Analysis of Segregated Boundary–Domain Integral Equations for Mixed Variable-Coefficient BVPs in Exterior Domains

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1 Introduction

The direct segregated boundary–domain integral equations for the mixed boundary-value problem for a scalar second order elliptic partial differential equation with variable coefficient in an exterior domain in $\mathbb{R}^3$ are analyzed in this paper. In the literature the boundary-value problems considered here have been investigated using variational methods in weighted Sobolev spaces, particularly in [Han71, NP73, GN78, Mäu83, Gir87, DL90, Néd01]. For some cases of the PDE with constant coefficients, when the fundamental solution is available, the Dirichlet and Neumann type boundary-value problems in exterior domains were also investigated by the classical potential (indirect boundary integral equation) method, see [NP73, GN78, Gir87, DL90, CC00, Néd01] and the references therein.

Our goal here is to show that the mixed problems with variable coefficients can be reduced to some systems of *boundary–domain integral equations* (BDIEs) and investigate equivalence of the reduction and invertibility of the corresponding boundary–domain integral operators in the weighted Sobolev spaces. To do this, we extend to the exterior domains and weighted spaces the methods developed in [CMN09a] for the interior domains and standard Sobolev (Bessel potential) spaces.

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2 Basic Notation and Spaces

Let $\Omega = \Omega^+$ be an unbounded (exterior) open three-dimensional region of $\mathbb{R}^3$ such that $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega}$ is a bounded open domain. For simplicity, we assume that the boundary $\partial\Omega = \partial\Omega^-$ is a simply connected, closed, infinitely smooth surface.

Let $\rho(x) := (1+|x|^2)^{1/2}$ be the weight function and $a \in C^\infty(\mathbb{R}^3)$ be such that

$$0 < a_0 < a(x) < a_1 < \infty, \qquad \rho(x)|\nabla a(x)| + \rho^2(x)|\Delta a(x)| < C < \infty, \quad x \in \mathbb{R}^3. \quad (1)$$

Let also $\partial_j = \partial_{x_j} := \partial/\partial x_j$ $(j = 1,2,3)$, $\nabla = \partial_x = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$.

We consider below some boundary–domain integral equations associated with the following scalar elliptic differential equation

$$Au(x) := A(x,\partial_x)\,u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(a(x)\,\frac{\partial u(x)}{\partial x_i}\right) = f(x), \quad x \in \Omega, \qquad (2)$$

where u is an unknown function and f is a given function in Ω.

In what follows, $H^s(\Omega) = H^s_2(\Omega)$, $H^s(\partial\Omega) = H^s_2(\partial\Omega)$ denote the Bessel potential spaces (coinciding with the Sobolev–Slobodetski spaces if $s \geq 0$), $H^s_{\partial\Omega} := \{g : \; g \in H^s(\mathbb{R}^3),\ \mathrm{supp}\, g \subset \partial\Omega\}$. For an open set Ω, we denote $\mathscr{D}(\Omega) = C^\infty_{comp}(\Omega)$, endowed with sequential continuity, $\mathscr{D}^*(\Omega)$ is the Schwartz space of sequentially continuous functionals on $\mathscr{D}(\Omega)$, while $\mathscr{D}(\bar{\Omega})$ is the set of restrictions on $\bar{\Omega}$ of functions from $\mathscr{D}(\mathbb{R}^3)$. We also denote $\widetilde{H}^s(S_1) = \{g : \; g \in H^s(S),\ \mathrm{supp}\, g \subset \overline{S_1}\}$, $H^s(S_1) = \{r_{S_1} g : \; g \in H^s(S)\}$, where S_1 is a proper submanifold of a closed surface S and r_{S_1} is the restriction operator on S_1.

To make solution of boundary-value problems for (2) in infinite domains unique, we will use weighted Sobolev spaces (see e.g. [Han71, NP73, GN78, Mäu83, Gir87, DL90, Néd01]). Let $L_2(\rho^{-1};\Omega) := \{g : \rho^{-1} g \in L_2(\Omega)\}$ and $\mathscr{H}^1(\Omega)$ be the Beppo–Levi space,

$$\mathscr{H}^1(\Omega) := \{g \in L_2(\rho^{-1};\Omega) \;:\; \nabla g \in L_2(\Omega)\}, \qquad \|g\|^2_{\mathscr{H}^1(\Omega)} := \|\rho^{-1} g\|^2_{L_2(\Omega)} + \|\nabla g\|^2_{L_2(\Omega)}.$$

Using the corresponding property for the space $H^1(\Omega)$, it is easy to prove that $\mathscr{D}(\overline{\Omega})$ is dense in $\mathscr{H}^1(\Omega)$, cf. [Han71, Theorem I.1], [Gir87, Theorem 2.2]. If Ω is unbounded, then the semi-norm $|g|_{\mathscr{H}^1(\Omega)} := \|\nabla g\|_{L_2(\Omega)}$ is equivalent to the norm $\|g\|_{\mathscr{H}^1(\Omega)}$ in $\mathscr{H}^1(\Omega)$, see e.g. [DL90, Ch. XI, Part B, §1]. If Ω is bounded, then $\mathscr{H}^1(\Omega) = H^1(\Omega)$. If Ω' is a bounded subdomain of an unbounded domain Ω and $g \in \mathscr{H}^1(\Omega)$, then $g \in H^1(\Omega')$.

Let us define $\widetilde{\mathscr{H}}^1(\Omega)$ as a completion of $\mathscr{D}(\Omega)$ in $\mathscr{H}^1(\mathbb{R}^3)$, $\widetilde{\mathscr{H}}^{-1}(\Omega) := [\mathscr{H}^1(\Omega)]^*$, $\mathscr{H}^{-1}(\Omega) := [\widetilde{\mathscr{H}}^1(\Omega)]^*$, $L_2(\rho;\Omega) := \{g : \rho g \in L_2(\Omega)\}$. Evidently $L_2(\rho;\Omega) \subset \mathscr{H}^{-1}(\Omega)$. Any distribution $g \in \widetilde{\mathscr{H}}^{-1}(\Omega)$ has a representation $g =$