \partial_l u_q)(y) = F_p(y) + \mathcal{P}_\Delta f_p(y) + [\mathcal{V} \psi(y)]_p - [\mathcal{W}(\varphi - \gamma^+ u)(y)]_p, \quad y \in \Omega \quad p = \overline{1,3}. \quad (2.38)$$

By Theorems B.1 and B.2, it follows that the right-hand side function in the equality belongs to the space

$$H^{1,0}(\Omega, \Delta) := \{v \in H^1(\Omega) : \Delta v \in H^0(\Omega)\},$$

because  $\gamma^+ u \in H^{\frac{1}{2}}(S)$ , and therefore

$$\partial_k \mathcal{P}_\Delta (a_{kl}^{pq} \partial_l u_q) \in H^{1,0}(\Omega, \Delta). \quad (2.39)$$

We have

$$\Delta_x P_\Delta(x-y) = \delta(x-y) + R_\Delta(x-y), \quad (2.40)$$

where

$$R_{\Delta}(x-y) := -\frac{1}{4\pi} \left\{ \frac{\Delta \chi(x-y)}{|x-y|} + 2 \frac{\partial \chi(x-y)}{\partial x_l} \frac{\partial}{\partial x_l} \frac{1}{|x-y|} \right\}. \quad (2.41)$$

Clearly,  $R_{\Delta}(x-y) = \mathcal{O}(|x-y|^{-2})$  as  $x \rightarrow y$  and by (2.40) and (2.41), one can establish that for arbitrary scalar test function  $\phi \in \mathcal{D}(\Omega)$ , there holds the relation (e.g. [26])

$$\Delta \mathcal{P}_{\Delta} \phi(y) = \phi(y) + \mathcal{R}_{\Delta} \phi(y), \quad y \in \Omega, \quad (2.42)$$

where

$$\mathcal{R}_{\Delta} \phi(y) := \int_{\Omega} R_{\Delta}(x-y) \phi(x) dx. \quad (2.43)$$

Evidently (2.42) remains true also for  $\phi \in H^0(\Omega)$ , because  $\mathcal{D}(\Omega)$  is dense in  $H^0(\Omega)$ . It is easy to see that [24]

$$\mathcal{R}_{\Delta} : H^0(\Omega) \rightarrow H^1(\Omega). \quad (2.44)$$

Consequently,

$$\begin{aligned} \Delta [\partial_k \mathcal{P}_{\Delta} (a_{kl}^{pq} \partial_l u_q)(y)] &= \partial_k [\Delta_y \mathcal{P}_{\Delta} (a_{kl}^{pq} \partial_l u_q)(y)] = \partial_k [a_{kl}^{pq}(y) \partial_l u_q(y)] + \partial_k \mathcal{R}_{\Delta} (a_{kl}^{pq} \partial_l u_q)(y) \\ &= [A u(y)]_p + \partial_k \mathcal{R}_{\Delta} (a_{kl}^{pq} \partial_l u_q)(y), \quad y \in \Omega. \end{aligned} \quad (2.45)$$

Hence, the embedding  $Au \in H^0(\Omega)$  follows from (2.38) due to (2.39) and (2.44).  $\square$

Actually, the continuity of operator in (2.44) and identity (2.45) in the proof of Lemma 2.2 imply by (2.34) the following assertion.

*Corollary 2.3*

If  $\chi \in X^3$ , then the following operator is bounded,

$$\beta + \mathcal{N} : H^{1,0}(\Omega, A) \rightarrow H^{1,0}(\Omega, \Delta).$$

### 3. Localized boundary-domain integral equation formulation of the Dirichlet problem and the equivalence theorem

Let  $u \in H^{1,0}(\Omega, A)$  be a solution to the Dirichlet BVP (2.4)–(2.5) with  $\varphi_0 \in H^{\frac{1}{2}}(S)$  and  $f \in H^0(\Omega)$ . As we have derived earlier, there holds relations (2.28) and (2.36), which now can be rewritten in the form

$$(\beta + \mathcal{N})u - V\psi = \mathcal{P}f - W\varphi_0 \quad \text{in } \Omega, \quad (3.1)$$

$$\mathcal{N}^+ u - \mathcal{V}\psi = \mathcal{P}^+ f - (\beta - \mu)\varphi_0 - \mathcal{W}\varphi_0 \quad \text{on } S, \quad (3.2)$$

where  $\psi := T^+ u \in H^{-\frac{1}{2}}(S)$  and  $\mu$  is defined by (B17). One can consider these relations as an LBDIE system with respect to the unknown vector functions  $u$  and  $\psi$ . Now, we prove the following equivalence theorem.

*Theorem 3.1*

Let  $\chi \in X^3_+$ ,  $\varphi_0 \in H^{\frac{1}{2}}(S)$  and  $f \in H^0(\Omega)$ .

- (i) If a vector function  $u \in H^{1,0}(\Omega, A)$  solves the Dirichlet BVP (2.4)–(2.5), then the solution is unique and the pair  $(u, \psi) \in H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(S)$  with

$$\psi = T^+ u, \quad (3.3)$$

solves the LBDIE system (3.1)–(3.2).

- (ii) Vice versa, if a pair  $(u, \psi) \in H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(S)$  solves the LBDIE system (3.1)–(3.2), then the solution is unique and the vector function  $u$  solves the Dirichlet BVP (2.4)–(2.5), and relation (3.3) holds.

*Proof*

(i) The first part of the theorem is trivial and directly follows from the relations (2.28), (2.36), (3.3) and Remark 2.1.

(ii) Now, let a pair  $(u, \psi) \in H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(S)$  solve the LBDIE system (3.1)–(3.2). Taking the trace of (3.1) on  $S$  and comparing it with (3.2), we get

$$\gamma^+ u = \varphi_0 \quad \text{on } S. \quad (3.4)$$

Further, because  $u \in H^{1,0}(\Omega, A)$ , we can write Green's third identity (2.28), which in view of (3.4) can be rewritten as

$$(\beta + \mathcal{N})u - V(T^+ u) = \mathcal{P}(Au) - W\varphi_0 \quad \text{in } \Omega. \quad (3.5)$$

From (3.1) and (3.5), it follows that

$$V(T^+ u - \psi) + \mathcal{P}(Au - f) = 0 \quad \text{in } \Omega. \quad (3.6)$$

Hence, by Lemma 6.3 in [24], we have

$$Au = f \text{ in } \Omega \text{ and } T^+u = \psi \text{ on } S.$$

Thus,  $u$  solves the Dirichlet BVP (2.4)–(2.5) and Equation (3.3) holds.

The uniqueness of solution to the LBDIE system (3.1)–(3.2) in the space  $H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(S)$  directly follows from the previously proved equivalence result and the uniqueness theorem for the Dirichlet problem (2.4)–(2.5) (Remark 2.1).  $\square$

#### 4. Symbols and invertibility of a domain operator in the half-space

In what follows in our analysis, we need the explicit expression of the principal homogeneous symbol matrix  $\mathfrak{S}(\mathcal{N})(y, \xi)$  of the singular integral operator  $\mathcal{N}$ , which due to (2.13), (2.14) and (2.18) reads as

$$\begin{aligned} [\mathfrak{S}(\mathcal{N})(y, \xi)]_{pq} &= [\mathfrak{S}(\mathbf{N})(y, \xi)]_{pq} = \mathcal{F}_{z \rightarrow \xi} \left[ -v.p. \frac{a_{kl}^{pq}(y)}{4\pi} \frac{\partial^2}{\partial z_k \partial z_l} \frac{1}{|z|} \right] = -\frac{a_{kl}^{pq}(y)}{4\pi} \mathcal{F}_{z \rightarrow \xi} \left[ v.p. \frac{\partial^2}{\partial z_k \partial z_l} \frac{1}{|z|} \right] \\ &= -\frac{a_{kl}^{pq}(y)}{4\pi} \mathcal{F}_{z \rightarrow \xi} \left[ \frac{4\pi}{3} \delta_{kl} \delta(z) + \frac{\partial^2}{\partial z_k \partial z_l} \frac{1}{|z|} \right] = -\beta_{pq}(y) - a_{kl}^{pq}(y)(-i\xi_k)(-i\xi_l) \mathcal{F}_{z \rightarrow \xi} \left[ \frac{1}{4\pi|z|} \right] \\ &= \frac{A_{pq}(y, \xi)}{|\xi|^2} - \beta_{pq}(y), \quad y \in \overline{\Omega}, \quad \xi \in \mathbb{R}^3, \end{aligned} \tag{4.1}$$

where

$$A_{pq}(y, \xi) = a_{kl}^{pq}(y) \xi_k \xi_l, \quad p, q = 1, 2, 3,$$

while the Fourier transform operator  $\mathcal{F}$  is defined as

$$\mathcal{F}g(\xi) = \mathcal{F}_{z \rightarrow \xi}[g(z)] = \int_{\mathbb{R}^3} g(z) e^{iz \cdot \xi} dz.$$

Here, we have applied that  $\mathcal{F}_{z \rightarrow \xi}[(4\pi|z|)^{-1}] = |\xi|^{-2}$  (e.g. [27]).

As we see, the entries of principal homogeneous symbol matrix  $\mathfrak{S}(\mathcal{N})(y, \xi)$  of the operator  $\mathcal{N}$  are even rational homogeneous functions in  $\xi$  of order 0. It can easily be verified that both the characteristic function of the singular kernel in (2.18) and the symbol (4.1) satisfy the Tricomi condition, that is, their integral averages over the unit sphere vanish (cf. [26]).

Relation (4.1) implies that the principal homogeneous symbols of the singular integral operators  $\mathbf{N}$  and  $\beta + \mathbf{N}$  read as

$$\mathfrak{S}(\mathbf{N})(y, \xi) = |\xi|^{-2}A(y, \xi) - \beta \quad \forall y \in \overline{\Omega}, \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}, \tag{4.2}$$

$$\mathfrak{S}(\beta + \mathbf{N})(y, \xi) = |\xi|^{-2}A(y, \xi) \quad \forall y \in \overline{\Omega}, \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}. \tag{4.3}$$

Because of (2.2), the symbol matrix (4.3) is positive definite,

$$[\mathfrak{S}(\beta + \mathbf{N})(y, \xi)\zeta] \cdot \bar{\zeta} = |\xi|^{-2} \bar{\zeta} \cdot A(y, \xi)\zeta \geq c_1 |\zeta|^2 \quad \forall y \in \overline{\Omega}, \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}, \quad \forall \zeta \in \mathbb{C}^3,$$

where  $c_1$  is the same positive constant as in (2.2).

Denote

$$\mathbf{B} := \beta + \mathbf{N}.$$

By (4.3), the principal homogeneous symbol matrix of the operator  $\mathbf{B}$  reads as

$$\mathfrak{S}(\mathbf{B})(y, \xi) = |\xi|^{-2}A(y, \xi) \quad \text{for } y \in \overline{\Omega}, \quad \xi \in \mathbb{R}^3 \setminus \{0\}, \tag{4.4}$$

is an even rational homogeneous matrix function of order 0 in  $\xi$  and due to (2.2) it is positive definite,

$$[\mathfrak{S}(\mathbf{B})(y, \xi)\zeta] \cdot \bar{\zeta} \geq c_1 |\zeta|^2 \quad \text{for all } y \in \overline{\Omega}, \quad \xi \in \mathbb{R}^3 \setminus \{0\} \text{ and } \zeta \in \mathbb{C}^3.$$

Consequently,  $\mathbf{B}$  is a strongly elliptic pseudo-differential operator of zero order (i.e. Cauchy-type singular integral operator) and the partial indices of factorization of the symbol (4.4) equal to zero (cf. [28–30]).

We need some auxiliary assertions in our further analysis. To formulate them, let  $\tilde{y} \in S = \partial\Omega$  be some fixed point and consider the frozen symbol  $\mathfrak{S}(\tilde{\mathbf{B}})(\tilde{y}, \xi) \equiv \mathfrak{S}(\tilde{\mathbf{B}})(\xi)$ , where  $\tilde{\mathbf{B}}$  denotes the operator  $\mathbf{B}$  written in chosen local co-ordinate system. Further, let  $\widehat{\tilde{\mathbf{B}}}$  denote the pseudo-differential operator with the symbol

$$\widehat{\tilde{\mathbf{B}}}(\xi', \xi_3) := \mathfrak{S}(\tilde{\mathbf{B}})\left((1 + |\xi'|)\omega, \xi_3\right), \quad \text{where } \omega = \frac{\xi'}{|\xi'|}, \quad \xi = (\xi', \xi_3), \quad \xi' = (\xi_1, \xi_2).$$



Then, the frozen principal homogeneous symbol matrix  $\mathfrak{S}(\tilde{\mathbf{B}})(\xi)$  is also the principal homogeneous symbol matrix of the operator  $\widehat{\mathbf{B}}$ . It can be factorized with respect to the variable  $\xi_3$  as

$$\mathfrak{S}(\tilde{\mathbf{B}})(\xi) = \mathfrak{S}^{(-)}(\tilde{\mathbf{B}})(\xi) \mathfrak{S}^{(+)}(\tilde{\mathbf{B}})(\xi), \tag{4.5}$$

where

$$\mathfrak{S}^{(\pm)}(\tilde{\mathbf{B}})(\xi) = \frac{1}{\Theta^{(\pm)}(\xi', \xi_3)} \tilde{A}^{(\pm)}(\xi', \xi_3). \tag{4.6}$$

Here,  $\Theta^{(\pm)}(\xi', \xi_3) := \xi_3 \pm i|\xi'|$  are the 'plus' and 'minus' factors of the symbol  $\Theta(\xi) := |\xi|^2$ , and  $\tilde{A}^{(\pm)}(\xi', \xi_3)$  are the 'plus' and 'minus' polynomial matrix factors of the first order in  $\xi_3$  of the positive definite polynomial symbol matrix  $\tilde{A}(\xi', \xi_3) \equiv \tilde{A}(\tilde{y}, \xi', \xi_3)$  corresponding to the frozen differential operator  $A(\tilde{y}, \partial_x)$  at the point  $\tilde{y} \in S$  [31–33], that is,

$$\tilde{A}(\xi', \xi_3) = \tilde{A}^{(-)}(\xi', \xi_3) \tilde{A}^{(+)}(\xi', \xi_3) \tag{4.7}$$

with  $\det \tilde{A}^{(+)}(\xi', \tau) \neq 0$  for  $\text{Im} \tau > 0$  and  $\det \tilde{A}^{(-)}(\xi', \tau) \neq 0$  for  $\text{Im} \tau < 0$ . Moreover, the entries of the matrices  $\tilde{A}^{(\pm)}(\xi', \xi_3)$  are homogeneous functions in  $\xi = (\xi', \xi_3)$  of order 1.

Denote, by  $a^{(\pm)}(\xi')$ , the coefficients at  $\xi_3^3$  in the determinants  $\det \tilde{A}^{(\pm)}(\xi', \xi_3)$ . Evidently,

$$a^{(-)}(\xi') a^{(+)}(\xi') = \det \tilde{A}(0, 0, 1) > 0 \quad \text{for } \xi' \neq 0. \tag{4.8}$$

It is easy to see that the factor matrices  $\tilde{A}^{(\pm)}(\xi', \xi_3)$  have the following structure:

$$\left( \left[ \tilde{A}^{(\pm)}(\xi', \xi_3) \right]^{-1} \right)_{ij} = \frac{1}{\det \tilde{A}^{(\pm)}(\xi', \xi_3)} p_{ij}^{(\pm)}(\xi', \xi_3), \quad i, j = 1, 2, 3,$$

where  $p_{ij}^{(\pm)}(\xi', \xi_3)$  are the co-factors of the matrix  $\tilde{A}^{(\pm)}(\xi', \xi_3)$ , which can be written in the form

$$p_{ij}^{(\pm)}(\xi', \xi_3) = c_{ij}^{(\pm)}(\xi') \xi_3^2 + b_{ij}^{(\pm)}(\xi') \xi_3 + d_{ij}^{(\pm)}(\xi'). \tag{4.9}$$

Here,  $c_{ij}^{(\pm)}$ ,  $b_{ij}^{(\pm)}$  and  $d_{ij}^{(\pm)}$ ,  $i, j = 1, 2, 3$ , are homogeneous functions in  $\xi'$  of order 0, 1 and 2, respectively.

From the previously mentioned, it follows that the entries of the factor-symbol matrices  $b_{kj}^{(\pm)}(\omega, r, \xi_3) := \mathfrak{S}_{kj}^{(\pm)}(\tilde{\mathbf{B}})(\xi', \xi_3)$ ,  $k, j = 1, 2, 3$ , with  $\omega = \xi' / |\xi'|$  and  $r = |\xi'|$ , satisfy the following relations:

$$\frac{\partial^l b_{kj}^{(\pm)}(\omega, 0, -1)}{\partial r^l} = (-1)^l \frac{\partial^l b_{kj}^{(\pm)}(\omega, 0, +1)}{\partial r^l}, \quad l = 0, 1, 2, \dots \tag{4.10}$$

These relations imply that the entries of the matrices  $\mathfrak{S}^{(\pm)}(\tilde{\mathbf{B}})(\xi', \xi_3)$  belong to the class of symbols  $D_0$  introduced in [27], Ch. III, § 10,

$$\mathfrak{S}^{(\pm)}(\tilde{\mathbf{B}})(\xi', \xi_3) \in D_0. \tag{4.11}$$

Denote by  $\Pi^\pm$  the Cauchy-type integral operators

$$\Pi^\pm h(\xi) := \pm \frac{i}{2\pi} \lim_{t \rightarrow 0^+} \int_{-\infty}^{+\infty} \frac{h(\xi', \eta_3) d\eta_3}{\xi_3 \pm it - \eta_3}, \tag{4.12}$$

which are well defined at any  $\xi \in \mathbb{R}^3$  for a bounded smooth function  $h(\xi', \cdot)$  satisfying the relation  $h(\xi', \eta_3) = \mathcal{O}(1 + |\eta_3|)^{-\kappa}$  with some  $\kappa > 0$ .

Let  $\hat{E}_+$  be the extension operator by zero from  $\mathbb{R}_+^3$  onto the whole space  $\mathbb{R}^3$  and  $r_+ := r|_{\mathbb{R}_+^3} : H^s(\mathbb{R}^3) \rightarrow H^s(\mathbb{R}_+^3)$  be the restriction operator to the half-space  $\mathbb{R}_+^3$ . First, we prove the following assertion.

**Lemma 4.1**

Let  $s \geq 0$  and  $\chi \in X_+^k$  with integer  $k \geq 2$ . The operator

$$r_+ \widehat{\mathbf{B}} \hat{E}_+ : H^s(\mathbb{R}_+^3) \rightarrow H^s(\mathbb{R}_+^3)$$

is invertible.

Moreover, for  $f \in H^s(\mathbb{R}_+^3)$ , the unique solution of the equation

$$r_+ \widehat{\mathbf{B}} \hat{E}_+ u = f \tag{4.13}$$



for  $u \in H^s(\mathbb{R}_+^3)$  can be represented in the form  $u = r_+ u_+$ , where

$$u_+ = \overset{\circ}{E} u = \mathcal{F}^{-1} \left\{ [\widehat{\mathfrak{G}}^{(+)}(\tilde{\mathbf{B}})]^{-1} \Pi^+ \left( [\widehat{\mathfrak{G}}^{(-)}(\tilde{\mathbf{B}})]^{-1} \mathcal{F}(f_*) \right) \right\},$$

and  $f_* \in H^s(\mathbb{R}^3)$  is an extension of  $f \in H^s(\mathbb{R}_+^3)$  (i.e.  $r_+ f_* = f$ ) such that  $\|f_*\|_{H^s(\mathbb{R}^3)} = \|f\|_{H^s(\mathbb{R}_+^3)}$ .

*Proof*

First, we show that if  $f \in H^0(\mathbb{R}_+^3)$ , then Equation (4.13) is uniquely solvable in the space  $H^0(\mathbb{R}_+^3)$ . Let  $u \in H^0(\mathbb{R}_+^3)$  be a solution of this equation, and let us denote

$$u_- := f_* - \widehat{\mathbf{B}} u_+, \tag{4.14}$$

where  $u_+ := \overset{\circ}{E} u \in \tilde{H}^0(\mathbb{R}_+^3)$  and  $f_* \in H^0(\mathbb{R}^3)$  is an arbitrary extension of  $f \in H^0(\mathbb{R}_+^3)$  onto  $\mathbb{R}^3$  such that  $\|f_*\|_{H^0(\mathbb{R}^3)} = \|f\|_{H^0(\mathbb{R}_+^3)}$ .

Because  $f_* \in H^0(\mathbb{R}^3)$  and  $\widehat{\mathbf{B}} u_+ \in H^0(\mathbb{R}^3)$ , we have  $u_- \in H^0(\mathbb{R}^3)$ . In addition,  $u_- \in \tilde{H}^0(\mathbb{R}_-^3)$ .

The Fourier transform of (4.14) leads to the following relation

$$\widehat{\mathfrak{G}}(\tilde{\mathbf{B}})(\xi) \mathcal{F}(u_+) + \mathcal{F}(u_-)(\xi) = \mathcal{F}(f_*)(\xi). \tag{4.15}$$

Because of (4.5), we have the following factorization

$$\widehat{\mathfrak{G}}(\tilde{\mathbf{B}})(\xi', \xi_3) = \widehat{\mathfrak{G}}^{(-)}(\tilde{\mathbf{B}})(\xi', \xi_3) \widehat{\mathfrak{G}}^{(+)}(\tilde{\mathbf{B}})(\xi', \xi_3), \tag{4.16}$$

where  $\widehat{\mathfrak{G}}^{(\pm)}(\tilde{\mathbf{B}})(\xi', \xi_3) = \mathfrak{G}^{(\pm)}(\tilde{\mathbf{B}}) \left( (1 + |\xi'|) \omega, \xi_3 \right)$  with  $\omega = \frac{\xi'}{|\xi'|}$ . Substituting (4.16) into (4.15) and multiplying both sides by  $[\widehat{\mathfrak{G}}^{(-)}(\tilde{\mathbf{B}})]^{-1}$ , we get

$$\widehat{\mathfrak{G}}^{(+)}(\tilde{\mathbf{B}})(\xi) \mathcal{F}(u_+)(\xi) + \left[ \widehat{\mathfrak{G}}^{(-)}(\tilde{\mathbf{B}})(\xi) \right]^{-1} \mathcal{F}(u_-)(\xi) = \left[ \widehat{\mathfrak{G}}^{(-)}(\tilde{\mathbf{B}})(\xi) \right]^{-1} \mathcal{F}(f_*)(\xi). \tag{4.17}$$

Introduce the notations

$$v_+(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left( \widehat{\mathfrak{G}}^{(+)}(\tilde{\mathbf{B}})(\xi) \mathcal{F}(u_+)(\xi) \right), \tag{4.18}$$

$$v_-(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left( \left[ \widehat{\mathfrak{G}}^{(-)}(\tilde{\mathbf{B}})(\xi) \right]^{-1} \mathcal{F}(u_-)(\xi) \right), \tag{4.19}$$

$$g(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left( \left[ \widehat{\mathfrak{G}}^{(-)}(\tilde{\mathbf{B}})(\xi) \right]^{-1} \mathcal{F}(f_*)(\xi) \right). \tag{4.20}$$

Then, we can conclude that ([27], Theorem 4.4 and Lemmas 20.2, 20.5)

$$v_+ \in \tilde{H}^0(\mathbb{R}_+^3), \quad v_- \in \tilde{H}^0(\mathbb{R}_-^3), \quad g \in H^0(\mathbb{R}^3), \tag{4.21}$$

because the degrees of homogeneity of  $\mathfrak{G}^{(+)}(\tilde{\mathbf{B}})(\xi)$  and  $\mathfrak{G}^{(-)}(\tilde{\mathbf{B}})(\xi)$  equal to 0.

In terms of notations (4.18)–(4.20), Equation (4.17) acquires the form

$$\mathcal{F}(v_+)(\xi) + \mathcal{F}(v_-)(\xi) = \mathcal{F}(g)(\xi). \tag{4.22}$$

In accordance with Lemma 5.4 in [27], we conclude that the representation of the vector function  $\mathcal{F}(g)(\xi)$  in the form (4.22) is unique in view of inclusions (4.21), which in turn leads to the following relations:

$$\mathcal{F}(v_+) = \Pi^+ \mathcal{F}(g), \quad \mathcal{F}(v_-) = \Pi^- \mathcal{F}(g). \tag{4.23}$$

Now, from (4.18), (4.20) and the first equation in (4.23), it follows that  $u_+ \in \tilde{H}^0(\mathbb{R}_+^3)$  is representable in the form

$$u_+ = \mathcal{F}^{-1} \left\{ [\widehat{\mathfrak{G}}^{(+)}(\tilde{\mathbf{B}})]^{-1} \Pi^+ \left( \left[ \widehat{\mathfrak{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) \right\}. \tag{4.24}$$

Evidently, for the solution  $u \in H^0(\mathbb{R}_+^3)$  of Equation (4.13), then we get the following representation

$$u = r_+ \mathcal{F}^{-1} \left\{ [\widehat{\mathfrak{G}}^{(+)}(\tilde{\mathbf{B}})]^{-1} \Pi^+ \left( \left[ \widehat{\mathfrak{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) \right\}. \tag{4.25}$$

Note that the representation (4.25) does not depend on the choice of the extension  $f_*$ . Indeed, let  $f_{*1} \in H^0(\mathbb{R}^3)$  be another extension of  $f \in H^0(\mathbb{R}^3_+)$ , that is,  $r_+ f_{*1} = f$ . Because  $f_- = f_* - f_{*1} \in \tilde{H}^0(\mathbb{R}^3_-)$ , it follows that ([27], Theorem 4.4, Lemmas 20.2 and 20.5)

$$\mathcal{F}^{-1} \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_-) \right) \in \tilde{H}^0(\mathbb{R}^3_-),$$

while

$$\Pi^+ \left\{ \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_-) \right\} = \mathcal{F} \left\{ \theta^+ \mathcal{F}^{-1} \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_-) \right) \right\} = 0$$

(cf. [27], Lemma 5.2). Here,  $\theta^+$  denotes the multiplication operator by the Heaviside step function  $\theta(x_3)$  that is equal to 1 for  $x_3 > 0$  and vanishes for  $x_3 < 0$ . Therefore,

$$\Pi^+ \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) = \Pi^+ \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_{*1}) \right)$$

and the claim follows. If, in particular,  $f = 0$ , then  $f_* = 0$ , and hence,  $u = 0$  by virtue of (4.24). Thus, Equation 4.13 possesses at most one solution in the space  $H^0(\mathbb{R}^3_+)$ .

Further, we show that the function

$$u = r_+ \mathcal{F}^{-1} \left\{ \left[ \hat{\mathcal{G}}^{(+)}(\tilde{\mathbf{B}}) \right]^{-1} \Pi^+ \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) \right\} \tag{4.26}$$

is a solution of Equation (4.13) for any  $f \in H^0(\mathbb{R}^3_+)$ . To this end, let us first note that for the vector function under the restriction operator in (4.26), the following embedding holds

$$\mathcal{F}^{-1} \left\{ \left[ \hat{\mathcal{G}}^{(+)}(\tilde{\mathbf{B}}) \right]^{-1} \Pi^+ \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) \right\} \in \tilde{H}^0(\mathbb{R}^3_+). \tag{4.27}$$

Indeed, by Lemma 5.2 in [27], we have

$$\mathcal{F}^{-1} \left\{ \left[ \hat{\mathcal{G}}^{(+)}(\tilde{\mathbf{B}}) \right]^{-1} \Pi^+ \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) \right\} = \mathcal{F}^{-1} \left\{ \left[ \hat{\mathcal{G}}^{(+)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F} \left[ \theta^+ \mathcal{F}^{-1} \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) \right] \right\}$$

and (4.27) follows from Theorem 4.4, Lemmas 20.2 and 20.5 in [27]. From (4.26) and (4.27), we obtain

$$u_+ := \mathring{E}_+ u = \mathcal{F}^{-1} \left\{ \left[ \hat{\mathcal{G}}^{(+)}(\tilde{\mathbf{B}}) \right]^{-1} \Pi^+ \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) \right\}. \tag{4.28}$$

By the relation

$$\Pi^+ \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) = \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) - \Pi^- \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right)$$

(cf. Lemma 5.4 in [27]), we get from equality (4.28),

$$\hat{\mathcal{G}}(\tilde{\mathbf{B}}) \mathcal{F}(u_+) = \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \Pi^+ \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) = \mathcal{F}(f_*) - \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \Pi^- \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right).$$

Because

$$\mathcal{F}^{-1} \left\{ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \Pi^- \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) \right\} \in \tilde{H}^0(\mathbb{R}^3_-),$$

(cf. [27], Theorems 4.4, 5.1, Lemmas 20.2, 20.5), we easily derive

$$r_+ \hat{\mathbf{B}} u_+ = r_+(f_*) - r_+ \mathcal{F}^{-1} \left\{ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \Pi^- \left( \left[ \hat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) \right\} = r_+(f_*) = f,$$

that is, the vector function (4.26) solves Equation (4.13) and belongs to the space  $H^0(\mathbb{R}^3_+)$  for  $f \in H^0(\mathbb{R}^3_+)$ .

In what follows, we prove that for  $f \in H^s(\mathbb{R}^3_+)$  and  $f_* \in H^s(\mathbb{R}^3)$  such that

$$\|f_*\|_{H^s(\mathbb{R}^3)} = \|f\|_{H^s(\mathbb{R}^3_+)} \text{ for } s \geq 0, \tag{4.29}$$

the vector function defined by (4.26) satisfies the inequality

$$\|u\|_{H^s(\mathbb{R}_+^3)} \leq C \|f\|_{H^s(\mathbb{R}_+^3)}, \tag{4.30}$$

and hence belongs to  $H^s(\mathbb{R}_+^3)$ . Indeed, because (by Lemma 5.2 and Theorem 5.1 in [27])

$$\Pi^+(\mathcal{F}g) = \mathcal{F}(\theta^+g) \text{ for all } g \in H^0(\mathbb{R}^3),$$

then representation (4.28) of  $u_+$  can be rewritten as

$$u_+ = \mathcal{F}^{-1} \left\{ \left[ \widehat{\mathcal{G}}^{(+)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F} \left[ \theta^+ \mathcal{F}^{-1} \left( \left[ \widehat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) \right] \right\}.$$

Therefore, using (4.29) and in view of (4.11), from Theorem 10.1, Lemmas 4.4, 20.2, and 20.5 in [27], we finally derive

$$\|u\|_{H^s(\mathbb{R}_+^3)} \leq c_1 \left\| \mathcal{F}^{-1} \left( \left[ \widehat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) \right\|_{H^s(\mathbb{R}^3)} \leq c_1 \left\| \mathcal{F}^{-1} \left( \left[ \widehat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(f_*) \right) \right\|_{H^s(\mathbb{R}^3)} \leq c_2 \|f_*\|_{H^s(\mathbb{R}^3)} = c_2 \|f\|_{H^s(\mathbb{R}_+^3)}$$

with some positive constants  $c_1$  and  $c_2$ , hence (4.30) follows. □

**Lemma 4.2**

Let the factor matrix  $\tilde{A}^{(+)}(\xi', \tau)$  be as in (4.7), and  $a^{(+)}$  and  $c_{ij}^{(+)}$  be as in (4.8) and (4.9), respectively. Then, the following equality holds

$$\frac{1}{2\pi i} \int_{\Gamma^-} \left[ \tilde{A}^{(+)}(\xi', \tau) \right]^{-1} d\tau = \frac{1}{a^{(+)}(\xi')} C^{(+)}(\xi'), \tag{4.31}$$

where  $C^{(+)}(\xi') = \left[ c_{ij}^{(+)}(\xi') \right]_{ij=1}^3$  and  $\det \left[ C^{(+)}(\xi') \right] \neq 0$  for  $\xi' \neq 0$ . Here  $\Gamma^-$  is a contour in the lower complex half-plane enclosing all the roots of the polynomial  $\det \tilde{A}^{(+)}(\xi', \tau)$  with respect to  $\tau$ .

*Proof*

Note that  $\det \tilde{A}^{(+)}(\xi', \tau)$  is a third order polynomial in  $\tau$ , while  $p_{ij}^{(+)}(\xi', \tau)$  is a second-order polynomial in  $\tau$  defined in (4.9).

Let  $\Gamma_R$  be a circle centred at the origin and having sufficiently large radius  $R$ . By the Cauchy theorem, then we derive

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma^-} \left\{ \left[ \tilde{A}^{(+)}(\xi', \tau) \right]^{-1} \right\}_{ij} d\tau &= \frac{1}{2\pi i} \int_{\Gamma^-} \frac{p_{ij}^{(+)}(\xi', \tau)}{\det \tilde{A}^{(+)}(\xi', \tau)} d\tau = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{p_{ij}^{(+)}(\xi', \tau)}{\det \tilde{A}^{(+)}(\xi', \tau)} d\tau \\ &= \frac{1}{2\pi i} \frac{c_{ij}^{(+)}(\xi')}{a^{(+)}(\xi')} \int_{\Gamma_R} \frac{1}{\tau} d\tau + \int_{\Gamma_R} Q_{ij}(\xi', \tau) d\tau = \frac{c_{ij}^{(+)}(\xi')}{a^{(+)}(\xi')} + \int_{\Gamma_R} Q_{ij}(\xi', \tau) d\tau, \end{aligned} \tag{4.32}$$

where  $Q_{ij}(\xi', \tau) = O(|\tau|^{-2})$  as  $|\tau| \rightarrow \infty$ .

It is clear that

$$\lim_{R \rightarrow \infty} \int_{\Gamma_R} Q_{ij}(\xi', \tau) d\tau = 0.$$

Therefore, by passing to the limit in (4.32) as  $R \rightarrow \infty$ , we obtain

$$\frac{1}{2\pi i} \int_{\Gamma^-} \left\{ \left[ \tilde{A}^{(+)}(\xi', \tau) \right]^{-1} \right\}_{ij} d\tau = \frac{c_{ij}^{(+)}(\xi')}{a^{(+)}(\xi')}.$$

Now, we show that  $\det \left[ C^{(+)} \right] \neq 0$ . We introduce the notations

$$P^{(+)}(\xi', \xi_3) = \left[ p_{ij}^{(+)}(\xi', \xi_3) \right]_{ij=1}^3 = C^{(+)}(\xi') \xi_3^2 + B^{(+)}(\xi') \xi_3 + D^{(+)}(\xi'),$$

where

$$B^{(+)}(\xi') = \left[ b_{ij}^{(+)}(\xi') \right]_{ij=1}^3 \text{ and } D^{(+)}(\xi') = \left[ d_{ij}^{(+)}(\xi') \right]_{ij=1}^3.$$

Because  $\det \left[ \tilde{A}^{(+)}(\xi', \xi_3) \right]^{-1} \neq 0$  for  $\xi = (\xi', \xi_3) \neq 0$ ; therefore,  $\det P^{(+)}(\xi', \xi_3) \neq 0$  for  $\xi = (\xi', \xi_3) \neq 0$ .

Let us introduce new coordinates  $r = |\xi'|$ ,  $\omega = \xi' / |\xi'|$  and denote

$$\mathcal{P}^{(+)}(\omega, r, \xi_3) := P^{(+)}(\xi', \xi_3) = P^{(+)}(\omega r, \xi_3).$$

Then, we have

$$\det \mathcal{P}^{(+)}(\omega, r, \xi_3) = \det P^{(+)}(\xi', \xi_3) = \det \left( C^{(+)}(\omega) \xi_3^2 + B^{(+)}(\omega) \xi_3 r + D^{(+)}(\omega) r^2 \right) \neq 0 \text{ for all } \xi_3 \neq 0. \quad (4.33)$$

Hence,

$$\lim_{r \rightarrow 0} \det \mathcal{P}^{(+)}(\omega, r, \xi_3) = \xi_3^6 \det C^{(+)}(\omega),$$

consequently  $\det C^{(+)}(\omega) \neq 0$  and Lemma 4.2 is proved. □

For further use, let us introduce the auxiliary operator  $\Pi'$  defined as

$$\Pi'(g)(\xi') := \lim_{x_3 \rightarrow 0+} r_{\mathbb{R}_+^3} \mathcal{F}_{\xi_3 \rightarrow x_3}^{-1} [g(\xi', \xi_3)] = \frac{1}{2\pi} \lim_{x_3 \rightarrow 0+} \int_{-\infty}^{+\infty} g(\xi', \xi_3) e^{-ix_3 \xi_3} d\xi_3 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\xi', \xi_3) d\xi_3 \text{ for } g(\xi', \cdot) \in L_1(\mathbb{R}).$$

The operator  $\Pi'$  can be extended to the class of functions  $g(\xi', \xi_3)$  that are rational in  $\xi_3$  with the denominator not vanishing for real non-zero  $\xi = (\xi', \xi_3) \in \mathbb{R}^3 \setminus \{0\}$ , homogeneous of order  $m \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$  in  $\xi$  and infinitely differentiable with respect to  $\xi$  for  $\xi' \neq 0$ . Then, one can show that (cf. Appendix C in [15])

$$\Pi'(g)(\xi') = \lim_{x_3 \rightarrow 0+} r_{\mathbb{R}_+} \mathcal{F}_{\xi_3 \rightarrow x_3}^{-1} [g(\xi', \xi_3)] = -\frac{1}{2\pi} \int_{\Gamma^-} g(\xi', \zeta) d\zeta, \quad (4.34)$$

where  $r_{\mathbb{R}_+}$  denotes the restriction operator onto  $\mathbb{R}_+ = (0, +\infty)$  with respect to  $x_3$ ,  $\Gamma^-$  is a contour in the lower complex half-plane in  $\zeta$ , orientated anticlockwise and enclosing all the poles of the rational function  $g(\xi', \cdot)$ . It is clear that if  $g(\xi', \zeta)$  is holomorphic in  $\zeta$  in the lower complex half-plane ( $\text{Im } \zeta < 0$ ), then  $\Pi'(g)(\xi') = 0$ .

### 5. Invertibility of the Dirichlet localized boundary-domain integral operator

From Theorem 3.1, it follows that the LBDIE system (3.1)–(3.2), which has a special right-hand side, is uniquely solvable in the space  $H^{1,0}(\Omega, A) \times H^{-1/2}(S)$ . Let us investigate the LBDIO, generated by the left-hand side expressions in (3.1)–(3.2), in appropriate functional spaces.

The LBDIE system (3.1)–(3.2) with an arbitrary right-hand side vector functions from the space  $H^1(\Omega) \times H^{1/2}(S)$  can be written as

$$\mathbf{B} \overset{\circ}{E} u - V \psi = F_1 \text{ in } \Omega, \quad (5.1)$$

$$\mathbf{N}^+ \overset{\circ}{E} u - \nu \psi = F_2 \text{ on } S, \quad (5.2)$$

where  $\mathbf{B} = \boldsymbol{\beta} + \mathbf{N}$ ,  $F_1 \in H^1(\Omega)$  and  $F_2 \in H^{1/2}(S)$ . Let us denote by  $\mathfrak{D}$  the LBDIO generated by the left-hand side expressions in LBDIE system (5.1)–(5.2),

$$\mathfrak{D} := \begin{bmatrix} r_{\Omega} \mathbf{B} \overset{\circ}{E} & -r_{\Omega} V \\ \mathbf{N}^+ \overset{\circ}{E} & -\nu \end{bmatrix}.$$

We would like to prove the following assertion.

*Theorem 5.1*

Let the localizing function  $\chi \in X_+^{\infty}$  and  $r > -\frac{1}{2}$ . Then, the operator

$$\mathfrak{D} : H^{r+1}(\Omega) \times H^{r-1/2}(S) \rightarrow H^{r+1}(\Omega) \times H^{r+1/2}(S) \quad (5.3)$$

is invertible.

We will reduce the theorem proof to several lemmas.

*Lemma 5.2*

Let  $\chi \in X^{\infty}$ . The operator  $r_{\Omega} \mathbf{B} \overset{\circ}{E} : H^s(\Omega) \rightarrow H^s(\Omega)$  for  $s \geq 0$  is Fredholm with zero index.

*Proof*

Because (4.4) is a rational function in  $\xi$ , we can apply the theory of pseudo-differential operators with symbol satisfying the transmission conditions [25, 27–29, 34]. Now, with the help of the local principle (Lemma 23.9 in [27]) and Lemma 4.1, we deduce that the operator

$$\mathcal{B} := r_{\Omega} \mathbf{B} \overset{\circ}{E} : H^s(\Omega) \rightarrow H^s(\Omega)$$

is Fredholm for all  $s \geq 0$ .

To show that  $\text{Ind } \mathcal{B} = 0$ , we use that the operators  $\mathcal{B}$  and

$$\mathcal{B}_t = r_{\Omega} (\boldsymbol{\beta} + t \mathbf{N}) \overset{\circ}{E},$$

where  $t \in [0, 1]$ , are homotopic. Note that  $\mathcal{B} = \mathcal{B}_1$ . The principal homogeneous symbol of the operator  $\mathcal{B}_t$  has the form

$$\mathfrak{S}(\mathcal{B}_t)(y, \xi) = \beta(y) + t \mathfrak{S}(\mathbf{N})(y, \xi) = (1 - t)\beta(y) + t\mathfrak{S}(\mathbf{B})(y, \xi).$$

It is easy to see that the symbol  $\mathfrak{S}(\mathcal{B}_t)(y, \xi)$  is positive definite,

$$[\mathfrak{S}(\mathcal{B}_t)(y, \xi)\zeta] \cdot \bar{\zeta} = (1 - t) [\beta(y)\zeta] \cdot \bar{\zeta} + t [\mathfrak{S}(\mathbf{B})(y, \xi)\zeta] \cdot \bar{\zeta} \geq c|\zeta|^2$$

for all  $y \in \bar{\Omega}$ ,  $\xi \neq 0$ ,  $\zeta \in \mathbb{C}^3$  and  $t \in [0, 1]$ , where  $c$  is some positive number.

Because  $\mathfrak{S}(\mathcal{B}_t)(y, \xi)$  is rational, even, and homogeneous of order zero in  $\xi$ , we conclude, as earlier, that the operator

$$\mathcal{B}_t : H^s(\Omega) \rightarrow H^s(\Omega)$$

is Fredholm for all  $s \geq 0$  and for all  $t \in [0, 1]$ . Therefore,  $\text{Ind } \mathcal{B}_t$  is the same for all  $t \in [0, 1]$ . On the other hand, due to the equality  $\mathcal{B}_0 = r_\Omega I$ , we get

$$\text{Ind } \mathcal{B} = \text{Ind } \mathcal{B}_1 = \text{Ind } \mathcal{B}_t = \text{Ind } \mathcal{B}_0 = 0.$$

□

*Lemma 5.3*

Let  $\chi \in X^\infty$ . The operator  $\mathcal{D}$  given by (5.3) is Fredholm.

*Proof*

To investigate Fredholm properties of the operator  $\mathcal{D}$ , we apply the local principle (cf. e.g. [27, 35], § 19 and § 22). Because of this principle, we have to show first that the operator  $\mathcal{D}$  is locally Fredholm at an arbitrary ‘frozen’ interior point  $\tilde{y} \in \Omega$ , and secondly that the so called generalized *Šapiro–Lopatinskiĭ condition* for the operator  $\mathcal{D}$  holds at an arbitrary ‘frozen’ boundary point  $\tilde{y} \in S$ . To obtain the explicit form of this condition, we proceed as follows. Let  $\tilde{U}$  be a neighbourhood of a fixed point  $\tilde{y} \in \bar{\Omega}$ , and let  $\tilde{\psi}_0, \tilde{\varphi}_0 \in \mathcal{D}(\tilde{U})$  such that

$$\text{supp } \tilde{\psi}_0 \cap \text{supp } \tilde{\varphi}_0 \neq \emptyset, \quad \tilde{y} \in \text{supp } \tilde{\psi}_0 \cap \text{supp } \tilde{\varphi}_0,$$

and consider the operator  $\tilde{\psi}_0 \mathcal{D} \tilde{\varphi}_0$ . We consider separately two possible cases, case (1):  $\tilde{y} \in \Omega$ , and case (2):  $\tilde{y} \in S$ .

*Case (1).* If  $\tilde{y} \in \Omega$ , then we can choose a neighbourhood  $\tilde{U}$  such that  $\bar{\tilde{U}} \subset \Omega$ . Therefore, the operator  $\tilde{\psi}_0 \mathcal{D} \tilde{\varphi}_0$  has the same Fredholm properties as the operator  $\tilde{\psi}_0 \mathbf{B} \tilde{\varphi}_0$  (see the similar arguments in the proof of Theorem 22.1 in [27]). Then by Lemma 5.2, we conclude that  $\tilde{\psi}_0 \mathcal{D} \tilde{\varphi}_0$  is a locally Fredholm operator at interior points of  $\Omega$ .

*Case (2).* If  $\tilde{y} \in S$ , then at this point we have to ‘freeze’ the operator  $\tilde{\psi}_0 \mathcal{D} \tilde{\varphi}_0$ , which means that we can choose a neighbourhood  $\tilde{U}$  sufficiently small such that at the local co-ordinate system with the origin at the point  $\tilde{y}$  and the third axis coinciding with the normal vector at the point  $\tilde{y} \in S$ , the following decomposition holds

$$\tilde{\psi}_0 \mathcal{D} \tilde{\varphi}_0 = \tilde{\psi}_0 \left( \hat{\mathcal{D}} + \tilde{\mathbf{K}} + \tilde{\mathbf{T}} \right) \tilde{\varphi}_0, \tag{5.4}$$

where

$$\tilde{\mathbf{K}} : H^{r+1}(\mathbb{R}_+^3) \times H^{r-1/2}(\mathbb{R}^2) \rightarrow H^{r+1}(\mathbb{R}_+^3) \times H^{r+1/2}(\mathbb{R}^2)$$

is a bounded operator with small norm, while

$$\tilde{\mathbf{T}} : H^{r+1}(\mathbb{R}_+^3) \times H^{r-1/2}(\mathbb{R}^2) \rightarrow H^{r+2}(\mathbb{R}_+^3) \times H^{r+3/2}(\mathbb{R}^2)$$

is a bounded operator. The operator

$$\hat{\mathcal{D}} := \begin{bmatrix} r_+ \hat{\mathbf{B}} \hat{\mathbf{E}} & -r_+ \hat{\mathbf{V}} \\ \hat{\mathbf{N}}^+ \hat{\mathbf{E}} & -\hat{\mathbf{Y}} \end{bmatrix}$$

with  $r_+ = r_{\mathbb{R}_+^3}$ , is defined in the upper half-space  $\mathbb{R}_+^3$  and possesses the following mapping property

$$\hat{\mathcal{D}} : H^{r+1}(\mathbb{R}_+^3) \times H^{r-1/2}(\mathbb{R}^2) \rightarrow H^{r+1}(\mathbb{R}_+^3) \times H^{r+1/2}(\mathbb{R}^2). \tag{5.5}$$

The operators involved in the expression of  $\hat{\mathcal{D}}$  are defined as follows: for the operator  $\tilde{M}$ , the operator  $\hat{M}$  denotes the operator in  $\mathbb{R}^n$  ( $n = 2, 3$ ) constructed by the symbol

$$\hat{\mathfrak{S}}(\tilde{M})(\xi) = \mathfrak{S}(\tilde{M})((1 + |\xi'|)\omega, \xi_3) \quad \text{if } n = 3$$

and

$$\hat{\mathfrak{S}}(\tilde{M})(\xi) = \mathfrak{S}(\tilde{M})((1 + |\xi'|)\omega) \quad \text{if } n = 2,$$

where  $\omega = \frac{\xi'}{|\xi'|}$ ,  $\xi = (\xi', \xi_n)$ ,  $\xi' = (\xi_1, \dots, \xi_{n-1})$ .

The generalized Šapiro–Lopatinskiĭ condition is related to the invertibility of the operator (5.5). Indeed, let us write the system corresponding to the operator  $\widehat{\mathcal{D}}$ :

$$r_+ \widehat{\mathbf{B}} \mathring{E} \tilde{u} - r_+ \widehat{\mathcal{V}} \tilde{\psi} = \tilde{F}_1 \text{ in } \mathbb{R}_+^3, \tag{5.6}$$

$$\widehat{\mathcal{N}}^+ \mathring{E} \tilde{u} - \widehat{\mathcal{V}} \tilde{\psi} = \tilde{F}_2 \text{ on } \mathbb{R}^2, \tag{5.7}$$

where  $\tilde{F}_1 \in H^1(\mathbb{R}_+^3)$ ,  $\tilde{F}_2 \in H^{1/2}(\mathbb{R}^2)$ .

Note that the operator  $r_+ \widehat{\mathbf{B}} \mathring{E}$  is a singular integral operator with even rational elliptic principal homogeneous symbol. Then, due to Lemma 4.1, the operator

$$r_+ \widehat{\mathbf{B}} \mathring{E} : H^{s+1}(\mathbb{R}_+^3) \rightarrow H^{s+1}(\mathbb{R}_+^3)$$

is invertible, we can determine  $\tilde{u}$  from Equation (5.6) and write

$$\mathring{E} \tilde{u} = \mathring{E} \left[ r_+ \widehat{\mathbf{B}} \mathring{E} \right]^{-1} \tilde{f} = \mathcal{F}^{-1} \left\{ \left[ \widehat{\mathcal{G}}^{(+)}(\tilde{\mathbf{B}}) \right]^{-1} \Pi^+ \left( \left[ \widehat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(\tilde{f}_*) \right) \right\}, \tag{5.8}$$

where  $\tilde{f}_* = \tilde{F}_1 + \widehat{\mathcal{V}} \tilde{\psi}$  is an extension of  $\tilde{f} = \tilde{F}_1 + r_+ \widehat{\mathcal{V}} \tilde{\psi}$  from  $\mathbb{R}_+^3$  to  $\mathbb{R}^3$  preserving the function space. The symbols  $\widehat{\mathcal{G}}^{(\pm)}(M)$  denote the so called ‘plus’ and ‘minus’ factors in the factorization of the symbol  $\widehat{\mathcal{G}}(M)$  with respect to the variable  $\xi_3$ . Note that the function  $\mathring{E} \tilde{u}$  in (5.8) does not depend on the chosen extension  $\tilde{f}_*$  of  $\tilde{f}$ .

Substituting (5.8) into (5.7) leads to the following pseudo-differential equation with respect to the unknown function  $\tilde{\psi}$ :

$$\widehat{\mathcal{N}}^+ \mathcal{F}^{-1} \left\{ \left[ \widehat{\mathcal{G}}^{(+)}(\tilde{\mathbf{B}}) \right]^{-1} \Pi^+ \left( \left[ \widehat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(\widehat{\mathcal{V}} \tilde{\psi}) \right) \right\} - \widehat{\mathcal{V}} \tilde{\psi} = \tilde{F}_2 \text{ on } \mathbb{R}^2, \tag{5.9}$$

where

$$\tilde{F}_2 = \tilde{F}_2 - \widehat{\mathcal{N}}^+ \mathring{E} \left[ r_+ \widehat{\mathbf{B}} \mathring{E} \right]^{-1} \tilde{F}_1.$$

It is easy to see that

$$\widehat{\mathcal{N}}^+ \nu(\tilde{y}') = \left[ \mathcal{F}_{\xi \rightarrow \tilde{y}'}^{-1} \left[ \mathcal{G}(\tilde{\mathcal{N}})(\xi) \mathcal{F}(\nu)(\xi) \right] \right]_{\tilde{y}_3=0^+} = \mathcal{F}_{\xi' \rightarrow \tilde{y}'}^{-1} \left[ \Pi' \left[ \mathcal{G}(\tilde{\mathcal{N}}) \mathcal{F}(\nu) \right] (\xi') \right].$$

In view of the relation (e.g. [21, Equation (4.1)], [15, Equations (B.5)–(B.6)])

$$\tilde{\mathcal{V}} \tilde{\psi}(y) = -\langle \gamma \tilde{P}(\cdot - y), \tilde{\psi} \rangle_S = -\langle \tilde{P}(\cdot - y), \gamma^* \tilde{\psi} \rangle_{\mathbb{R}^3} = -\tilde{\mathbf{P}}(\gamma^* \tilde{\psi})(y),$$

where the operator  $\gamma^*$  is dual to the trace operator  $\gamma$ . When the surface  $S$  coincides with  $\mathbb{R}^2 = \partial \mathbb{R}_+^3$ , then we have  $\gamma^* \tilde{\psi} = \tilde{\psi}(\tilde{\mathcal{V}}) \otimes \delta_3$  with  $\delta_3$  being the one-dimensional Dirac distribution in the  $\tilde{y}_3$  direction. Then, we arrive at the equality

$$\begin{aligned} \widehat{\mathcal{N}}^+ \mathcal{F}_{\xi \rightarrow \tilde{y}'}^{-1} \left\{ \left[ \widehat{\mathcal{G}}^{(+)}(\tilde{\mathbf{B}})(\xi) \right]^{-1} \Pi^+ \left( \left[ \widehat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{F}(\widehat{\mathcal{V}} \tilde{\psi}) \right) (\xi) \right\} (\tilde{y}') = \\ - \mathcal{F}_{\xi' \rightarrow \tilde{y}'}^{-1} \left\{ \Pi' \left[ \mathcal{G}(\tilde{\mathcal{N}}) \left[ \widehat{\mathcal{G}}^{(+)}(\tilde{\mathbf{B}}) \right]^{-1} \Pi^+ \left( \left[ \widehat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \widehat{\mathcal{G}}(\tilde{\mathbf{P}}) \right) \right] (\xi') \mathcal{F}_{\xi' \rightarrow \tilde{y}'}^{-1} \tilde{\psi} \right\}. \end{aligned}$$

With the help of these relations Equation (5.9) can be rewritten in the following form

$$\mathcal{F}_{\xi' \rightarrow \tilde{y}'}^{-1} \left[ \widehat{\mathcal{E}}(\xi') \mathcal{F}(\tilde{\psi})(\xi') \right] = \tilde{F}(\tilde{y}') \text{ on } \mathbb{R}^2, \tag{5.10}$$

where

$$\widehat{\mathcal{E}}(\xi') = e \left( (1 + |\xi'|) \omega \right), \quad \omega = \frac{\xi'}{|\xi'|}, \tag{5.11}$$

with  $e$  being a homogeneous function of order  $-1$  given by the equality

$$e(\xi') = -\Pi' \left\{ \mathcal{G}(\tilde{\mathcal{N}}) \left[ \widehat{\mathcal{G}}^{(+)}(\tilde{\mathbf{B}}) \right]^{-1} \Pi^+ \left( \left[ \widehat{\mathcal{G}}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathcal{G}(\tilde{\mathbf{P}}) \right) \right\} (\xi') - \mathcal{G}(\tilde{\mathcal{V}})(\xi'), \quad \forall \xi' \neq 0. \tag{5.12}$$

If the function  $\det e(\xi')$  is different from zero for all  $\xi' \neq 0$ , then  $\det \widehat{\mathcal{E}}(\xi') \neq 0$  for all  $\xi' \in \mathbb{R}^2$ , and the corresponding pseudo-differential operator

$$\widehat{\mathbf{E}} : H^s(\mathbb{R}) \rightarrow H^{s+1}(\mathbb{R}) \text{ for all } s \in \mathbb{R}$$

generated by the left-hand side expression in (5.10) is invertible. In particular, it follows that the system of Equation 5.6–(5.7) is uniquely solvable with respect to  $(\tilde{u}, \tilde{\psi})$  in the space  $H^1(\mathbb{R}_+^3) \times H^{-1/2}(\mathbb{R}^2)$  for arbitrary right-hand sides  $(\tilde{F}_1, \tilde{F}_2) \in H^1(\mathbb{R}_+^3) \times H^{1/2}(\mathbb{R}^2)$ . Consequently, the operator  $\hat{\mathcal{D}}$  in (5.5) is invertible, which implies that the operator (5.4) possesses a left and right regularizer. In turn, this yields that the operator (5.3) possesses a left and right regularizer as well. Thus, the operator (5.3) is Fredholm if

$$\det e(\xi') \neq 0 \quad \forall \xi' \neq 0.$$

This condition is called the Šapiro–Lopatinskiĭ condition (cf. [27], Theorems 12.2 and 23.1, and also formulas (12.27) and (12.25)). Let us show that in our case the Šapiro–Lopatinskiĭ condition holds. To this end, let us note that the principal homogeneous symbols  $\mathfrak{S}(\tilde{N})$ ,  $\mathfrak{S}(\tilde{B})$ ,  $\mathfrak{S}(\tilde{P})$  and  $\mathfrak{S}(\tilde{\mathcal{V}})$  of the operators  $\mathbf{N}$ ,  $\mathbf{B}$ ,  $\mathbf{P}$ , and  $\mathcal{V}$  in the chosen local co-ordinate system involved in formula (5.12) read as

$$\mathfrak{S}(\tilde{N})(\xi) = |\xi|^{-2} \tilde{A}(\xi) - \tilde{\beta}, \quad \mathfrak{S}(\tilde{B})(\xi) = |\xi|^{-2} \tilde{A}(\xi), \quad \mathfrak{S}(\tilde{P})(\xi) = -|\xi|^{-2} l, \quad \mathfrak{S}(\tilde{\mathcal{V}})(\xi') = \frac{1}{2|\xi'|} l, \quad \xi = (\xi', \xi_3), \quad \xi' = (\xi_1, \xi_2),$$

where  $\tilde{\beta}$  denotes the matrix  $\beta$  written in chosen local co-ordinate system. Rewrite (5.12) in the form

$$e(\xi') = -\Pi' \left\{ (\mathfrak{S}(\tilde{B}) - \tilde{\beta}) [\mathfrak{S}^{(+)}(\tilde{B})]^{-1} \Pi^+ \left( [\mathfrak{S}^{(-)}(\tilde{B})]^{-1} \mathfrak{S}(\tilde{P}) \right) \right\} (\xi') - \mathfrak{S}(\tilde{\mathcal{V}})(\xi') = e_1(\xi') + e_2(\xi') - \mathfrak{S}(\tilde{\mathcal{V}})(\xi'), \quad (5.13)$$

where

$$e_1(\xi') = -\Pi' \left\{ \mathfrak{S}(\tilde{B}) [\mathfrak{S}^{(+)}(\tilde{B})]^{-1} \Pi^+ \left( [\mathfrak{S}^{(-)}(\tilde{B})]^{-1} \mathfrak{S}(\tilde{P}) \right) \right\} (\xi'), \quad (5.14)$$

$$e_2(\xi') = \tilde{\beta} \Pi' \left\{ [\mathfrak{S}^{(+)}(\tilde{B})]^{-1} \Pi^+ \left( [\mathfrak{S}^{(-)}(\tilde{B})]^{-1} \mathfrak{S}(\tilde{P}) \right) \right\} (\xi'), \quad (5.15)$$

$$\mathfrak{S}(\tilde{\mathcal{V}})(\xi') = \frac{1}{2|\xi'|} l. \quad (5.16)$$

Direct calculations give

$$\begin{aligned} \Pi^+ \left( [\mathfrak{S}^{(-)}(\tilde{B})]^{-1} \mathfrak{S}(\tilde{P}) \right) (\xi') &= \frac{i}{2\pi} \lim_{t \rightarrow 0^+} \int_{-\infty}^{+\infty} \frac{\left( [\mathfrak{S}^{(-)}(\tilde{B})]^{-1} \mathfrak{S}(\tilde{P}) \right) (\xi', \eta_3) d\eta_3}{\xi_3 + it - \eta_3} \\ &= -\frac{i}{2\pi} \lim_{t \rightarrow 0^+} \int_{-\infty}^{+\infty} \frac{[\mathfrak{S}^{(-)}(\tilde{B})]^{-1} (\xi', \eta_3) d\eta_3}{(\xi_3 + it - \eta_3) (|\xi'|^2 + \eta_3^2)} = \frac{i}{2\pi} \lim_{t \rightarrow 0^+} \int_{\Gamma^-} \frac{[\mathfrak{S}^{(-)}(\tilde{B})]^{-1} (\xi', \tau) d\tau}{(\xi_3 + it - \tau) (|\xi'|^2 + \tau^2)} \\ &= \frac{i}{2\pi} \lim_{t \rightarrow 0^+} \frac{2\pi i [\mathfrak{S}^{(-)}(\tilde{B})]^{-1} (\xi', -i|\xi'|)}{(\xi_3 + it + i|\xi'|) 2(-i|\xi'|)} = -\frac{i [\mathfrak{S}^{(-)}(\tilde{B})]^{-1} (\xi', -i|\xi'|)}{2|\xi'| \Theta^{(+)}(\xi', \xi_3)}. \end{aligned} \quad (5.17)$$

Now, from (5.14) with the help of (5.17), we derive

$$\begin{aligned} e_1(\xi') &= -\Pi' \left\{ \mathfrak{S}^{(-)}(\tilde{B}) \mathfrak{S}^{(+)}(\tilde{B}) [\mathfrak{S}^{(+)}(\tilde{B})]^{-1} \Pi^+ \left( [\mathfrak{S}^{(-)}(\tilde{B})]^{-1} \mathfrak{S}(\tilde{P}) \right) \right\} (\xi') \\ &= -\Pi' \left\{ \mathfrak{S}^{(-)}(\tilde{B}) \Pi^+ \left( [\mathfrak{S}^{(-)}(\tilde{B})]^{-1} \mathfrak{S}(\tilde{P}) \right) \right\} (\xi') = \Pi' \left\{ \frac{\mathfrak{S}^{(-)}(\tilde{B})}{\Theta^{(+)}} \right\} (\xi') \left( \frac{i [\mathfrak{S}^{(-)}(\tilde{B})]^{-1} (\xi', -i|\xi'|)}{2|\xi'|} \right) \\ &= -\frac{1}{2\pi} \int_{\Gamma^-} \frac{\mathfrak{S}^{(-)}(\tilde{B})(\xi', \tau)}{\tau + i|\xi'|} d\tau \left( \frac{i [\mathfrak{S}^{(-)}(\tilde{B})]^{-1} (\xi', -i|\xi'|)}{2|\xi'|} \right) \\ &= -i \mathfrak{S}^{(-)}(\tilde{B})(\xi', -i|\xi'|) \frac{i [\mathfrak{S}^{(-)}(\tilde{B})]^{-1} (\xi', -i|\xi'|)}{2|\xi'|} = \frac{1}{2|\xi'|} l. \end{aligned} \quad (5.18)$$



Quite similarly, from (5.15) with the help of (5.17), we get

$$\begin{aligned} e_2(\xi') &= \tilde{\beta} \Pi' \left\{ \left[ \mathfrak{G}^{(+)}(\tilde{\mathbf{B}}) \right]^{-1} \Pi^+ \left( \left[ \mathfrak{G}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1} \mathfrak{G}(\tilde{\mathbf{P}}) \right) \right\} (\xi') = -\tilde{\beta} \Pi' \left\{ \frac{\left[ \mathfrak{G}^{(+)}(\tilde{\mathbf{B}}) \right]^{-1}}{\Theta^{(+)}} \right\} (\xi') \left( \frac{i \left[ \mathfrak{G}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1}(\xi', -i|\xi'|)}{2|\xi'|} \right) \\ &= -\frac{i\tilde{\beta}}{2|\xi'|} \left( -\frac{1}{2\pi} \int_{\Gamma^-} \frac{\left[ \mathfrak{G}^{(+)}(\tilde{\mathbf{B}}) \right]^{-1}(\xi', \tau)}{\tau + i|\xi'|} d\tau \right) \left[ \mathfrak{G}^{(-)}(\tilde{\mathbf{B}}) \right]^{-1}(\xi', -i|\xi'|) \\ &= \frac{i\tilde{\beta}}{4\pi|\xi'|} \int_{\Gamma^-} \left[ \tilde{\mathbf{A}}^{(+)}(\xi', \tau) \right]^{-1} d\tau (-i|\xi'|) \left[ \tilde{\mathbf{A}}^{(-)}(\xi', -i|\xi'|) \right]^{-1} = i\tilde{\beta} \left\{ \frac{1}{2\pi i} \int_{\Gamma^-} \left[ \tilde{\mathbf{A}}^{(+)}(\xi', \tau) \right]^{-1} d\tau \right\} \left[ \tilde{\mathbf{A}}^{(-)}(\xi', -i|\xi'|) \right]^{-1}. \end{aligned}$$

Therefore, due to (5.13), (5.16), (5.18) and Lemma 4.2, we have

$$e_2(\xi') = \frac{i}{a^{(+)}(\xi')} \tilde{\beta} C^{(+)}(\xi') \left[ \tilde{\mathbf{A}}^{(-)}(\xi', -i|\xi'|) \right]^{-1}, \tag{5.19}$$

where  $\det \tilde{\beta} \neq 0$ ,  $\det C^{(+)}(\xi') \neq 0$  and  $\det \tilde{\mathbf{A}}^{(-)}(\xi', -i|\xi'|) \neq 0$  for all  $\xi' \neq 0$ . Then, it is clear that

$$\det e(\xi') = -\frac{i}{(a^{(+)}(\xi'))^3} \det \tilde{\beta} \det C^{(+)}(\xi') \det \left[ \tilde{\mathbf{A}}^{(-)}(\xi', -i|\xi'|) \right]^{-1} \neq 0$$

for all  $\xi' \neq 0$ .

Thus, we have obtained that for the operator  $\mathfrak{D}$  the Šapiro–Lopatinskiĭ condition holds. Therefore, the operator

$$\mathfrak{D} : H^{r+1}(\Omega) \times H^{r-1/2}(S) \rightarrow H^{r+1}(\Omega) \times H^{r+1/2}(S)$$

is Fredholm for  $r > -\frac{1}{2}$ . □

**Lemma 5.4**

Let  $\chi \in X^\infty$ . The operator  $\mathfrak{D}$  given by (5.3) is Fredholm with zero index.

*Proof*

For  $t \in [0, 1]$ , let us consider the operator

$$\mathfrak{D}_t := \begin{bmatrix} r_\Omega \mathbf{B}_t \overset{\circ}{E} & -r_\Omega V \\ t \mathbf{N}^+ \overset{\circ}{E} & -\mathcal{V} \end{bmatrix}$$

with  $\mathbf{B}_t = \beta + t \mathbf{N}$  and establish that it is homotopic to the operator  $\mathfrak{D} = \mathfrak{D}_1$ . We have to check that for the operator  $\mathfrak{D}_t$  the Šapiro–Lopatinskiĭ condition is satisfied for all  $t \in [0, 1]$ . Indeed, in this case the Šapiro–Lopatinskiĭ condition reads as

$$\det e_t(\xi') \neq 0 \quad \text{for all } \xi' \neq 0,$$

where (cf. (5.12))

$$e_t(\xi') = -\Pi' \left\{ (\mathfrak{G}(\tilde{\mathbf{B}}_t) - \tilde{\beta}) \left[ \mathfrak{G}^{(+)}(\tilde{\mathbf{B}}_t) \right]^{-1} \Pi^+ \left( \left[ \mathfrak{G}^{(-)}(\tilde{\mathbf{B}}_t) \right]^{-1} \mathfrak{G}(\tilde{\mathbf{P}}) \right) \right\} (\xi') - \mathfrak{G}(\tilde{\mathcal{V}})(\xi') = e_t^{(1)}(\xi') + e_t^{(2)}(\xi') - \mathfrak{G}(\tilde{\mathcal{V}})(\xi') \tag{5.20}$$

with

$$e_t^{(1)}(\xi') = -\Pi' \left\{ \mathfrak{G}(\tilde{\mathbf{B}}_t) \left[ \mathfrak{G}^{(+)}(\tilde{\mathbf{B}}_t) \right]^{-1} \Pi^+ \left( \left[ \mathfrak{G}^{(-)}(\tilde{\mathbf{B}}_t) \right]^{-1} \mathfrak{G}(\tilde{\mathbf{P}}) \right) \right\} (\xi') = \frac{1}{2|\xi'|} l, \tag{5.21}$$

$$e_t^{(2)}(\xi') = \tilde{\beta} \Pi' \left\{ \left[ \mathfrak{G}^{(+)}(\tilde{\mathbf{B}}_t) \right]^{-1} \Pi^+ \left( \left[ \mathfrak{G}^{(-)}(\tilde{\mathbf{B}}_t) \right]^{-1} \mathfrak{G}(\tilde{\mathbf{P}}) \right) \right\} (\xi'), \tag{5.22}$$

$$\mathfrak{G}(\tilde{\mathcal{V}})(\xi') = \frac{1}{2|\xi'|} l. \tag{5.23}$$

By direct calculations, we get

$$\begin{aligned}
 e_t^{(2)}(\xi') &= \tilde{\beta} \Pi' \left\{ \left[ \mathfrak{G}^{(+)}(\tilde{\mathbf{B}}_t) \right]^{-1} \Pi^+ \left( \left[ \mathfrak{G}^{(-)}(\tilde{\mathbf{B}}_t) \right]^{-1} \mathfrak{G}(\tilde{\mathbf{P}}) \right) \right\} (\xi') \\
 &= -\tilde{\beta} \Pi' \left\{ \frac{\left[ \mathfrak{G}^{(+)}(\tilde{\mathbf{B}}_t) \right]^{-1}}{\Theta^{(+)}} \right\} (\xi') \left( \frac{i \left[ \mathfrak{G}^{(-)}(\tilde{\mathbf{B}}_t) \right]^{-1} (\xi', -i|\xi'|)}{2|\xi'|} \right) \\
 &= -\frac{i\tilde{\beta}}{2|\xi'|} \left( -\frac{1}{2\pi} \int_{\Gamma^-} \frac{\left[ \mathfrak{G}^{(+)}(\tilde{\mathbf{B}}_t) \right]^{-1} (\xi', \tau)}{\tau + i|\xi'|} d\tau \right) \left[ \mathfrak{G}^{(-)}(\tilde{\mathbf{B}}_t) \right]^{-1} (\xi', -i|\xi'|) \\
 &= \frac{i\tilde{\beta}}{4\pi|\xi'|} \int_{\Gamma^-} \left[ \tilde{A}_t^{(+)}(\xi', \tau) \right]^{-1} d\tau (-2i|\xi'|) \left[ \tilde{A}_t^{(-)}(\xi', -i|\xi'|) \right]^{-1} \\
 &= i\tilde{\beta} \left\{ \frac{1}{2\pi i} \int_{\Gamma^-} \left[ \tilde{A}_t^{(+)}(\xi', \tau) \right]^{-1} d\tau \right\} \left[ \tilde{A}_t^{(-)}(\xi', -i|\xi'|) \right]^{-1},
 \end{aligned} \tag{5.24}$$

where  $\tilde{A}_t(\xi) = (1-t)|\xi|^2 \tilde{\beta} + t\tilde{A}(\xi)$ ,  $\tilde{A}_t(\xi', \xi_3) = \tilde{A}_t^{(-)}(\xi', \xi_3)\tilde{A}_t^{(+)}(\xi', \xi_3)$  and  $\tilde{A}_t^{(\pm)}(\xi', \xi_3)$  are the 'plus' and 'minus' polynomial matrix factors in  $\xi_3$  of the polynomial symbol matrix  $\tilde{A}_t(\xi', \xi_3)$ . Because of (5.20), (5.21), (5.23), (5.24) and Lemma 4.2, we have

$$e_t^{(2)}(\xi') = \frac{i}{a_t^{(+)}(\xi')} \tilde{\beta} C_t^{(+)}(\xi') \left[ \tilde{A}_t^{(-)}(\xi', -i|\xi'|) \right]^{-1},$$

where  $C_t^{(+)}(\xi') = \left[ c_{ij}^{(+)}(\xi') \right]_{ij=1}^3$  and  $c_{ij}^{(+)}$ ,  $i, j = 1, 2, 3$ , are main coefficients of the co-factors  $p_{ij}^{(+)}(\xi', \tau)$  of the polynomial matrix  $\tilde{A}_t^{(+)}(\xi', \tau)$  and  $a^{(+)}$  the coefficient at  $\tau^3$  in the determinant  $\det \tilde{A}_t^{(+)}(\xi', \tau)$ . In addition,

$$\det \tilde{\beta} \neq 0, \quad \det C_t^{(+)}(\xi') \neq 0, \quad \det \tilde{A}_t^{(-)}(\xi', -i|\xi'|) \neq 0$$

for all  $\xi' \neq 0$  and  $t \in [0, 1]$ .

Then, it is clear that

$$\det e_t(\xi') = -\frac{i}{\left( a_t^{(+)}(\xi') \right)^3} \det \tilde{\beta} \det C_t^{(+)}(\xi') \det \left[ \tilde{A}_t^{(-)}(\xi', -i|\xi'|) \right]^{-1} \neq 0$$

for all  $\xi' \neq 0$  and for all  $t \in [0, 1]$ , which implies that for the operator  $\mathfrak{D}_t$  the Šapiro–Lopatinskiĭ condition is satisfied.

Therefore, the operator

$$\mathfrak{D}_t : H^{r+1}(\Omega) \times H^{r-1/2}(S) \rightarrow H^{r+1}(\Omega) \times H^{r+1/2}(S)$$

is Fredholm for all  $r > -\frac{1}{2}$  and for all  $t \in [0, 1]$ . Consequently,

$$\text{Ind } \mathfrak{D} = \text{Ind } \mathfrak{D}_1 = \text{Ind } \mathfrak{D}_t = \text{Ind } \mathfrak{D}_0 = 0.$$

□

#### Theorem 5.1 Proof

Because by Lemma 5.4, the operator  $\mathfrak{D}$  is Fredholm with zero index, its injectivity implies the invertibility. Thus, it remains to prove that the null space of the operator  $\mathfrak{D}$  is trivial for  $r > -\frac{1}{2}$ . Assume that  $U = (u, \psi)^T \in H^{r+1}(\Omega) \times H^{r-1/2}(S)$  is a solution to the homogeneous equation

$$\mathfrak{D} U = 0. \tag{5.25}$$

The operator

$$\mathfrak{D} : H^{r+1}(\Omega) \times H^{r-1/2}(S) \rightarrow H^{r+1}(\Omega) \times H^{r+1/2}(S)$$

is Fredholm with index zero for all  $r > -\frac{1}{2}$ . It is well known that then there exists a left regularizer  $\mathfrak{L}$  of the operator  $\mathfrak{D}$ ,

$$\mathfrak{L} : H^{r+1}(\Omega) \times H^{r+1/2}(S) \rightarrow H^{r+1}(\Omega) \times H^{r-1/2}(S), \tag{5.26}$$

such that

$$\mathfrak{L} \mathfrak{D} = I + \mathfrak{T},$$

where  $\mathfrak{T}$  is the operator of order  $-1$  (cf. proofs of Theorems 22.1 and 23.1 in [27]), that is,

$$\mathfrak{T} : H^{r+1}(\Omega) \times H^{r-1/2}(S) \rightarrow H^{r+2}(\Omega) \times H^{r+1/2}(S). \tag{5.27}$$

Therefore, from (5.25), we have

$$\mathfrak{L} \mathfrak{D} U = U + \mathfrak{T} U = 0. \tag{5.28}$$

From (5.27), we see that

$$\mathfrak{T} U \in H^{r+2}(\Omega) \times H^{r+1/2}(S).$$

Consequently, in view of (5.28)

$$U = (u, \psi)^\top \in H^{r+2}(\Omega) \times H^{r+1/2}(S). \tag{5.29}$$

If  $r \geq 0$ , this implies  $u \in H^{1,0}(\Omega, A)$ . If  $-\frac{1}{2} < r < 0$ , we iterate the earlier reasoning for  $U$  satisfying (5.29) to obtain

$$U = (u, \psi)^\top \in H^{r+3}(\Omega) \times H^{r+3/2}(S), \tag{5.30}$$

which again implies  $u \in H^{1,0}(\Omega, A)$ . Then, we can apply the equivalence Theorem 3.1 to conclude that a solution  $U = (u, \psi)^\top$  to the homogeneous equation (5.25) is trivial, that is,

$$u = 0 \text{ in } \Omega, \quad \psi = 0 \text{ on } S.$$

Thus,  $\text{Ker } \mathfrak{D} = \{0\}$  in the class  $H^{r+1}(\Omega) \times H^{r-1/2}(S)$ , and therefore, the operator

$$\mathfrak{D} : H^{r+1}(\Omega) \times H^{r-1/2}(S) \rightarrow H^{r+1}(\Omega) \times H^{r+1/2}(S)$$

is invertible for all  $r > -\frac{1}{2}$ . □

For localizing function  $\chi$  of finite smoothness, we have the following result.

*Corollary 5.5*

Let a localizing function  $\chi \in X_+^3$ . Then, the operator

$$\mathfrak{D} : H^1(\Omega) \times H^{-1/2}(S) \rightarrow H^1(\Omega) \times H^{1/2}(S)$$

is invertible.

*Proof*

It can be performed by word for word arguments employed in the proofs of Lemmas 5.2–5.4 and Theorem 5.1, with  $r = 0$  and using the mapping properties of the localized potentials for a localizing function of finite smoothness (Appendix B). □

Lemma 2.2, Theorem 3.1 and Corollaries 2.3 and 5.5 imply the following assertion.

*Corollary 5.6*

Let a localizing function  $\chi \in X_+^3$ . Then, the operator

$$\mathfrak{D} : H^{1,0}(\Omega, A) \times H^{-1/2}(S) \rightarrow H^{1,0}(\Omega, \Delta) \times H^{1/2}(S)$$

is invertible.

## APPENDIX A. Classes of localizing functions.

Here, we present the classes of localizing functions used in the main text (see [24] for details).

*Definition 1*

We say  $\chi \in X^k$  for integer  $k \geq 0$  if  $\chi(x) = \check{\chi}(|x|)$ ,  $\check{\chi} \in W_1^k(0, \infty)$  and  $\varrho \check{\chi}(\varrho) \in L_1(0, \infty)$ . We say  $\chi \in X_+^k$  for integer  $k \geq 1$  if  $\chi \in X^k$ ,  $\chi(0) = 1$  and  $\sigma_\chi(\omega) > 0$  for all  $\omega \in \mathbb{R}$ , where

$$\sigma_\chi(\omega) := \begin{cases} \hat{\chi}_s(\omega) > 0 & \text{for } \omega \in \mathbb{R} \setminus \{0\}, \\ \int_0^\infty \varrho \check{\chi}(\varrho) d\varrho & \text{for } \omega = 0, \end{cases} \tag{A1}$$

and  $\hat{\chi}_s(\omega)$  denotes the sine-transform of the function  $\check{\chi}$

$$\hat{\chi}_s(\omega) := \int_0^\infty \check{\chi}(\varrho) \sin(\varrho \omega) d\varrho. \tag{A2}$$

Evidently, we have the following imbeddings:  $X^{k_1} \subset X^{k_2}$  and  $X_+^{k_1} \subset X_+^{k_2}$  for  $k_1 > k_2$ . The class  $X_+^k$  is defined in terms of the sine-transform. The following lemma from [24] provides an easily verifiable sufficient condition for non-negative non-increasing functions to belong to this class.

**Lemma A.2**

Let  $k \geq 1$ . If  $\chi \in X^k$ ,  $\check{\chi}(0) = 1$ ,  $\check{\chi}(\varrho) \geq 0$  for all  $\varrho \in (0, \infty)$ , and  $\check{\chi}$  is a non-increasing function on  $[0, +\infty)$ , then  $\chi \in X^k_+$ .

The following (and other) examples for  $\chi$  are presented in [24],

$$\chi_{1,k}(x) = \begin{cases} \left[1 - \frac{|x|}{\varepsilon}\right]^k & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon, \end{cases} \tag{A3}$$

$$\chi_2(x) = \begin{cases} \exp\left[\frac{|x|^2}{|x|^2 - \varepsilon^2}\right] & \text{for } |x| < \varepsilon, \\ 0 & \text{for } |x| \geq \varepsilon, \end{cases} \tag{A4}$$

One can observe that  $\chi_{1,k} \in X^k_+$  for  $k \geq 1$ , while  $\chi_2 \in X^\infty_+$  due to Lemma A.2.

**APPENDIX B. Properties of localized potentials.**

Here, we collect some assertions describing mapping properties of the localized potentials. The proofs coincide with or are similar to the ones in [24] and [15, Appendix B] (see also [1], Chapter 8 and the references therein).

Let us introduce the boundary operators generated by the localized layer potentials associated with the localized parametrix  $P(x-y) \equiv P_\chi(x-y)$

$$\mathcal{V}g(y) := - \int_S P(x-y)g(x) dS_x, \quad y \in S, \tag{B1}$$

$$\mathcal{W}g(y) := - \int_S [T(x, \partial_x)P(x-y)]^\top g(x) dS_x, \quad y \in S, \tag{B2}$$

$$\mathcal{W}'g(y) := - \int_S [T(y, \partial_y)P(x-y)] g(x) dS_x, \quad y \in S, \tag{B3}$$

$$\mathcal{L}^\pm g(y) := T^\pm(y, \partial_y)Wg(y), \quad y \in S. \tag{B4}$$

**Theorem B.1**

The following operators are continuous

$$\mathcal{P} : \tilde{H}^s(\Omega) \rightarrow H^{s+2,s}(\Omega; \Delta), \quad -\frac{1}{2} < s < \frac{1}{2}, \quad \chi \in X^1, \tag{B5}$$

$$: H^s(\Omega) \rightarrow H^{s+2,s}(\Omega; \Delta), \quad -\frac{1}{2} < s < \frac{1}{2}, \quad \chi \in X^1, \tag{B6}$$

$$: H^s(\Omega) \rightarrow H^{\frac{5}{2}-\varepsilon, \frac{1}{2}-\varepsilon}(\Omega; \Delta), \quad \frac{1}{2} \leq s < \frac{3}{2}, \quad \forall \varepsilon \in (0, 1), \quad \chi \in X^2, \tag{B7}$$

where  $\Delta$  is the Laplace operator.

**Theorem B.2**

The following operators are continuous

$$V : H^{s-\frac{3}{2}}(S) \rightarrow H^s(\mathbb{R}^3), \quad s < \frac{3}{2}, \quad \text{if } \chi \in X^1, \tag{B8}$$

$$: H^{s-\frac{3}{2}}(S) \rightarrow H^{s,s-1}(\Omega^\pm; \Delta), \quad \frac{1}{2} < s < \frac{3}{2}, \quad \text{if } \chi \in X^2, \tag{B9}$$

$$W : H^{s-\frac{1}{2}}(S) \rightarrow H^s(\Omega^\pm), \quad s < \frac{3}{2}, \quad \text{if } \chi \in X^2, \tag{B10}$$

$$: H^{s-\frac{1}{2}}(S) \rightarrow H^{s,s-1}(\Omega^\pm; \Delta), \quad \frac{1}{2} < s < \frac{3}{2}, \quad \text{if } \chi \in X^3. \tag{B11}$$

**Theorem B.3**

If  $\chi \in X^k$  has a compact support and  $-\frac{1}{2} \leq s \leq \frac{1}{2}$ , then the following localized operators are continuous

$$V : H^s(S) \rightarrow H^{s+\frac{3}{2}}(\Omega^\pm) \quad \text{for } k = 2, \tag{B12}$$

$$W : H^{s+1}(S) \rightarrow H^{s+\frac{3}{2}}(\Omega^\pm) \quad \text{for } k = 3. \tag{B13}$$

**Theorem B.4**

Let  $\psi \in H^{-\frac{1}{2}}(S)$  and  $\varphi \in H^{\frac{1}{2}}(S)$ . Then the following jump relations hold on  $S$ :

$$\gamma^\pm V\psi = \mathcal{V}\psi, \quad \chi \in X^1, \tag{B14}$$

$$\gamma^\pm W\varphi = \mp \mu \varphi + \mathcal{W}\varphi, \quad \chi \in X^2, \tag{B15}$$

$$T^\pm V\psi = \pm \mu \psi + \mathcal{W}'\psi, \quad \chi \in X^2, \tag{B16}$$

where

$$\mu(y) = [\mu^{pq}(y)]_{p,q=1}^3 := \frac{1}{2} \left[ d_{kj}^{pq}(y) n_k(y) n_j(y) \right]_{p,q=1}^3, \quad y \in S, \tag{B17}$$

and  $\mu(y)$  is positive definite because of (2.2).

**Theorem B.5**

Let  $-\frac{1}{2} \leq s \leq \frac{1}{2}$ . The following operators

$$\mathcal{V} : H^s(S) \rightarrow H^{s+1}(S), \quad \chi \in X^2, \tag{B18}$$

$$\mathcal{W} : H^{s+1}(S) \rightarrow H^{s+1}(S), \quad \chi \in X^3, \tag{B19}$$

$$\mathcal{W}' : H^s(S) \rightarrow H^s(S), \quad \chi \in X^3, \tag{B20}$$

$$\mathcal{L}^\pm : H^{s+1}(S) \rightarrow H^s(S), \quad \chi \in X^3, \tag{B21}$$

are continuous.

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