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Characterization of linear differential systems (in several variables)

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ABSTRACT

We show that a set of smooth trajectories is the solution set of a linear constant coefficient partial differential equation if and only if it is linear, shift-invariant and complete. (By completeness, we mean exactly what Willems called jet-completeness in his Automatica paper in 1986.)

1. Introduction

In all that follows n is the number of coordinates. We shall write ∂_p $(p = 1, \ldots, n)$ to denote the partial differentiation operators acting on $C^\infty(\mathbb{R}^n)$. Recall that a multi-index is an n-tuple of nonnegative integers, i.e., an element of $\mathbb{Z}_+^n$. For a multi-index $i = (i_1, \ldots, i_n)$, it is usual to write ∂^i for $\partial_1^{i_1} \ldots \partial_n^{i_n}$. If k is a nonnegative integer, we let $\Delta(k)$ denote the set of multi-indices of order less than or equal to k. (The order of $i = (i_1, \ldots, i_n)$ is defined to be $|i| = i_1 + \cdots + i_n$.)

Given a trajectory $w \in C^\infty(\mathbb{R}^n)$, a time $t \in \mathbb{R}^n$ and a nonnegative integer $k \in \mathbb{Z}_+$, one defines the k-jet $J_t^k(w)$ of w at t as follows:

$$J_t^k(w)(i) = \partial^i w(t), i \in \Delta(k).$$

For t and k as above, we thus have a mapping

$$J_t^k : C^\infty(\mathbb{R}^n) \to \mathbb{R}^{\Delta(k)}.$$

Let $\mathcal{B}$ be a subset of $C^\infty(\mathbb{R}^n)^q$, a (continuous-time) dynamical system with the signal number q. For every time $t \in \mathbb{R}^n$ and every nonnegative integer $k \in \mathbb{Z}_+$, we let $\mathcal{B}|_{t,k}$ denote the image of $\mathcal{B}$ under

$$J_t^k : C^\infty(\mathbb{R}^n)^q \to (\mathbb{R}^{\Delta(k)})^q.$$

The sets $\mathcal{B}|_{t,k}$ seem to be interesting invariants, and a natural idea, due to Willems [1], is to try to get information about the trajectories of $\mathcal{B}$ looking at all of them. This idea leads to what Willems called jet-completeness (see Section 6 in Willems [1]), but that we just call *completeness* here. The definition of this important notion runs like this. The dynamical system $\mathcal{B}$ is complete if it satisfies the following condition

$$\forall w \in C^\infty(\mathbb{R}^n)^q, \quad (w \in \mathcal{B} \Leftrightarrow J_t^k(w) \in \mathcal{B}|_{t,k} \quad \forall t, k).$$

By definition, completeness of $\mathcal{B}$ means that the jet-spaces $\mathcal{B}|_{t,k}$ contain complete information about the trajectories of $\mathcal{B}$.

In this article, we are going to show that the solution sets of linear constant coefficient partial differential equations are exactly those dynamical systems that are linear, shift-invariant and complete.

It should be interesting to compare the result with the existing discrete-time counterpart due to Willems. Let $C(\mathbb{Z}_+^n)$ be the space of all real-valued functions on $\mathbb{Z}_+^n$, and let $\sigma_p (p = 1, \ldots, n)$ be the partial shift operators in it. We remind that a dynamical system $\mathcal{B} \subseteq C(\mathbb{Z}_+^n)^q$ is said to be complete if

$$\forall w \in C(\mathbb{Z}_+^n)^q, \quad (w \in \mathcal{B} \Leftrightarrow w|_{\Delta(k)} \in \mathcal{B}|_{\Delta(k)} \forall k).$$

The Willems theorem states that $\mathcal{B}$ is the solution set of a linear constant coefficient partial difference equation if and only if it is linear, shift-invariant and complete. (Willems [1] proved the theorem for dimension 1, and it was extended then to higher dimensions by Oberst [2] and Rocha [3]; however, this extension is easy.) Notice that, for $w \in C(\mathbb{Z}_+^n)$ and $i \in \mathbb{Z}_+^n$, we have:

$$w(i) = \sigma^i w(0).$$

(Here $\sigma^i = \sigma_1^{i_1} \ldots \sigma_n^{i_n}$, where $i_1, \ldots, i_n$ are the components of i.) In view of this, $w|_{\Delta(k)}$ can be interpreted as the k-jet of w at 0. We therefore can regard the truncated spaces $\mathcal{B}|_{\Delta(k)}$ as the jet-spaces of $\mathcal{B}$ at 0. Thus, in the definition of completeness jet-spaces at 0 only are involved.

The point of the continuous-time case is that in $C^\infty(\mathbb{R}^n)$ there are a lot of flat functions. (A function $w \in C^\infty(\mathbb{R}^n)$ is said to be flat at time t, if all its derivatives $\partial^i w$ vanish at t.) A priori is clear that if

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$\mathcal{B} \subseteq C^\infty(\mathbb{R}^n)^q$, then the trajectories of $\mathcal{B}$ that are flat at time 0 by no means can be recovered from the knowledge of the jet-spaces $\mathcal{B}|_{0,k}$. For this reason, in order to define the completeness property, it is necessary to bring into play jet-spaces at *all times*.

We equip $C(\mathbb{Z}_+^n) = \mathbb{R}^{\mathbb{Z}_+^n}$ with the product topology. It is worth noting that this topology coincides with the pointwise convergence topology. (On the field of real numbers we consider the ordinary topology defined by the absolute value $|\ |$.)

We let $s_1, \dots, s_n$ be indeterminates. We put $s = (s_1, \dots, s_n)$ and write $\mathbb{R}[s]$ for the ring of polynomials in $s_1, \dots, s_n$. Likewise, we put

$$\partial = (\partial_1, \dots, \partial_n) \quad \text{and} \quad \sigma = (\sigma_1, \dots, \sigma_n).$$

We let $A(\mathbb{Z}_+^n)$ denote the set of all $a \in C(\mathbb{Z}_+^n)$ such that the power series (in $\mathbb{R}^n$)

$$L(a)(x) = \sum_{i \in \mathbb{Z}_+^n} a(i) \frac{x^i}{i!}$$

is uniformly convergent on compact subsets of $\mathbb{R}^n$. (Here $x^i = x_1^{i_1} \cdots x_r^{i_n}$ and $i! = i_1! \cdots i_n!$.) By the very definition, the functions $L(a)$ are entire analytic functions. It is clear that $A(\mathbb{Z}_+^n)$ is invariant with respect to the partial shift operators. (Hence these operators make it a module over $\mathbb{R}[s]$.)

We shall use "tr" for the transpose. For any topological vector space V, V^* will denote the space of continuous linear functionals on V. For any integer $k \geq 0$, $\mathbb{R}[s]_{\leq k}$ will denote the set of polynomials of degree $\leq k$.

The prerequisite for this article is Oberst's theorem (see Oberst [4]), which is a basic fact and which says that $A(\mathbb{Z}_+^n)$ is a cogenerator module. We remind that a module $\mathcal{U}$ is called a cogenerator module if for every module M and every $0 \neq x \in M$, there exists a homomorphism $\phi : M \to \mathcal{U}$ such that $\phi(x) \neq 0$. (A proof of Oberst's theorem can be found also in [5].) We shall make use of also the well-known Hahn–Banach theorem in the following formulation. If X is a subspace of a Hausdorff locally convex space V, then an element $v \in V$ belongs to the closure $\overline{X}$ of X if and only if there is no continuous linear functional f on V such that $f|_X = 0$ but $f(v) \neq 0$. (See Theorem 3.5 in [6].)

Throughout, q is a fixed positive integer number.

2. Preliminaries

Consider the canonical bilinear form

$$\mathbb{R}[s]^q \times C(\mathbb{Z}_+^n)^q \to \mathbb{R},$$

given by

$$\langle f, g \rangle = (f^{tr}(\sigma)g)(0).$$

This bilinear form is important as it permits us to identify the space of continuous linear functionals on $C(\mathbb{Z}_+^n)^q$ with $\mathbb{R}[s]^q$. Let "$\perp$" denote the orthogonal complement with respect to this bilinear form.

For every $u \in \mathbb{R}[s]^l$ and $g \in C(\mathbb{Z}_+^n)^q$, we have

$$\langle R^{tr} u, g \rangle = (u^{tr}(\sigma)R(\sigma)g)(0). \tag{1}$$

Using this formula, it is easy to see that

$$(R^{tr}\mathbb{R}[s]^l)^\perp = Ker\ R(\sigma). \tag{2}$$

Lemma 1 (*Duality Theorem*)**.** *Let R be a polynomial matrix of size $l \times q$. Then*

$$(Ker\ R(\sigma))^\perp = R^{tr}\mathbb{R}[s]^l = (Ker\ R(\sigma) \cap A(\mathbb{Z}_+^n)^q)^\perp.$$

Proof. Let E be either $C(\mathbb{Z}_+^n)^q$ or $A(\mathbb{Z}_+^n)^q$. We have to show that

$$(Ker\ R(\sigma) \cap E)^\perp = R^{tr}\mathbb{R}[s]^l.$$

By (1), the inclusion "$\supseteq$" is immediate. To show "$\subseteq$", take any $h \in \mathbb{R}[s]^q$ that does not belong to $R^{tr}\mathbb{R}[s]^l$. Then, by the cogenerator property of E, there exists a homomorphism

$$\mathbb{R}[s]^q / R^{tr}\mathbb{R}[s]^l \to E$$

taking the coset of h to a nonzero element. In other words, there is a homomorphism $\phi : \mathbb{R}[s]^q \to E$ that is zero everywhere on $R^{tr}\mathbb{R}[s]^l$, but not on h. Multiplying ϕ, if necessary, by some power $s^i = s_1^{i_1} \dots s_r^{i_r}$, we may assume that $(\phi(h))(0) \neq 0$. Any homomorphism $\mathbb{R}[s]^q \to E$ is of the form $f \mapsto f^{tr}(\sigma)g$ with $g \in E$. So that there is $g \in E$ such that

$$\phi(f) = f^{tr}(\sigma)g \quad \forall f \in \mathbb{R}[s]^q.$$

For every $u \in \mathbb{R}[s]^l$, we have

$$0 = \phi(R^{tr}u) = u^{tr}(\sigma)R(\sigma)g.$$

It follows that $R(\sigma)g = 0$, and hence $g \in Ker\ R(\sigma) \cap E$. On the other hand,

$$0 \neq \phi(h)(0) = (h^{tr}(\sigma)g)(0) = \langle h, g \rangle,$$

that is, h is not orthogonal to $Ker\ R(\sigma) \cap E$.

The lemma is proved. □

One important consequence of the duality theorem is the Willems theorem. We shall carry out its proof, for the reader's convenience (and to make the text self-contained).

Lemma 2 (*Willems Theorem*)**.** *Let X be a subset of $C(\mathbb{Z}_+^n)^q$. For X to be the solution set of a linear constant coefficient partial difference equation it is necessary and sufficient that X be linear, shift-invariant and closed.*

Proof. Suppose that X is linear, shift-invariant and closed subspace in $C(\mathbb{Z}_+^n)^q$. Then $X^\perp$ is a submodule of $\mathbb{R}[s]^q$ and hence has the form $R^{tr}\mathbb{R}[s]^l$, where l is an integer and R is a polynomial matrix of size $l \times q$. By the duality theorem, we have

$$X^\perp = (Ker\ R(\sigma))^\perp.$$

This implies that

$$X^* = (Ker\ R(\sigma))^*.$$

Using the duality theorem and (2), we get

$$X \subseteq X^{\perp\perp} = (Ker\ R(\sigma))^{\perp\perp} = (R^{tr}\mathbb{R}[s]^l)^\perp = Ker\ R(\sigma).$$

Because X is closed and has the same continuous linear functionals as $Ker\ R(\sigma)$, by the Hahn–Banach theorem, we must have $X = Ker\ R(\sigma)$.

The necessity is obvious. □

Another important consequence of the duality theorem is the approximation theorem saying that the "A-solutions" of a linear constant coefficient partial difference equation are dense in the set of all solutions. This is the discrete-time analog of the Malgrange approximation theorem (see Theorem 7.14 in Hörmander [7]).

Lemma 3 (*Approximation Theorem*)**.** *Let R be a polynomial matrix. Then*

$$\overline{Ker\ R(\sigma) \cap A(\mathbb{Z}_+^n)^q} = Ker\ R(\sigma).$$

Proof. Let $l \times q$ be the size of R. By the duality theorem,

$$Ker\ R(\sigma)^\perp = (Ker\ R(\sigma) \cap A(\mathbb{Z}_+^n)^q)^\perp.$$

This yields

$$(Ker\ R(\sigma))^* = (Ker\ R(\sigma) \cap A(\mathbb{Z}_+^n)^q)^*.$$

If the lemma were false, by the Hahn–Banach theorem, we would have a nontrivial continuous linear functional on $Ker\ R(\sigma)$ that vanishes on $Ker\ R(\sigma) \cap A(\mathbb{Z}_+^n)^q$. But this is in contradiction with what we have obtained. □

Remark. The topological arguments in Lemmas 2 and 3 can be found in the proof of Theorem 2.23 in Bourlès and Oberst [8], where considerably more general situations were considered.

If $w \in C^\infty(\mathbb{R}^n)$, the (infinite) jet $J_t(w)$ of w at time $t \in \mathbb{R}^n$ is defined by

$$J_t(w)(i) = \partial^i w(t) \quad (i \in \mathbb{Z}_+^n).$$

For every time t, we thus have a linear map

$$J_t : C^\infty(\mathbb{R}^n) \to C(\mathbb{Z}_+^n).$$

One easily checks that

$$J_t \circ \partial_p = \sigma_p \circ J_t$$

for $p = 1, \ldots, n$. It is clear from this that if R is a polynomial matrix, then

$$J_t \circ R(\partial) = R(\sigma) \circ J_t, \tag{3}$$

and consequently

$$J_t(Ker\ R(\partial)) \subseteq Ker\ R(\sigma).$$

We come now to the lemma, which will permit us to derive the main result of the article from the Willems theorem.

Lemma 4. *Let R be a polynomial matrix, and let t be an arbitrary time. Then*

$$\overline{J_t(Ker\ R(\partial))} = Ker\ R(\sigma).$$

Proof. Without loss of generality, we may assume that $t = 0$. In view of the approximation theorem, it suffices to show that

$$Ker\ R(\sigma) \cap A(\mathbb{Z}_+^n)^q \subseteq J_0(Ker\ R(\partial)).$$

For $a \in A(\mathbb{Z}_+^n)^q$, we have

$$a \in Ker\ R(\sigma) \Leftrightarrow L(a) \in Ker\ R(\partial).$$

Next, for a as above,

$$a = J_0(L(a)).$$

The lemma follows. □

Remark. We believe that, in fact, there holds

$$J_t(Ker\ R(\partial)) = Ker\ R(\sigma).$$

But this stronger formula, in our opinion, is very hard to prove. (This is so in dimension 1 (see [9,10]).) The statement is a generalization of Borel's beautiful theorem, which says that

$$J_t(C^\infty(\mathbb{R}^n)) = C(\mathbb{Z}_+^n).$$

(See, for instance, Theorem 1.5.4 in Narasimhan [11].)

3. Characterization of LTID systems

We begin by remarking that a fundamental system of neighborhoods of 0 in $C(\mathbb{Z}_+^n)$ is given by the sets

$$U_{k,\varepsilon} = \{g \in C(\mathbb{Z}_+^n) : |g(i)| < \varepsilon\ \forall i \in \Delta(k)\},$$

where k runs over all nonnegative integer numbers and ε over all positive real numbers.

Lemma 5. *Let $X \subseteq C(\mathbb{Z}_+^n)^q$ be a linear subspace, and let $c \in C(\mathbb{Z}_+^n)^q$. Then*

$$c|_{\Delta(k)} \in X|_{\Delta(k)}\ \forall k \geq 0 \Leftrightarrow c \in \overline{X}.$$

Proof. "⇒" Consider any $U_{k,\varepsilon}$ with $k \in \mathbb{Z}_+$ and $\varepsilon > 0$. Take $x \in X$ such that $c|_{\Delta(k)} = x|_{\Delta(k)}$. Then clearly $x \in c + U_{k,\varepsilon}$. Because sets of the form $c + U_{k,\varepsilon}$ form a fundamental system of neighborhoods of c, we conclude that $c \in \overline{X}$.

"⇐" Suppose that there is k such that $c|_{\Delta(k)} \notin X|_{\Delta(k)}$. Identify $\mathbb{R}[s]^q_{\leq k}$ with the dual of $(\mathbb{R}^{\Delta(k)})^q$ via the canonical bilinear form

$$\mathbb{R}[s]^q_{\leq k} \times (\mathbb{R}^{\Delta(k)})^q \to \mathbb{R}$$

defined by

$$\left(\sum a_i s^i, b\right) \mapsto \sum_i a_i^{tr} b(i).$$

We can find $f \in \mathbb{R}[s]^q_{\leq k}$ that vanishes on $X|_{\Delta(k)}$, but $f(c|_{\Delta(k)}) \neq 0$. Notice that

$$\forall g \in C(\mathbb{Z}_+^n), \quad f(g|_{\Delta(k)}) = \langle f, g\rangle.$$

We therefore have:

$$f|_X = 0 \quad \text{and} \quad \langle f, c\rangle \neq 0.$$

The Hahn–Banach theorem implies that $c \notin \overline{X}$.

The proof is complete. □

Remark. It is immediate from this lemma that, for linear subspaces in $C(\mathbb{Z}_+^n)^q$, completeness and closedness are equivalent properties. This is a fact, which is well-known of course (see Willems [1,12]).

The previous lemma permits us to define completeness in the following manner. Let $\mathcal{B}$ be a dynamical system in $C^\infty(\mathbb{R}^n)^q$. For each time t, set

$$\mathcal{B}|_t = J_t(\mathcal{B}).$$

The system $\mathcal{B}$ is complete if it satisfies the following condition:

$$\forall w \in C^\infty(\mathbb{R}^n)^q, \quad (w \in \mathcal{B} \Leftrightarrow J_t(w) \in \overline{\mathcal{B}|_t}\ \forall t).$$

Recall that an LTID (linear time-invariant differential) system (with signal number q) is a subset of $C^\infty(\mathbb{R}^n)^q$ having the form $Ker\ R(\partial)$, where $R \in \mathbb{R}[s]^{\bullet\times q}$. Any such polynomial matrix is called a kernel representation. (See Polderman and Willems [13].)

Proposition 1. *An LTID system is complete.*

Proof. Let $\mathcal{B}$ be an LTID system, and let R be its kernel representation. Take any $w \in C^\infty(\mathbb{R}^n)^q$. By Lemma 4, if t is a time, then saying that $J_t(w) \in \overline{\mathcal{B}|_t}$ is equivalent to saying that $J_t(w) \in KerR(\sigma)$, that is, to saying that $R(\sigma)J_t(w) = 0$. By (3), we have

$$R(\sigma)J_t(w) = 0 \Leftrightarrow J_t(R(\partial)w) = 0 \Rightarrow (R(\partial)w)(t) = 0.$$

Hence

$$\forall t, \quad J_t(w) \in \overline{\mathcal{B}|_t} \Rightarrow \forall t, \qquad (R(\partial)w)(t) = 0 \Rightarrow w \in \mathcal{B}.$$

Thus, $\mathcal{B}$ is complete. □

Theorem 1. *A dynamical system is an LTID system if and only if it is linear, shift-invariant, and complete.*

Proof. We only need to prove the "If" part. For this, suppose that $\mathcal{B} \subseteq C^\infty(\mathbb{R}^n)^q$ is a linear, shift-invariant and complete dynamical system.

First, notice that $\mathcal{B}$ is a closed subset of $C^\infty(\mathbb{R}^n)^q$ (with respect to the standard C^∞-topology). Indeed, the condition that $\mathcal{B}$ is complete means that the kernel of the continuous linear map

$$C^\infty(\mathbb{R}^n)^q \to \prod_t C(\mathbb{Z}_+^n)^q/\overline{\mathcal{B}|_t}$$

coincides with $\mathcal{B}$. For every time t, the quotient space $C(\mathbb{Z}_+, \mathbb{R})^q/\overline{\mathcal{B}|_t}$ is a Hausdorff topological vector space, since $\overline{\mathcal{B}_t}$ is closed. Therefore, the product of all these quotient spaces is also Hausdorff. Hence $\mathcal{B}$, being the preimage of the closed set $\{0\}$, must be closed.

Now, it is a standard result of the theory of distributions (see p. 161 in Schwartz [14]) that a closed shift-invariant subspace of $C^\infty(\mathbb{R}^n)^q$ is differentiation-invariant. (A straightforward proof of this fact is easy. Indeed, let S^t denote the shift operator by a time t. From closedness and shift-invariance of $\mathcal{B}$, we have

$$\forall\, w \in \mathcal{B}, \quad \partial_p w = \lim_{\varepsilon \to 0} \frac{S^{\varepsilon e_p} w - w}{\varepsilon} \in \mathcal{B}.$$

Here e_p denotes the p-th unit vector in $\mathbb{R}^n$.)

Differentiation-invariance of $\mathcal{B}$ implies that the discrete-time linear dynamical systems $\mathcal{B}|_t$ are shift-invariant. So are the systems $\overline{\mathcal{B}|_t}$. In view of shift-invariance, all these systems are independent of t. By the Willems theorem, there exists a polynomial matrix R such that

$$\forall t, \quad \overline{\mathcal{B}|_t} = Ker\ R(\sigma).$$

From this, in view of Lemma 4, we have that

$$\overline{\mathcal{B}|_t} = \overline{Ker\ R(\partial)|_t}$$

for all t. We conclude that

$$\mathcal{B} = Ker\ R(\partial),$$

since both of these dynamical systems are complete.

The proof is complete. □

4. Concluding remarks

It was shown by Willems [1], Oberst [2] and Rocha [3] that a discrete-time dynamical system can be represented by a linear constant coefficient partial difference equation if and only if it is linear, shift-invariant and complete.

In this article, we have extended this result to the continuous-time case. It turned out that the same properties (linearity, shift-invariance and completeness) characterize as well LTID systems, the solution sets of linear constant coefficient differential equations.

The continuous-time version of the completeness property was introduced by Willems [1] (under the name of "jet-completeness"). The concept of continuous-time completeness is a bit more complicated due to the presence of flat functions in the space of C^∞-functions. It takes into account jet-spaces at all times, but not at one time only. The result has been derived without difficulty from its discrete-time counterpart by using the following formula

$$\overline{J_t(Ker\ R(\partial))} = Ker\ R(\sigma),$$

obtained in Lemma 4.

This article has something in common with Lomadze [15], where Taylor polynomials at 0 were employed to characterize LTID systems. (Needless to say that finite jets and Taylor polynomials are equivalent objects.) It is proved in the mentioned paper that a dynamical system $\mathcal{B} \subseteq C^\infty(\mathbb{R}^n)^q$ is an LTID system if and only if it is linear, differentiation-invariant, and satisfies the following two conditions:

(1) $\forall w \in A(\mathbb{R}^n)^q$, $(w \in \mathcal{B} \Leftrightarrow J_0^k(w) \in \mathcal{B}|_{0,k}\ \forall k)$;

(2) $\mathcal{B} = \overline{\mathcal{B} \cap A(\mathbb{R}^n)^q}$.

(Here $A(\mathbb{R}^n)$ denotes the space of entire analytic functions on $\mathbb{R}^n$.) Condition (1) means that the analytic trajectories of $\mathcal{B}$ are completely determined via the jet-spaces of $\mathcal{B}$ at time 0; condition (2) means that the analytic trajectories of $\mathcal{B}$ have density property (and therefore they, in turn, determine all other trajectories).

We remind that the first characterization of LTID systems was obtained by Oberst [2], which is as follows. LTID systems are exactly $\mathbb{R}[s]$-submodules of $C^\infty(\mathbb{R}^n)^q$ that are finitely generated as modules over the ring $\mathcal{E} = End_{\mathbb{R}[s]}(C^\infty(\mathbb{R}^n))$.

For the one-dimensional case, various different characterizations are presented in Delvenne [16], Lomadze [17,18,10] and Soethoudt [19].

We want to note that the result of the present article cannot be viewed as a generalization of that of Lomadze [10]. The latter states that a (one-dimensional) dynamical system is an LTID system if and only if it is linear, shift-invariant, jet-closed and jet-determined. (We recall from Willems [1] that a set $\mathcal{B} \subseteq C^\infty(\mathbb{R})^q$ is jet-closed if $J_t(\mathcal{B})$ is a closed subset of $C(\mathbb{Z}_+, \mathbb{R})^q$ for every t and jet-determined if

$$w \in \mathcal{B} \Leftrightarrow \forall t, \quad J_t(w) \in \mathcal{B}|_t).$$

The properties of jet-closedness and jet-determinedness can be obviously generalized to the nD case. However, the generalization of the above characterization requires the formula

$$J_t(Ker\ R(\partial)) = Ker\ R(\sigma),$$

which, as we have remarked, seems to be very hard to prove.

Appendix. Jet complexes

If $\mathcal{B}$ is a linear shift-invariant dynamical system, then the jet-spaces $\mathcal{B}|_{t,k}$ are independent of t and therefore we may denote them by $\mathcal{B}|_k$. What are the properties that characterize the sequences $(\mathcal{B}|_k)_{k\geq 0}$ obtained this way? This is a question that arises naturally.

For $1 \leq p \leq n$ and $k \geq 1$, let

$$\sigma_{p,k} : \mathbb{R}^{\Delta(k)} \to \mathbb{R}^{\Delta(k-1)}$$

be the map that is induced by the shift operator σ_p.

Call a jet complex (of size q) any sequence $(B_k)_{k\geq 0}$ of linear subspaces $B_k \subseteq (\mathbb{R}^{\Delta(k)})^q$ satisfying the following two simple conditions:

(1) $\sigma_{p,k}(B_k) \subseteq B_{k-1}\ \forall 1 \leq p \leq n,\ k \geq 1$;

(2) $B_k|_{\Delta(k-1)} = B_{k-1}\ \forall k \geq 1$.

One can easily check that, for every linear shift-invariant dynamical system $\mathcal{B}$, the sequence $(\mathcal{B}|_k)_{k\geq 0}$ is a jet complex.

The following theorem tells us, in particular, that any jet complex can be realized via an LTID system.

Theorem 2. *The mapping*

$$\mathcal{B} \mapsto (\mathcal{B}|_k)_{k\geq 0}$$

induces a one-to-one correspondence between LTID systems and jet complexes.

Sketched Proof. Injectivity is obvious by the completeness property. To show surjectivity, take any jet complex $(B_k)_{k\geq 0}$. For each $k \geq 0$, let C_k denote the orthogonal of B_k with respect to the bilinear form that we have defined in the proof of Lemma 5. The sequence $(C_k)_{k\geq 0}$ satisfies the following two conditions:

(1) $s_p C_{k-1} \subseteq C_k\ \forall 1 \leq p \leq n,\ k \geq 1$;

(2) $C_{k-1} = C_k \cap \mathbb{R}[s]^q_{\leq k-1}\ \forall k \geq 1$.

Define C as the union of all C_k. This is a submodule of $\mathbb{R}[s]^q$ and hence has the form $R^{tr}\mathbb{R}[s]^l$, where l is an integer and R is a polynomial matrix of size $l \times q$. We leave to the reader to show that the jet spaces of $Ker\ R(\partial)$ are precisely B_k $(k = 0, 1, 2, \ldots)$.

Say that two linear shift-invariant dynamical systems are jet-equivalent if they produce the same jet complex.

Corollary 1. *LTID systems are precisely those linear shift-invariant dynamical systems that are maximal in jet-equivalence classes.*

Proof. It is immediate from the previous theorem that every jet-equivalence class contains one and only one LTID system. By the completeness property, this system is maximal. □

Closing Appendix, we remark that all what is said above holds mutatis mutandis in the discrete-time case. In particular, the mapping

$$\mathcal{B} \mapsto (\mathcal{B}|_{\Delta(k)})_{k\geq 0}$$

induces a one-to-one correspondence between linear time-invariant difference systems and jet complexes.

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