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The PBH test for multidimensional LTID systems ☆

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Abstract

In this article, the classical Popov–Belevitch–Hautus controllability test is generalized to higher dimensions.

Introduction

The well-known PBH (Popov–Belevitch–Hautus) controllability test gives a necessary and sufficient condition for controllability of a classical linear system. Namely, there holds the following. The linear system $\dot{x} = Ax + Bu$ is controllable if and only if $rk\,[\lambda I - AB] = d \quad \forall \lambda \in \mathscr{C}$, where d is the dimension of states. This statement has been generalized then by Willems (1991) as follows. The linear differential system$R\,(d/dt)\,w = 0$ is controllable if and only if $R\,(\lambda)$ has full row rank for all $\lambda \in \mathscr{C}$. (Here R is a full row rank polynomial matrix in one indeterminate s.)

Our goal in this article is to give two generalizations of the PBH test to higher dimensions.

To motivate our approach, we reformulate the one-dimensional statement in the following manner. Let R be as above, and let G denote the transpose of R. The matrix G has full column rank, and if $p \times q$ is the size of R, then $0 \to \mathbb{R}[s]^p \xrightarrow{G} \mathbb{R}[s]^q$ is an exact sequence. For an irreducible polynomial $\mathfrak{p}$, let $G/\mathfrak{p}$ denote the matrix G modulo $\mathfrak{p}$. This is a matrix with entries in the field

$\mathbb{R}(\mathfrak{p}) = \mathbb{R}[s]/\mathfrak{p}$. The PBH test asserts that the system $R(\partial)w = 0$ is controllable if and only if the sequence $0 \to \mathbb{R}(\mathfrak{p})^p \overset{G/\mathfrak{p}}{\longrightarrow} \mathbb{R}(\mathfrak{p})^q$ is exact for all irreducible polynomials $\mathfrak{p}$.

We now describe the general setting for our approach.

Throughout, $\mathbb{F}$ is a field, $s_1, \ldots, s_n$ are indeterminates, and q is a fixed nonnegative integer. Let $\overline{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}$. We write $s = (s_1, \ldots, s_n)$, and so $\mathbb{F}[s]$ will be a polynomial ring in n indeterminates.

We assume that given is an $\mathbb{F}$-linear space $\mathscr{U}$ together with endomorphisms $\partial_1 : \mathscr{U} \to \mathscr{U}, \ldots, \partial_n : \mathscr{U} \to \mathscr{U}$ that commute with each other. These endomorphisms permit to view $\mathscr{U}$ as an $\mathbb{F}[s]$-module. (The indeterminate s_i acts on $\mathscr{U}$ as the operator ∂_i.) We require that as such $\mathscr{U}$ be an *injective cogenerator* module (see Section 19A in Lam, 1999 for the notion of injective cogenerator). Important examples of $\mathscr{U}$ are the spaces of infinitely differentiable functions (or distributions) and the spaces of sequences.

Put $\partial = (\partial_1, \ldots, \partial_n)$. By an LTID (linear time-invariant dynamical) system with signal number q, we understand a subset of $\mathscr{U}^q$ that can be represented as the solution set of an equation of the form $R(\partial)w = 0, \quad w \in \mathscr{U}^q$, where R is a polynomial matrix with q columns. Associated with an LTID system $\mathscr{B}$ there is an important submodule $\mathbf{Ann}(\mathscr{B}) \subseteq \mathbb{F}[s]^q$, the annihilator of $\mathscr{B}$. It is defined by the formula $\mathbf{Ann}(\mathscr{B}) = \{f \in \mathbb{F}[s]^q | f^{tr}(\partial)w = 0 \forall w \in \mathscr{B}\}$. (The "$\mathbf{tr}$" stands for "transpose".) The map $\mathscr{B} \mapsto \mathbf{Ann}(\mathscr{B})$ establishes a one-to-one correspondence between the set of LTID systems in $\mathscr{U}^q$ and the set of submodules in $\mathbb{F}[s]^q$ (see Oberst, 1990). We remind that if an LTID system $\mathscr{B}$ is given by the equation $R(\partial)w = 0$, then $\mathbf{Ann}(\mathscr{B}) = R^{tr}\mathbb{F}[s]^p$, where p is the row number of R.

Let $\mathscr{B}$ be an LTID system with annihilator A. Then, $\mathscr{B}$ is called controllable (or weakly controllable) if the module $\mathbb{F}[s]^q/A$ is torsion free; the system is said to be strongly controllable if this quotient module is free. (Needless to say that in dimension 1, the two types of controllability coincide.)

Remark

As far as we know, the above definitions of controllability are due to M. Fliess and U. Oberst. The system-theoretic interpretations of these two types of controllability can be found in Lomadze, 2011, Lomadze, 2012, Pillai and Shankar (1999), Rocha (2012), Rocha and Willems (1999), Rocha and Wood (2001), Zerz (2001) and Zerz and Rocha (2006).

By a polynomial complex, we understand a sequence $(G_1, \ldots, G_n)$ of polynomial matrices, where G_1 has q rows and all products $G_1G_2, \ldots, G_{n-1}G_n$ (are defined and) are zero. The size is defined to be $(p_1, \ldots, p_n)$, where $p_1, \ldots, p_n$ are the column numbers of $G_1, \ldots, G_n$, respectively. Say that $(G_1, \ldots, G_n)$ is a polynomial resolution if $0 \to \mathbb{F}[s]^{p_n} \overset{G_n}{\to} \mathbb{F}[s]^{p_{n-1}} \to \cdots \to \mathbb{F}[s]^{p_1} \overset{G_1}{\to} \mathbb{F}[s]^q$ is exact. If A

is the image of G_1, then $(G_1, \ldots, G_n)$ determines a free resolution of A in the sense of homological algebra, and we say that $(G_1, \ldots, G_n)$ is a polynomial resolution of A.

Polynomial resolutions (of length n) should be considered as analogs of full column rank polynomial matrices in dimension n. (See Lomadze, in press.) By Hilbert's syzygy theorem (see Corollary 19.7 in Eisenbud, 1995), any submodule of $\mathbb{F}[s]^q$ has a polynomial resolution.

We define a polynomial resolution of an LTID system as a polynomial resolution of its annihilator.

Section snippets

The PBH test for (weak) controllability

We begin with the following technical lemma.

Lemma 1

Let R be a (commutative) ring, and let λ be a non zero-divisor element of R . Suppose that $0 \to L \to F \to M \to 0$ is a short exact sequence of modules, where F is projective. Then the following three conditions are equivalent:

(a) *λ is not a zero-divisor on M ;...*

(b) *the homomorphism $L/\lambda L \to F/\lambda F$ is injective....*

(c) *the sequence $0 \to L/\lambda L \to F/\lambda F \to M/\lambda M \to 0$ is exact....*

...

Proof

Consider the diagram $$\begin{array}{ccccccc} 0 \to & L & \to & F & \to & M & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & L & \to & F & \to & M & \to 0, \end{array}$$ where the downward arrows stand for the homomorphisms defined by multiplication by λ. The...

...

The PBH test for strong controllability

We begin with the following well-known lemma (for which, however, we do not know a reference).

Lemma 2

Let R be a Noetherian (commutative) ring, and let $u : L \to M$ be a homomorphism of finitely generated projective R-modules. Then the following two conditions are equivalent:

(a) *for each maximal ideal $\mathfrak{m}$the homomorphism$L/\mathfrak{m}L \to M/\mathfrak{m}M$is injective;…*

(b) *u is injective and $\mathrm{Coker}\,(u)$ is projective.…*

…

Proof

By Theorem 1 in Section II.5.2 in Bourbaki (1989), a finitely generated R-module is projective if and only if all its localizations…

…

Concluding remarks

One defines the affine space $\mathbb{A}^n$ to be the set of equivalence classes in $\overline{\mathbb{F}}^n$. Two n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are equivalent if there exists an $\mathbb{F}$-automorphism σ of the field $\overline{\mathbb{F}}$ such that $y_i = \sigma(x_i)$ for $i = 1, \ldots, n$. (By Hilbert's Nullstellensatz (see Proposition 2 in Section V.3.3 in Bourbaki, 1989), these equivalence classes are in one-to-one correspondence with maximal ideals of $\mathbb{F}[s]$.)

If f is a polynomial in $\mathbb{F}[s]$, then $f(\sigma(x_1), \ldots, \sigma(x_n)) = \sigma f(x_1, \ldots, x_n)$, and so it does make sense to say that f is zero …

Acknowledgments

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