Proceedings of the 8th UK Conference on Boundary Integral Methods, (edited by D. Lesnic), Leeds University Press, Leeds, UK, ISBN 978 0 85316 2957, (2011) 49-56.

LOCALIZED BOUNDARY-DOMAIN INTEGRAL EQUATIONS METHOD FOR AN INTERFACE CRACK PROBLEM

O.CHKADUA[1], S.E.MIKHAILOV[2], and D.NATROSHVILI[3]
[1] *Razmadze Mathematical Institute, 2 University str., 0186 Tbilisi, Georgia*
e-mail: chkadua@rmi.acnet.ge
[2] *Brunel University West London, Uxbridge, Middlesex, UB83PH, UK*
e-mail: sergey.mikhailov@brunel.ac.uk
[3] *Georgian Technical University, 77 M.Kostava, Tbilisi 0175, Georgia*
e-mail: natrosh@hotmail.com

Abstract. In a composite domain consisting of adjacent domains having a common interface surface, we consider a crack-type transmission problem for a scalar second order elliptic partial differential equation. The matrix of coefficients of the differential operator in each of the adjacent domains under consideration is represented as the product of a constant matrix by a smooth scalar function. These coefficients may have jumps across the interface surface. On the exterior boundary of the composite domain the Dirichlet boundary condition is prescribed, while the Neumann boundary conditions on the crack faces and the transmission conditions on the remaining part of the interface are given. By the localized potential method we reduce the problem to the localized boundary-domain integral equations. We investigate the corresponding localized boundary-domain integral operators, establish their Fredholm properties and prove their invertibility in appropriate function spaces.

1. INTRODUCTION

For simplicity we consider the case when two adjacent domains under consideration, Ω_1 and Ω_2, are such that the boundary of Ω_1 is also the common simply connected *interface surface* S_i. The matrix of coefficients of the elliptic scalar operator in each domain is represented as the product of a constant matrix by a smooth scalar function. These coefficients are discontinuous across the interface surface.

We deal with the case when on the exterior boundary S_e of the composite domain $\overline{\Omega}_1 \cup \overline{\Omega}_2$ the Dirichlet boundary condition is prescribed, while on the interface surface S_i, the Neumann boundary conditions on the crack faces and the transmission conditions on the remaining part of the interface are given.

Our goal here is to show that the transmission problems in question can be equivalently reduced to some *localized boundary-domain integral equations* (LBDIE) and that the corresponding *localized boundary-domain integral operators* (LBDIO) are invertible, which beside a pure mathematical interest may have also some applications in numerical analysis for construction of efficient numerical algorithms (see, e.g., [1]-[3] and the references therein).

In our case, the localized parametrix $P_{q\chi}(x-y,y)$, $q=1,2$, is represented as the product of a Levi function $P_{q1}(x-y,y)$ of the differential operator under consideration by an appropriately chosen cut-off function $\chi_q(x-y)$ supported on some neighbourhood of the origin. Clearly, the kernels of the corresponding localized potentials are supported in some neighbourhood of the reference point y (assuming that x is an integration variable) and they do not solve the original differential equation (cf. [3]-[5]).

By means of the direct approach based on Green's representation formula we reduce the transmission problem to the *localized boundary-domain integral equations (LBDIE) system.* First we establish the equivalence between the original transmission problems and the corresponding LBDIE systems which proved to be a quite nontrivial problem and plays a crucial role in our analysis. Afterwards we investigate Fredholm properties of the LBDIO and prove their invertibility in appropriate function spaces.

2. FORMULATION OF THE INTERFACE CRACK PROBLEM

Let Ω and Ω_1 be bounded domains in $\mathbb{R}^3$ and $\overline{\Omega}_1 \subset \Omega$. Denote $\Omega_2 := \Omega \setminus \overline{\Omega}_1$ and $S_i := \partial\Omega_1$, $S_e := \partial\Omega$. Clearly, $\partial\Omega_2 = S_i \cup S_e$. We assume that the *interface surface* S_i and the *exterior boundary* S_e of the composite body $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ are sufficiently smooth, say C^∞-regular if not otherwise stated. Throughout the paper $n^{(q)} = n^{(q)}(x)$ denotes the unit normal vector to $\partial\Omega_q$ directed outward the respective domains Ω_q. Clearly, $n^{(1)}(x) = -n^{(2)}(x)$ for $x \in S_i$.

Further, let the interface S_i is decomposed as a sum of smooth disjoint proper submanifolds, $S_i^{(c)}$ (the interface crack part) and $S_i^{(t)}$ (the transmission part): $S_i = \overline{S_i^{(c)}} \cup \overline{S_i^{(t)}}$ and $S_i^{(c)} \cap S_i^{(t)} = \varnothing$.

By $H^r(\Omega) = H_2^r(\Omega)$ and $H^r(S) = H_2^r(S)$, $r \in \mathbb{R}$, we denote the Bessel potential spaces on a domain Ω and on a closed manifold S without boundary. For a smooth proper submanifold $\mathcal{M} \subset S$ we denote by $\widetilde{H}^r(\mathcal{M})$ the subspace of $H^r(S)$, $\widetilde{H}^r(\mathcal{M}) := \{g : g \in H^r(S),\ \operatorname{supp} g \subset \overline{\mathcal{M}}\}$, while $H^r(\mathcal{M})$ denotes the space of restrictions on $\mathcal{M}$ of functions from $H^r(S)$, $H^r(\mathcal{M}) := \{r_{\mathcal{M}} f \ : \ f \in H^r(S)\}$, where $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$.

Let us consider the differential operator in the domain Ω_q

$$A_q(x, \partial_x)\, u(x) := \sum_{j,k=1}^{3} \partial_{x_k} \big[a_{kj}^{(q)}(x)\, \partial_{x_j} u(x) \big], \quad q = 1, 2, \tag{1}$$

where $\partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial_{x_j} = \partial/\partial x_j$, $j = 1, 2, 3$, and

$$a_{kj}^{(q)}(x) = a_{jk}^{(q)}(x) = a_q(x)\, a_{kj\star}^{(q)}, \quad \mathbf{a}_q(x) := [\, a_{kj}^{(q)}(x)\,]_{3\times 3} = a_q(x)\, [a_{kj\star}^{(q)}]_{3\times 3}, \quad \mathbf{a}_{q\star} := [a_{kj\star}^{(q)}]_{3\times 3}.$$

Here $a_{kj\star}^{(q)}$ are constants and the matrix $\mathbf{a}_{q\star} := [\, a_{kj\star}^{(q)}\,]_{3\times 3}$ is positive definite. Moreover, we assume that $a_q \in C^\infty(\mathbb{R}^3)$, $0 < c_0 \le a_q(x) \le c_1 < \infty$, $q = 1, 2$.

Further, for a sufficiently smooth function $u \in H^2(\Omega_q)$ we introduce the co-normal derivative operator on $\partial\Omega_q$, $q = 1, 2$, in the usual trace sense,

$$T_q^{\pm}(x, \partial_x)\, u(x) := \sum_{k,j=1}^{3} a_{kj}^{(q)}(x)\, n_k^{(q)}(x)\, \gamma_q^{\pm} [\partial_{x_j} u(x)], \quad x \in \partial\Omega_q, \tag{2}$$

where the symbols $\gamma_q \equiv \gamma_q^+$ and γ_q^- denote the trace operators on $\partial\Omega_q$ from the interior and exterior of Ω_q respectively. We use the notation $T_q(x, \partial_x)\, u(x) \equiv T_q^+(x, \partial_x)\, u(x)$.

We set

$$H^{1,0}(\Omega_q; A_q) := \{ v \in H^1(\Omega_q) \ : \ A_q v \in H^0(\Omega_q) \}, \quad q = 1, 2. \tag{3}$$

The above introduced co-normal derivative operators can be extended to functions from the space $H^{1,0}(\Omega_q; A_q)$ by defining the generalized (canonical) co-normal derivatives $T_q^{\pm} u \in H^{-\frac{1}{2}}(\partial\Omega_q)$ with the help of the first Green identity (cf., e.g., [6] and the references therein).

Now we formulate the following crack type Dirichlet transmission problem (CTD):
Find functions $u_1 \in H^{1,0}(\Omega_1; A_1)$ and $u_2 \in H^{1,0}(\Omega_2; A_2)$ satisfying the differential equations

$$A_q(x, \partial)\, u_q = f_q \quad \text{in} \quad \Omega_q, \quad q = 1, 2, \tag{4}$$

the transmission conditions on $S_i^{(t)}$

$$\gamma_1 u_1 - \gamma_2 u_2 = \varphi_{0i}^{(t)}, \qquad T_1 u_1 + T_2 u_2 = \psi_{0i}^{(t)} \quad \text{on} \quad S_i^{(t)}, \tag{5}$$

the crack type conditions on $S_i^{(c)}$

$$T_1 u_1 = \psi'_{0i}, \qquad T_2 u_2 = \psi''_{0i} \quad \text{on} \quad S_i^{(c)}, \tag{6}$$

and the Dirichlet boundary condition on S_e

$$\gamma_2 u_2 = \varphi_{0e} \quad \text{on} \quad S_e. \tag{7}$$

For the data in the above formulated problem we assume

$$\begin{array}{lll} f_q \in H^0(\Omega_q), \ \ q = 1,2, & \varphi_{0i}^{(t)} \in H^{\frac{1}{2}}(S_i^{(t)}), & \psi_{0i}^{(t)} \in H^{-\frac{1}{2}}(S_i^{(t)}), \\ \psi'_{0i} \in H^{-\frac{1}{2}}(S_i^{(c)}), & \psi''_{0i} \in H^{-\frac{1}{2}}(S_i^{(c)}), & \varphi_{0e} \in H^{\frac{1}{2}}(S_e). \end{array} \tag{8}$$

Moreover, for the function

$$\psi_{0i} := \begin{cases} \psi_{0i}^{(t)} & \text{on} \ \ S_i^{(t)}, \\ \psi'_{0i} + \psi''_{0i} & \text{on} \ \ S_i^{(c)}, \end{cases} \tag{9}$$

we require that the embedding $\psi_{0i} \in H^{-1/2}(S_i)$ holds which is a necessary compatibility condition for the problem to be solvable in the space $H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)$. We will refer the formulated crack type boundary transmission problem as (CTD) problem.

On the basis of the Lax-Milgram theorem one can prove the following proposition.

Theorem 1 *The crack type Dirichlet transmission problems (CTD) is uniquely solvable in the space* $H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)$.

We recall that our goal here is to show that the above formulated transmission problem can be equivalently reduced to some segregated LBDIE and to perform full analysis of the corresponding LBDIO.

3. PROPERTIES OF LOCALIZED POTENTIALS

It is well known that the fundamental solution-function of the elliptic operator with constant coefficients

$$A_{q\star}(\partial) := \sum_{i,j=1}^{3} a_{kj\star}^{(q)} \partial_k \partial_j \tag{10}$$

is written as

$$P_{q1\star}(x) = \frac{\alpha_q}{(x \cdot \mathbf{a}_{q\star}^{-1} x)^{\frac{1}{2}}} \ \ \text{with} \ \ \alpha_q = -\frac{1}{4\pi \, [\det \mathbf{a}_{q\star}]^{\frac{1}{2}}}, \ \ \mathbf{a}_{q\star} = [a_{kj\star}^{(q)}]_{3\times 3}. \tag{11}$$

Here $\mathbf{a}_{q\star}^{-1}$ stands for the inverse matrix to $\mathbf{a}_{q\star}$. Clearly, $\mathbf{a}_{q\star}^{-1}$ is symmetric and positive definite. Therefore there is a symmetric positive definite matrix $\mathbf{d}_{q\star}$ such that $\mathbf{a}_{q\star}^{-1} = \mathbf{d}_{q\star}^2$ and

$$(x \cdot \mathbf{a}_{q\star}^{-1} x) = |\mathbf{d}_{q\star} x|^2, \quad \det \mathbf{d}_{q\star} = [\det \mathbf{a}_{q\star}]^{-\frac{1}{2}}. \tag{12}$$

Note that $A_{q\star}(\partial_x) P_{q1\star}(x-y) = \delta(x-y)$, where $\delta(\cdot)$ is the Dirac distribution. Now we introduce the *localized parametrix* (*localized Levi function*) for the operator A_q,

$$P_q(x-y,y) \equiv P_{q\chi}(x-y,y) := a_q^{-1}(y)\, \chi_q(x-y)\, P_{q1\star}(x-y), \qquad q = 1,2, \tag{13}$$

where χ_q is a localizing cut-off function of the class X^3_{1+} introduced in [4]

$$\chi_q(x) := \chi(\mathbf{d}_{q\star}x) = \breve{\chi}(\,|\mathbf{d}_{q\star}x|\,) = \breve{\chi}\big((x\cdot \mathbf{a}_{q\star}^{-1}x)^{1/2}\big), \quad \chi \in X^3_{1+}. \tag{14}$$

One can easily check the following relations

$$A_q(x,\partial_x)\,P_q(x-y,y) = \delta(x-y) + R_q(x,y), \quad q=1,2, \tag{15}$$

where

$$\begin{aligned} R_q(x,y) =& \frac{a_q(x)}{a_q(y)}\big[P_{q1\star}(x-y)\,A_{q\star}(\partial_x)\chi_q(x-y) + 2\,\nabla_x\chi_q(x-y)\cdot\,\mathbf{a}_{q\star}\nabla_x P_{q1\star}(x-y)\big] \\ &+ \frac{1}{a_q(y)}\big(\,\nabla_x a_q(x)\cdot\mathbf{a}_{q\star}\nabla_x[\chi_q(x-y)P_{q1\star}(x-y)]\big). \end{aligned} \tag{16}$$

The function $R_q(x,y)$ possesses a weak singularity of type $\mathcal{O}(|x-y|^{-2})$ as $x\to y$.
Let us introduce the localized potentials, based on the localized parametrix P_q,

$$\begin{aligned} &V_S^{(q)}g(y) := -\int_S P_q(x-y,y)g(x)dS_x, \qquad &&W_S^{(q)}g(y) := -\int_S \big[\,T_q(x,\partial_x)\,P_q(x-y,y)\,\big]g(x)dS_x, \\ &\mathcal{P}_q\,f(y) := \int_{\Omega_q} P_q(x-y,y)\,f(x)\,dx, &&\mathcal{R}_q\,f(y) := \int_{\Omega_q} R_q(x,y)\,f(x)\,dx. \end{aligned}$$

Here $S\in\{S_i,S_e,\partial\Omega_2\}$. Note that for layer potentials we drop the subindex S when $S=\partial\Omega_q$, i.e., $V^{(q)} := V^{(q)}_{\partial\Omega_q}$, $W^{(q)} := W^{(q)}_{\partial\Omega_q}$.

Let us also define the corresponding boundary operators generated by the direct values of the localized single and double layer potentials and their co-normal derivatives on S,

$$\begin{aligned} &\mathcal{V}_S^{(q)}g(y) := -\int_S P_q(x-y,y)\,g(x)\,dS_x, \quad &&\mathcal{W}_S^{(q)}g(y) := -\int_S \big[\,T_q(x,\partial_x)P_q(x-y,y)\,\big]\,g(x)\,dS_x, \\ &\mathcal{W}_S'^{(q)}g(y) := -\int_S \big[\,T_q(y,\partial_y)\,P_q(x-y,y)\,\big]\,g(x)\,dS_x, &&\mathcal{L}_S^{\pm(q)}g(y) := T_q^{\pm}(y,\partial_y)\,W_S^{(q)}g(y). \end{aligned}$$

We employ also the notation $\mathcal{L}_S^{(q)} := \mathcal{L}_S^{+(q)}$.

Note that the kernel functions of the operators $\mathcal{W}_S^{(q)}$ and $\mathcal{W}_S'^{(q)}$ are at most weakly singular if the surface S is $C^{1,\alpha}$ smooth with $\alpha>0$. Before we go over to the localized boundary-domain integral formulation of the problem (CTD), we present some basic properties of the localized layer and volume potentials in the form of the following theorems (cf. [4]).

Theorem 2 *The following operators are continuous*

$$\begin{aligned} \mathcal{P}_q \ &:\ H^s(\Omega_q)\to H^{s+2}(\Omega_q), \quad -\tfrac12 < s < \tfrac52\,, \\ \mathcal{R}_q \ &:\ H^s(\Omega_q)\to H^{\frac52-\varepsilon}(\Omega_q), \quad \tfrac12 \le s\,, \end{aligned}$$

where ε is an arbitrarily small positive number.

Theorem 3 *The localized single and double layer potentials possess the mapping properties*

$$\begin{aligned} &V_S^{(q)}\ :\ H^{-\frac12}(S)\to H^{1,0}(\Omega_q;A_q), \qquad &&W_S^{(q)}\ :\ H^{\frac12}(S)\to H^{1,0}(\Omega_q;A_q), \\ &\mathcal{V}_S^{(q)}\ :\ H^{-\frac12}(S)\to H^{\frac12}(S), &&\mathcal{W}_S'^{(q)}\ :\ H^{-\frac12}(S)\to H^{-\frac12}(S), \\ &\mathcal{W}_S^{(q)}\ :\ H^{\frac12}(S)\to H^{\frac12}(S), &&\mathcal{L}_S^{\pm(q)}\ :\ H^{\frac12}(S)\to H^{-\frac12}(S). \end{aligned}$$

By the same arguments as in [4] one can easily show the following jump relations for localized layer potentials.

Theorem 4 *Let $g \in H^{-\frac{1}{2}}(S)$ and $h \in H^{\frac{1}{2}}(S)$. Then*

$$\gamma_q^+ V_S^{(q)} g = \gamma_q^- V_S^{(q)} g = \mathcal{V}_S^{(q)} g, \qquad T_q^{\pm} V_S^{(q)} g = \pm\tfrac{1}{2}\, g + \mathcal{W}_S'^{(q)} g,$$
$$\gamma_q^{\pm} W_S^{(q)} h = \mp\tfrac{1}{2}\, h + \mathcal{W}_S^{(q)} h, \qquad T_q^{+} W_S^{(q)} h - T_q^{-} W_S^{(q)} h = -(T_q a_q) g.$$

Throughout the rest of the paper we assume that the following relation holds on S_i

$$a_2(x) = \varkappa\, a_1(x) \quad \text{for} \quad x \in S_i \tag{17}$$

with some positive constant $\varkappa > 0$. We essentially apply the following assertion in our analysis.

Theorem 5 *Let $\chi \in X^3_{1+}$, the condition* (17) *hold, and*

$$G_q \in H^0(\Omega_q), \;\; g_{i1} \in \widetilde{H}^{-\frac{1}{2}}(S_i^{(t)}), \;\; g_{i2}, g_{i3} \in H^{\frac{1}{2}}(S_i), \;\; g_{i2} - g_{i3} \in \widetilde{H}^{\frac{1}{2}}(S_i^{(c)}), \;\; g_e \in H^{-\frac{1}{2}}(S_e).$$

Further let

$$V_{S_i}^{(1)}(g_{i1}) + W_{S_i}^{(1)}(g_{i2}) + \mathcal{P}_1(G_1) = 0 \;\text{ in }\; \Omega_1, \tag{18}$$
$$-V_{S_i}^{(2)}(g_{i1}) + W_{S_i}^{(2)}(g_{i3}) + V_{S_e}^{(2)}(g_e) + \mathcal{P}_2(G_2) = 0 \;\text{ in }\; \Omega_2. \tag{19}$$

Then $g_{i1} = g_{i2} = g_{i3} = 0$ on S_i, $g_e = 0$ on S_e and $G_q = 0$ in Ω_q, $q = 1, 2$.

4. BASIC LBDIE RELATIONS

We recall that the second Green's identity for the operator $A_q(x, \partial)$,

$$\int_{\Omega_q} [v\, A_q\, u - u\, A_q\, v]\, dx = \int_{\partial\Omega_q} [(\gamma_q v) T_q u - (\gamma_q u) T_q v]\, dS, \;\; q = 1, 2, \tag{20}$$

holds for $u, v \in H^{1,0}(\Omega_q; A_q)$ (cf., e.g., [6]).

Then for $v(x) := P_q(x - y, y)$ and $u = u_q \in H^{1,0}(\Omega_q; A_q)$ in (20), by the standard limiting procedure, we obtain the following parametrix-based third Green identity,

$$u_q + \mathcal{R}_q u_q - V^{(q)} T_q u_q + W^{(q)} \gamma_q u_q = \mathcal{P}_q A_q u_q \;\text{ in }\; \Omega_q. \tag{21}$$

Recall that for the layer potentials we drop the subindex S when $S = \partial\Omega_q$.
Taking into account the properties of localized potentials, the trace and co-normal derivative of (21) have the following form,

$$\frac{1}{2}\, \gamma_q u_q + \gamma_q \mathcal{R}_q u_q - \mathcal{V}^{(q)} T_q u_q + \mathcal{W}^{(q)} \gamma_q u_q = \gamma_q \mathcal{P}_q A_q u_q \;\text{ on }\; \partial\Omega_q, \tag{22}$$

$$\frac{1}{2}\, T_q u_q + T_q \mathcal{R}_q u_q - \mathcal{W}'^{(q)} T_q u_q + \mathcal{L}^{(q)} \gamma_q u_q = T_q \mathcal{P}_q A_q u_q \;\text{ on }\; \partial\Omega_q. \tag{23}$$

With the help of these relations one can construct various types of localized boundary domain integral equation systems for the above formulated interface crack problem.

5. LBDIE FORMULATION OF PROBLEM (CTD) AND BASIC RESULTS

Let a pair $(u_1, u_2) \in H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)$ be a solution to the problem (CTD). Denote by $\Psi_{0i} \in H^{-\frac{1}{2}}(S_i)$ some fixed extension of the function $\psi'_{0i} - \psi''_{0i}$ from $S_i^{(c)}$ onto the whole of S_i preserving the function space. Analogously, let $\Phi_{0i} \in H^{\frac{1}{2}}(S_i)$ be some fixed extension of the function $\varphi_{0i}^{(t)}$ from $S_i^{(t)}$ onto the whole of S_i preserving the function space.

Then we can write the following relations on S_i

$$T_1u_1 = \frac{1}{2}[T_1u_1 + T_2u_2] + \frac{1}{2}[T_1u_1 - T_2u_2] = \frac{1}{2}\psi_{0i} + \frac{1}{2}\Psi_{0i} + \widetilde{\psi}_i, \tag{24}$$

$$T_2u_2 = \frac{1}{2}[T_1u_1 + T_2u_2] - \frac{1}{2}[T_1u_1 - T_2u_2] = \frac{1}{2}\psi_{0i} - \frac{1}{2}\Psi_{0i} - \widetilde{\psi}_i, \tag{25}$$

$$\gamma_1u_1 = \frac{1}{2}[\gamma_1u_1 + \gamma_2u_2] + \frac{1}{2}[\gamma_1u_1 - \gamma_2u_2] = \frac{1}{2}\Phi_{0i} + \varphi_i + \widetilde{\varphi}_i, \tag{26}$$

$$\gamma_2u_2 = \frac{1}{2}[\gamma_1u_1 + \gamma_2u_2] - \frac{1}{2}[\gamma_1u_1 - \gamma_2u_2] = -\frac{1}{2}\Phi_{0i} + \varphi_i - \widetilde{\varphi}_i, \tag{27}$$

where ψ_{0i} is given by (9), while

$$\widetilde{\psi}_i := \frac{1}{2}[T_1u_1 - T_2u_2] - \frac{1}{2}\Psi_{0i} \in \widetilde{H}^{-1/2}(S_i^{(t)}), \quad \varphi_i := \frac{1}{2}[\gamma_1u_1 + \gamma_2u_2] \in H^{1/2}(S_i), \tag{28}$$

$$\widetilde{\varphi}_i := \frac{1}{2}[\gamma_1u_1 - \gamma_2u_2] - \frac{1}{2}\Phi_{0i} \in \widetilde{H}^{1/2}(S_i^{(c)}), \tag{29}$$

are unknown functions. Let us introduce one more unknown function defined on S_e

$$\psi_e := T_2u_2 \in H^{-1/2}(S_e)\,. \tag{30}$$

Then after substituting the notation (24)-(27) and (30) into equations (21), (22) and (23), and taking into consideration the relations (4)-(7) we arrive at the following system of direct segregated LBDIE for the components of unknown vector $U^{(CTD)} = (u_1, u_2, \widetilde{\psi}_i, \varphi_i, \widetilde{\varphi}_i, \psi_e)$,

$$u_1 + \mathcal{R}_1u_1 - V_{S_i}^{(1)}\widetilde{\psi}_i + W_{S_i}^{(1)}\varphi_i + W_{S_i}^{(1)}\widetilde{\varphi}_i = F_1^{(CTD)} \ \text{ in } \ \Omega_1, \tag{31}$$

$$u_2 + \mathcal{R}_2u_2 + V_{S_i}^{(2)}\widetilde{\psi}_i + W_{S_i}^{(2)}\varphi_i - W_{S_i}^{(2)}\widetilde{\varphi}_i - V_{S_e}^{(2)}\psi_e = F_2^{(CTD)} \ \text{ in } \ \Omega_2, \tag{32}$$

$$\gamma_1\mathcal{R}_1u_1 - \gamma_2\mathcal{R}_2u_2 - [\mathcal{V}_{S_i}^{(1)} + \mathcal{V}_{S_i}^{(2)}]\widetilde{\psi}_i + [\mathcal{W}_{S_i}^{(1)} - \mathcal{W}_{S_i}^{(2)}]\varphi_i + [\mathcal{W}_{S_i}^{(1)} + \mathcal{W}_{S_i}^{(2)}]\widetilde{\varphi}_i + \gamma_2V_{S_e}^{(2)}\psi_e$$
$$= \gamma_1F_1^{(CTD)} - \gamma_2F_2^{(CTD)} - \Phi_{0i} \ \text{ on } \ S_i^{(t)}, \tag{33}$$

$$T_1\mathcal{R}_1u_1 + T_2\mathcal{R}_2u_2 - [\mathcal{W}'^{(1)}_{S_i} - \mathcal{W}'^{(2)}_{S_i}]\widetilde{\psi}_i + [\mathcal{L}_{S_i}^{(1)} + \mathcal{L}_{S_i}^{(2)}]\varphi_i + [\mathcal{L}_{S_i}^{(1)} - \mathcal{L}_{S_i}^{(2)}]\widetilde{\varphi}_i - T_2V_{S_e}^{(2)}\psi_e$$
$$= T_1F_1^{(CTD)} + T_2F_2^{(CTD)} - \psi_{0i} \ \text{ on } \ S_i, \tag{34}$$

$$T_1\mathcal{R}_1u_1 - T_2\mathcal{R}_2u_2 - [\mathcal{W}'^{(1)}_{S_i} + \mathcal{W}'^{(2)}_{S_i}]\widetilde{\psi}_i + [\mathcal{L}_{S_i}^{(1)} - \mathcal{L}_{S_i}^{(2)}]\varphi_i + [\mathcal{L}_{S_i}^{(1)} + \mathcal{L}_{S_i}^{(2)}]\widetilde{\varphi}_i + T_2V_{S_e}^{(2)}\psi_e$$
$$= T_1F_1^{(CTD)} - T_2F_2^{(CTD)} - \Psi_{0i} \ \text{ on } \ S_i^{(c)}, \tag{35}$$

$$\gamma_2\mathcal{R}_2u_2 + \gamma_2V_{S_i}^{(2)}\widetilde{\psi}_i + \gamma_2W_{S_i}^{(2)}\varphi_i - \gamma_2W_{S_i}^{(2)}\widetilde{\varphi}_i - \mathcal{V}_{S_e}^{(2)}\psi_e = \gamma_2F_2^{(TM)} - \varphi_{0e} \ \text{ on } \ S_e, \tag{36}$$

where

$$F_1^{(CTD)} = \mathcal{P}_1f_1 + \tfrac{1}{2}V_{S_i}^{(1)}\psi_{0i} + \tfrac{1}{2}V_{S_i}^{(1)}\Psi_{0i} - \tfrac{1}{2}W_{S_i}^{(1)}\Phi_{0i} \quad \text{in} \quad \Omega_1,$$
$$F_2^{(CTD)} = \mathcal{P}_2f_2 + \tfrac{1}{2}V_{S_i}^{(2)}\psi_{0i} - \tfrac{1}{2}V_{S_i}^{(2)}\Psi_{0i} + \tfrac{1}{2}W_{S_i}^{(2)}\Phi_{0i} - W_{S_e}^{(2)}\varphi_{0e} \quad \text{in} \quad \Omega_2.$$

There holds the following equivalence theorem.

Theorem 6 *Let conditions* (8), (17) *be satisfied and* $\psi_{0i} \in H^{-1/2}(S_i)$ *with* ψ_{0i} *defined in* (9), *and* Ψ_{0i} *and* Φ_{0i} *be the above introduced extensions of the functions* $\psi_{0i}^{(t)}$ *and* $\varphi_{0i}^{(t)}$ *respectively.*
(i) *If a pair* $(u_1, u_2) \in H^{1,0}(\Omega_1; A_1) \times H^{1,0}(\Omega_2; A_2)$ *solves the interface crack problem (CTD), then the vector* $U^{(CTD)} = (u_1, u_2, \widetilde{\psi}_i, \varphi_i, \widetilde{\varphi}_i, \psi_e)$, *where* $\widetilde{\psi}_i$, φ_i, $\widetilde{\varphi}_i$ *and* ψ_e *are defined by relations* (28)-(30), *solves the LBDIE system* (31)-(36), *and vice versa,*

(ii) *if a vector*

$$U^{(CTD)}=(u_1,u_2,\widetilde{\psi}_i,\varphi_i,\widetilde{\varphi}_i,\psi_e)\in\mathbb{H}^{(TD)}, \tag{37}$$

$$\mathbb{H}^{(CTD)}:=H^{1,0}(\Omega_1;L_1)\times H^{1,0}(\Omega_2;L_2)\times\widetilde{H}^{-\frac{1}{2}}(S_i^{(t)})\times H^{\frac{1}{2}}(S_i)\times\widetilde{H}^{\frac{1}{2}}(S_i^{(c)})\times H^{-\frac{1}{2}}(S_e), \tag{38}$$

solves the LBDIE system (31)-(36)*, then the pair* (u_1,u_2) *solves the problem (CTD) and the relations* (24)-(30) *hold true.*

Due to this equivalence theorem and Theorem 1 we conclude that the LBDIE system (31)-(36) with the special right hand side functions, which belong to the space

$$\mathbb{F}^{(CTD)}:=H^{1,0}(\Omega_1;A_1)\times H^{1,0}(\Omega_2;A_2)\times H^{\frac{1}{2}}(S_i^{(t)})\times H^{-\frac{1}{2}}(S_i)\times H^{-\frac{1}{2}}(S_i^{(c)})\times H^{\frac{1}{2}}(S_e)\,, \tag{39}$$

is uniquely solvable in the space $\mathbb{H}^{(CTD)}$ defined in (38). In particular, the corresponding homogeneous LBDIE system possesses only the trivial solution. However, these results do not lead to invertibility of the corresponding localized boundary-domain integral operator and some additional analysis is needed.

Our main goal is to establish that the matrix operator $\mathcal{K}^{(CTD)}:=[\mathcal{K}_{kj}^{(CTD)}]_{6\times 6}:=\mathbf{R}\,\mathbf{K}$ generated by the left hand side expressions in the LBDIE system (31)-(36) is invertible in appropriate function spaces. Here

$$\mathbf{R}:=\begin{bmatrix} r_{\Omega_1} & 0 & 0 & 0 & 0 & 0\\ 0 & r_{\Omega_2} & 0 & 0 & 0 & 0\\ 0 & 0 & r_{S_i^{(t)}} & 0 & 0 & 0\\ 0 & 0 & 0 & r_{S_i} & 0 & 0\\ 0 & 0 & 0 & 0 & r_{S_i^{(c)}} & 0\\ 0 & 0 & 0 & 0 & 0 & r_{S_e}\end{bmatrix},$$

$$\mathbf{K}:=\begin{bmatrix} I+\mathcal{R}_1 & 0 & -V_{S_i}^{(1)} & W_{S_i}^{(1)} & W_{S_i}^{(1)} & 0\\ 0 & I+\mathcal{R}_2 & V_{S_i}^{(2)} & W_{S_i}^{(2)} & -W_{S_i}^{(2)} & -V_{S_e}^{(2)}\\ \gamma_1\mathcal{R}_1 & -\gamma_2\mathcal{R}_2 & -\mathcal{V}_{S_i}^{(1)}-\mathcal{V}_{S_i}^{(2)} & \mathcal{W}_{S_i}^{(1)}-\mathcal{W}_{S_i}^{(2)} & \mathcal{W}_{S_i}^{(1)}+\mathcal{W}_{S_i}^{(2)} & \gamma_2V_{S_e}^{(2)}\\ T_1\mathcal{R}_1 & T_2\mathcal{R}_2 & -\mathcal{W}'^{(1)}_{S_i}+\mathcal{W}'^{(2)}_{S_i} & \mathcal{L}_{S_i}^{(1)}+\mathcal{L}_{S_i}^{(2)} & \mathcal{L}_{S_i}^{(1)}-\mathcal{L}_{S_i}^{(2)} & -T_2V_{S_e}^{(2)}\\ T_1\mathcal{R}_1 & -T_2\mathcal{R}_2 & -\mathcal{W}'^{(1)}_{S_i}-\mathcal{W}'^{(2)}_{S_i} & \mathcal{L}_{S_i}^{(1)}-\mathcal{L}_{S_i}^{(2)} & \mathcal{L}_{S_i}^{(1)}+\mathcal{L}_{S_i}^{(2)} & T_2V_{S_e}^{(2)}\\ 0 & \gamma_2\mathcal{R}_2 & \gamma_2V_{S_i}^{(2)} & \gamma_2W_{S_i}^{(2)} & -\gamma_2W_{S_i}^{(2)} & -\mathcal{V}_{S_e}^{(2)}\end{bmatrix}.$$

Introduce the function spaces

$$\mathbb{X}^{(CTD)}:=H^1(\Omega_1)\times H^1(\Omega_2)\times\widetilde{H}^{-\frac{1}{2}}(S_1^{(t)})\times H^{\frac{1}{2}}(S_1)\times\widetilde{H}^{\frac{1}{2}}(S_1^{(c)})\times H^{-\frac{1}{2}}(S_2)\,, \tag{40}$$

$$\mathbb{Y}^{(CTD)}:=H^1(\Omega_1)\times H^1(\Omega_2)\times H^{\frac{1}{2}}(S_1^{(t)})\times H^{-\frac{1}{2}}(S_1)\times H^{-\frac{1}{2}}(S_1^{(c)})\times H^{\frac{1}{2}}(S_2)\,. \tag{41}$$

By Theorems 2 and 3 we see that the operator $\mathcal{K}^{(CTD)}$ has the following mapping property

$$\mathcal{K}^{(CTD)}\,:\,\mathbb{X}^{(CTD)}\to\mathbb{Y}^{(CTD)}. \tag{42}$$

A counterpart of the LBDIE system (31)-(36) can be written then as

$$\mathcal{K}^{(CTD)}U^{(CTD)}=\Psi, \tag{43}$$

where $U^{(CTD)}=(u_1,u_2,\widetilde{\psi}_i,\varphi_i,\widetilde{\varphi}_i,\psi_e)\in\mathbb{X}^{(CTD)}$ is the unknown vector and Ψ is an arbitrary vector function from the space $\mathbb{Y}^{(CTD)}$. Now we formulate our basic result.

Theorem 7 *The operators*

$$\mathcal{K}^{(CTD)} : \mathbb{X}^{(CTD)} \to \mathbb{Y}^{(CTD)},$$
$$\mathcal{K}^{(CTD)} : \mathbb{H}^{(CTD)} \to \mathbb{F}^{(CTD)},$$

are invertible.

From this theorem, in particular, it follows that equation (43) is uniquely solvable in the space $\mathbb{X}^{(CTD)}$ for arbitrary right hand side vector $\Psi \in \mathbb{Y}^{(CTD)}$.

Acknowledgements
This research was supported by the EPSRC grant No. EP/H020497/1: "Mathematical analysis of Localized Boundary-Domain Integral Equations for Variable-Coefficient Boundary Value Problems" and partly by the Georgian Technical University grant in the case of the third author.

REFERENCES

1. T. Zhu, J.-D. Zhang, and S.N. Atluri (1998) A local boundary integral equation (LBIE) method in computational mechanics, and a meshless discretization approach, *Comput. Mech.* **21**, 223–235.

2. T. Zhu, J.-D. Zhang, and S.N. Atluri (1999), A meshless numerical method based on the local boundary integral equation (LBIE) to solve linear and non-linear boundary value problems, *Eng. Anal. Bound. Elem.* **23**, 375–389.

3. S.E. Mikhailov (2002) Localized boundary-domain integral formulation for problems with variable coefficients, *Int. J. Engineering Analysis with Boundary Elements* **26**, 681–690.

4. O. Chkadua, S.E. Mikhailov, and D. Natroshvili (2009) Analysis of some localized boundary-domain integral equations, *J. Integral Equations Appl.* **21**, 407–447.

5. O. Chkadua, S.E. Mikhailov, and D. Natroshvili (2011) Analysis of segregated boundary-domain integral equations for variable-coefficient problems with cracks, *Numer. Meth. for PDEs*, **27**, 121-140

6. S. E. Mikhailov (2011) Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains. *J. Math. Analysis and Appl.*, doi:10.1016/j.jmaa.2010.12.027.