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Linear Algebra and its Applications 425 (2007) 534–547
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

State and internal variables for linear systems

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Received 3 July 2006; accepted 4 March 2007
Available online 25 March 2007
Submitted by E. Zerz

Abstract

In this paper we study the question of how to define state representations of a linear system in terms of its trajectories.

Keywords: Linear system; Transfer function; State model; Behavior

1. Introduction

In this paper we are concerned with construction state space realizations for linear dynamical systems. The problem is well-studied and understood, of course; numerous papers are written about it. (Major contributions have been made, as is known, by Fuhrmann [2] and Kalman et al. [3].) However, we believe that the present paper still will be of interest, as it is provided here the definitions of state and internal variables (and various linear maps between them) in intrinsic terms, i.e., in terms of system trajectories. We remind that such definitions have been given for the discrete-time case only (see [8]).

Following Willems [6,8,9], we shall deal with linear systems in which external variables are not classified into inputs and outputs. As is known (see, for example, [4,6–9]), associated with such a system there are state models of two types

$$\begin{cases} Gy' = Fy \\ w = Hy \end{cases} \quad \text{and} \quad Kx' - Lx + Mw = 0.$$

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0024-3795/$ - see front matter
doi:10.1016/j.laa.2007.03.006

We shall obtain both these representations in a simple natural way and then show that they are related by $KF - LG + MH = 0$. (It is interesting to note that this relation is an immediate consequence of the Newton–Leibniz formula.)

Let us fix a few notations and recall some definitions and facts from [5]. Throughout k is an arbitrary field, s an indeterminate, and q a signal number. By O we denote the ring of proper rational functions in $k(s)$, and t will stand for its distinguished element s^{-1}. Given a proper rational function f, we shall denote by $f(\infty)$ the value at infinity of f and by f^σ the backward shift of f. So, if $f = a_0 + a_1 t + a_2 t^2 + \cdots$, then

$$f(\infty) = a_0 \quad \text{and} \quad f^\sigma = a_1 + a_2 t + \cdots = s(f - f(\infty)).$$

We extend these two notations in the obvious way to arbitrary proper rational matrices.

Assume that there are given: A torsion free module $\mathscr{U}$ over O and a k-linear imbedding $\mathscr{L} : k \to \mathscr{U}$, satisfying the "Newton–Leibniz axiom"

$$\mathscr{U} = t\mathscr{U} \oplus \mathscr{L}(k).$$

The two most important examples of $\mathscr{U}$ are $C^\infty(\mathbb{R}_+)$ (when $k = \mathbb{R}$ or $\mathbb{C}$) and $k^{\mathbb{Z}_+}$ $(= k[[t]])$. (In both examples $\mathscr{L}$ is the evident canonical map.) Throughout we shall keep in mind the continuous-time case, and therefore we shall view elements of $\mathscr{U}$ as smooth functions (of the nonnegative real variable). Multiplication by t will be interpreted as the integration operator on these functions, and we shall denote it by $\int$. If $a \in k$, then $\mathscr{L}(a)$ will be regarded as the constant function associated with a. The axiom above allows us to define, for each $\xi \in \mathscr{U}$, the "derivative" ξ' and the "value" $\xi(0)$ at 0. By definition, we thus have

$$\xi = \int \xi' + \mathscr{L}(\xi(0)).$$

For the "differentiation" operator $\xi \mapsto \xi'$ we shall use also the symbol ∂. It is very helpful to introduce the fraction space of $\mathscr{U}$, denoted by $\mathscr{M}$. We call its elements generalized or Mikusinski functions. The linear map $\mathscr{L}$ can be extended to a $k(s)$-linear map $k(s) \to \mathscr{M}$, which we denote also by $\mathscr{L}$ and call the (inverse) Laplace transform. It immediately follows from our axiom that $\mathscr{M} = \mathscr{U} \oplus \mathscr{L}(sk[s])$. Elements from $\mathscr{L}(O)$ $(\subseteq \mathscr{U})$ should be interpreted as exponential functions and elements from $\mathscr{L}(sk[s])$ as purely impulsive functions.

Given a submodule $T \subseteq O^q$, we define $T\mathscr{U}$ to be the submodule in $\mathscr{U}^q$ consisting of finite sums of trajectories of the form $g\xi$, where $g \in T$ and $\xi \in \mathscr{U}$. It should be noted that if $T = GO^m$, where $m \geqslant 0$ and G is a full column rank proper rational matrix of size $q \times m$, then $T\mathscr{U} = G\mathscr{U}^m$. One can prove without difficulty that the correspondence $T \mapsto T\mathscr{U}$ is one-to-one. (This is an easy consequence of the facts that $\mathscr{U}$ is torsion free and $t\mathscr{U} \neq \mathscr{U}$.) A submodule $T \subseteq O^q$ is called a transfer function if O^q/T is without torsion (equivalently, if its generator matrix G is left invertible). It is easy to see that given a transfer function T there exists a componentwise partition $k^q \simeq k^m \oplus k^p$ that allows to represent T as the graph of a usual classical transfer function (and this justifies the term).

If $\mathscr{S}$ is a linear subspace of $\mathscr{U}^q$, then the set $T = \{g \in O^q | g\mathscr{U} \subseteq \mathscr{S}\}$ is easily seen to be a submodule of O^q. We consider it as the most important invariant of $\mathscr{S}$. Clearly $T\mathscr{U} \subseteq \mathscr{S}$, and we call $\dim(\mathscr{S}/T\mathscr{U})$ the McMillan degree (or the relative dimension) of $\mathscr{S}$. The space $\mathscr{S}/T\mathscr{U}$ itself is called the state (or the initial condition) space, and trajectories belonging to $T\mathscr{U}$ are called zero initial condition trajectories. The rank of T (as an O-module) is called the input number of $\mathscr{S}$ and the signal number q minus the input number is called the output number. It can be shown easily that if $\mathscr{S}$ is differentiation-invariant, then T necessarily is a transfer function; it is easy also to show that if the McMillan degree of $\mathscr{S}$ is finite, then $\mathscr{S}$ has sufficiently many exponential trajectories in the sense that $\mathscr{S} \subseteq T\mathscr{U} + \mathscr{L}(O^q)$. The main result of [5] states: $\mathscr{S}$ can be described via an equation of the form $R(\partial)w = 0$, where R is a polynomial matrix with q columns, if and only if

(a) $\mathscr{S}$ is differentiation-invariant;
(b) $\mathscr{S}$ has finite McMillan degree.

We can define therefore a linear (dynamical) system as any linear subspace in $\mathscr{U}^q$ that satisfies the above two conditions. It is the goal of this paper to construct state representation theory taking this definition as a starting point.

2. Preliminaries on state models and their behaviors

A left state model is a quintuple (X, Y, F, G, H), where X, Y are finite-dimensional linear spaces and $F, G : Y \to X$, $H : Y \to k^q$ linear maps such that G is surjective and $\begin{bmatrix} G \\ H \end{bmatrix}$ is injective. The space X is called the state space and Y the (left) internal variable space. The number $\dim X$ is called the dimension, the number $m = \dim Y - \dim X$ the input number, and the number $p = q - m$ the output number. The model is called observable if $\begin{bmatrix} sG - F \\ H \end{bmatrix}$ is left unimodular, and controllable if $sG - F$ is right unimodular.

A right state model is a quintuple (X, Z, K, L, M), where X, Z are finite-dimensional linear spaces and $K, L : X \to Z$, $M : k^q \to Z$ linear maps such that K is injective and $[K \ \ M]$ is surjective. The space X is called the state space and Z the (right) internal variable space. The number $\dim X$ is called the dimension, the number $p = \dim Z - \dim X$ the output number, and the number $m = q - p$ the input number. The model is called observable if $sK - L$ is left unimodular, and controllable if $[sK - LM]$ is right unimodular.

Two left state models $(X_1, Y_1, F_1, G_1, H_1)$ and $(X_2, Y_2, F_2, G_2, H_2)$ are said to be equivalent if there exist bijective linear maps $\alpha : X_1 \to X_2$ and $\beta : Y_1 \to Y_2$ such that

$$\alpha F_1 = F_2\beta, \quad \alpha G_1 = G_2\beta \quad \text{and} \quad H_1 = H_2\beta.$$

Similarly, two right state models $(X_1, Z_1, K_1, L_1, M_1)$ and $(X_2, Z_2, K_2, L_2, M_2)$ are said to be equivalent if there exist bijective linear maps $\alpha : X_1 \to X_2$ and $\beta : Z_1 \to Z_2$ such that

$$\beta K_1 = K_2\alpha, \quad \beta L_1 = L_2\alpha \quad \text{and} \quad M_2 = \beta M_1.$$

We say that (X, Y, F, G, H) and (X, Z, K, L, M) form an exact couple, if the sequence

$$0 \to Y \overset{\begin{bmatrix} F \\ G \\ H \end{bmatrix}}{\to} X \oplus X \oplus k^q \overset{[K \ -L \ M]}{\to} Z \to 0 \tag{1}$$

is exact (equivalently, if both models have the same input and output numbers and satisfy the equality

$$KF - LG + MH = 0).$$

Example 1. Let (X, A, B, C, D) be a usual classical linear system with state space X, with m inputs and p outputs. (Recall that this just means that X is a finite-dimensional linear space, and $A : X \to X$, $B : k^m \to X$, $C : X \to k^p$, $D : k^m \to k^p$ are linear maps.) Then

$$\left(X, X \oplus k^m, [A \quad B], [I \quad 0], \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}\right) \text{ and } \left(X, X \oplus k^p, \begin{bmatrix} I \\ 0 \end{bmatrix}, \begin{bmatrix} A \\ C \end{bmatrix}, \begin{bmatrix} -B & 0 \\ -D & I \end{bmatrix}\right)$$

are left and right state models, respectively. It is easy to check that these two state models form an exact couple. Notice that both these models are controllable if and only if $sI - A$ and B are right coprime and observable if and only if $sI - A$ and C are left coprime.

Lemma 1. *Let X, Y and Z be finite-dimensional linear spaces and $F, G : Y \to X$, $H : Y \to k^q$, $K, L : X \to Z$, $M : k^q \to Z$ linear maps such that the sequence* (1) *is exact. If one of the quintuples (X, Y, F, G, H) and (X, Z, K, L, M) is a state model, then so is the other.*

Proof. Straightforward and easy. □

If two state models form an exact couple, we shall say also that they are adjoint to each other. Using the previous lemma, one can show easily that for each state model there exists an adjoint one. Certainly, the latter is uniquely determined up to equivalence. We thus have the following obvious:

Proposition 1. *There is a canonical one-to-one correspondence between the equivalence classes of left state models and the equivalence classes of right state models.*

We have

Lemma 2. *Every exact couple of state models is "equivalent" to a couple coming from a classical linear system.*

Proof. Let (X, Y, F, G, H) and (X, Z, K, L, M) be state models forming an exact couple, and let m and p be their input and output numbers.

The condition that $\begin{bmatrix} G \\ H \end{bmatrix}$ is injective implies that H induces an injective linear map $\mathrm{Ker}(G) \to k^q$. Clearly, there exists a componentwise partition $k^q \simeq k^m \oplus k^p$ leading to a representation $H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ with H_1 inducing an isomorphism $\mathrm{Ker}(G) \simeq k^m$. The map $\begin{bmatrix} G \\ H_1 \end{bmatrix} : Y \to X \oplus k^m$ must be bijective, and therefore we can define $A : X \to X$, $B : k^m \to X$, $C : X \to k^p$, $D : k^m \to k^p$ so that

$$[A \quad B] \begin{bmatrix} G \\ H_1 \end{bmatrix} = F \quad \text{and} \quad [C \quad D] \begin{bmatrix} G \\ H_1 \end{bmatrix} = H_2.$$

Clearly, the bijective linear maps $I : X = X$ and $\begin{bmatrix} G \\ H_1 \end{bmatrix} : Y \simeq X \oplus k^m$ determine an isomorphism of (X, Y, F, G, H) onto the left state model of Example 1.

Further, write $M = [M_1 \quad M_2]$ according to the above partition of k^q. We claim that $[K \quad M_2] : X \oplus k^p \simeq Z$ is bijective. Indeed, assume that $x \in X$ and $y \in k^p$ are such that $Kx + M_2 y = 0$. Using the exact sequence

$$0 \to X \oplus k^m \overset{\begin{bmatrix} A & B \\ I & 0 \\ 0 & I \\ C & D \end{bmatrix}}{\to} X \oplus X \oplus k^m \oplus k^p \overset{[K \; -L \; M_1 \; M_2]}{\to} Z \to 0,$$

we can see that there exist $x_1 \in X$ and $u \in k^m$ such that

$$\begin{pmatrix} x \\ 0 \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} Ax_1 + Bu \\ x_1 \\ u \\ Cx_1 + Du \end{pmatrix}.$$

It immediately follows that $x = 0$ and $y = 0$. So, our map is injective. Take now an arbitrary element $z \in Z$. Because $[K \;\; M_1 \;\; M_2]$ is surjective, we can find $x \in X$, $u \in k^m$ and $y \in k^p$ such that $Kx + M_1u + M_2y = z$. By the exact sequence above, $KB + M_1 + M_2D = 0$, and therefore

$$z = Kx + M_1u + M_2y = Kx - KBu - M_2Du + M_2y = K(x - Bu) + M_2(y - Du).$$

So, the map is surjective as well. Applying again the exact sequence above, we obtain

$$[K \;\; M_2]\begin{bmatrix} I \\ 0 \end{bmatrix} = K, \quad [K \;\; M_2]\begin{bmatrix} A \\ C \end{bmatrix} = L, \quad [M_1 \;\; M_2] = [K \;\; M_2]\begin{bmatrix} -B & 0 \\ -D & I \end{bmatrix}.$$

This means that $I : X = X$ and $[K \;\; M_2] : X \oplus k^p \simeq Z$ determine an isomorphism of the right state model of Example 1 onto (X, Z, K, L, M).

The proof is complete. □

Using the lemma above, we can easily see that that a state model is observable (resp. controllable) if and only if its adjoint is observable (resp. controllable).

From now on we shall be interested exclusively in observable state models. The behavior of an observable left state model (X, Y, F, G, H) is defined by the equation

$$\begin{cases} Gy' = Fy, \\ w = Hy, \end{cases}$$

i.e., as the set

$$\{w \in \mathscr{U}^q \,|\, \exists y \in Y \otimes \mathscr{U} \text{ such that } Gy' = Fy, w = Hy\}.$$

The behavior of an observable right state model (X, Z, K, L, M) is defined by the equation

$$Kx' - Lx + Mw = 0;$$

i.e., as the set

$$\{w \in \mathscr{U}^q \,|\, \exists x \in X \otimes \mathscr{U} \text{ such that } Kx' - Lx + Mw = 0\}.$$

Two equivalent (observable) state models clearly have the same behavior.

Example 2. The behavioral equations for the state models in Example 1 are

$$\begin{cases} [I \;\; 0]\begin{pmatrix} x' \\ u' \end{pmatrix} = t[A \;\; B]\begin{pmatrix} x \\ u \end{pmatrix} \\ \begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}\begin{pmatrix} x \\ u \end{pmatrix} \end{cases} \quad \text{and} \quad \begin{bmatrix} I \\ 0 \end{bmatrix} x' - \begin{bmatrix} A \\ C \end{bmatrix} x + \begin{bmatrix} -B & 0 \\ -D & I \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0,$$

respectively. Both these equations can be rewritten in the classical form

$$\begin{cases} x' = Ax + Bu, \\ v = Cx + Du. \end{cases}$$

Proposition 2. *The behavior of an observable state model is a linear system.*

Proof. In view of Lemma 2 and Example 2, it suffices to show that

$$\mathscr{S} = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \middle\| \, u \in \mathscr{U}^m, v \in \mathscr{U}^p \text{ and } \exists x \in X \otimes \mathscr{U} \text{ such that } x' = Ax + Bu, v = Cx + Du \right\}$$

is a linear system.

It is clear that $\mathscr{S}$ is differentiation-invariant. Let $T = C(sI - A)^{-1}B + D$ and, for each $x \in X$ and $u \in \mathscr{U}^m$, define

$$w_{x,u} = \begin{pmatrix} u \\ C(I - tA)^{-1}x + Tu \end{pmatrix}.$$

As is known (and as it can be seen easily), $(x, u) \mapsto w_{x,u}$ establishes an isomorphism $X \oplus \mathscr{U}^m \simeq \mathscr{S}$. Letting $\overline{T} = \begin{bmatrix} I \\ T \end{bmatrix}$, we obviously have $\overline{T}\mathscr{U}^m \subseteq \mathscr{S}$ and $\dim(\mathscr{S}/\overline{T}\mathscr{U}^m) < +\infty$. In view of Lemma 9 in [5], $\overline{T}O^m$ necessarily is the transfer function of $\mathscr{S}$ (in the sense of this paper). Thus, the McMillan degree of $\mathscr{S}$ is finite. The proof is complete. □

Proposition 3. *Two (observable) models that are adjoint to each other have the same behavior.*

Proof. Follows easily from the previous lemma and the examples above. □

3. From a linear system to state models

We begin with the following:

Lemma 3. *Let T be a transfer function. Then*

$$T\mathscr{U} \cap t\mathscr{U}^q = tT\mathscr{U}.$$

Proof. We claim that $T \cap tO^q = tT$. Indeed, let tg, where $g \in O^q$, belongs to T. Then, letting V denote the fraction space of T, we have $g \in V \cap O^q = T$, and so $tg \in tT$. We see that $T \cap tO^q \subseteq tT$. The inclusion "⊇" is obvious.

It follows that the canonical sequence

$$0 \to tT \to T \oplus tO^q \to O^q$$

is exact. Tensoring this by $\mathscr{U}$, we obtain an exact sequence

$$0 \to tT\mathscr{U} \to T\mathscr{U} \oplus t\mathscr{U}^q \to \mathscr{U}^q,$$

which completes the proof. □

We remark that $t\mathscr{U}^q = \{w \in \mathscr{U}^q | w(0) = 0\}$. So, the lemma above says that the set of trajectories with initial state 0 and initial value 0 is equal to the set of the integrals of trajectories with initial state 0.

Assume we have a linear system $\mathscr{S}$. Set $\overline{\mathscr{S}} = t\mathscr{S} + \mathscr{L}(k^q)$, which is the least subspace in $\mathscr{U}^q$ containing the integrals of trajectories in $\mathscr{S}$ and the constant functions. Notice that, by the Newton–Leibniz formula, $\mathscr{S} \subseteq \overline{\mathscr{S}}$. We have canonical linear maps

$$\mathscr{S} \to \mathscr{S} \oplus \mathscr{S} \oplus k^q, \quad w \mapsto (w', w, w(0))$$

and

$$\mathscr{S} \oplus \mathscr{S} \oplus k^q \to \overline{\mathscr{S}}, \quad (w_1, w_2, a) \mapsto tw_1 - w_2 + \mathscr{L}(a).$$

Obviously, the first one is injective and the second surjective.

Lemma 4. *The sequence*

$$0 \to \mathscr{S} \to \mathscr{S} \oplus \mathscr{S} \oplus k^q \to \overline{\mathscr{S}} \to 0 \tag{2}$$

is exact.

Proof. We only need to show exactness at the middle term. But this immediately follows from the Newton–Leibniz formula. □

Let T be the transfer function of our system $\mathscr{S}$. We have already introduced in Introduction the state space $X = \mathscr{S}/T\mathscr{U}$. Set also

$$Y = \mathscr{S}/tT\mathscr{U} \quad \text{and} \quad Z = \overline{\mathscr{S}}/T\mathscr{U}.$$

Elements of Y are called left internal variables of $\mathscr{S}$ and elements of Z right internal variables. There are canonical linear maps $F, G : Y \to X$, $H : Y \to k^q$ defined by the formulas

$$F(w \bmod tT\mathscr{U}) = w' \bmod T\mathscr{U}, \quad G(w \bmod tT\mathscr{U}) = w \bmod T\mathscr{U}, \quad H(w \bmod tT\mathscr{U}) = w(0)$$

and canonical linear maps $K, L : X \to Z$, $M : k^q \to Z$ defined by the formulas

$$K(w \bmod T\mathscr{U}) = tw \bmod T\mathscr{U}, \quad L(w \bmod T\mathscr{U}) = w \bmod T\mathscr{U}, \quad Ma = \mathscr{L}(a) \bmod T\mathscr{U}.$$

Proposition 4. *The quintuples* (X, Y, F, G, H) *and* (X, Z, K, L, M) *are state models.*

Proof. Assume that $w \bmod tT\mathscr{U}$ goes to zero under G and H, i.e., $w \in T\mathscr{U}$ and $w(0) = 0$. By Lemma 3, then $w \in tT\mathscr{U}$, and hence $w \bmod tT\mathscr{U} = 0$. It follows that $\begin{bmatrix} G \\ H \end{bmatrix}$ is injective.

Next, assume that $w \bmod T\mathscr{U}$ goes to zero under K, i.e., $tw \in T\mathscr{U}$. Using again Lemma 3, we see that $tw \in tT\mathscr{U}$. It follows that $w \in T\mathscr{U}$, and hence $w \bmod T\mathscr{U} = 0$. So, K is injective.

Finally, it is obvious that G and $[K \ M]$ are surjective. □

The state models that we have constructed will be called the canonical state representations of $\mathscr{S}$.

Theorem 1. *The canonical state representations form an exact couple.*

Proof. Consider the sequence

$$0 \to tT\mathscr{U} \to T\mathscr{U} \oplus T\mathscr{U} \to T\mathscr{U} \to 0,$$

where the second and third arrows are defined respectively by

$$\xi \mapsto (s\xi, \xi) \quad \text{and} \quad (\xi_1, \xi_2) \mapsto t\xi_1 - \xi_2.$$

The sequence is exact, and together with (2) gives the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & tT\mathscr{U} & \to & T\mathscr{U} \oplus T\mathscr{U} & \to & T\mathscr{U} & \to & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{S} & \to & \mathscr{S} \oplus \mathscr{S} \oplus k^q & \to & \overline{\mathscr{S}} & \to & 0, \end{array}$$

where the vertical arrows are inclusion maps. Applying now the snake lemma (see [1, Proposition 2.10]), we conclude the proof. □

Proposition 5. *The dimension of the canonical state representations is equal to the McMillan degree of* $\mathscr{S}$; *their input and output numbers are equal to those of* $\mathscr{S}$.

Proof. The first statement is true by definition.

To prove the second one, let m denote the input number of $\mathscr{S}$. By definition, there is an isomorphism $T \simeq O^m$. Hence, $T\mathscr{U} \simeq \mathscr{U}^m$, and therefore we have

$$T\mathscr{U}/tT\mathscr{U} \simeq \mathscr{U}^m/t\mathscr{U}^m \simeq k^m.$$

(We remark that the isomorphism $\mathscr{U}/t\mathscr{U} \simeq k$ holds by the Newton–Leibniz axiom.) Using this and the tower $tT\mathscr{U} \subseteq T\mathscr{U} \subseteq \mathscr{S}$, we can see that the statement is true for the left state model. Dimension count in (1) shows that the statement must be true for the right state model as well. □

4. Realization and uniqueness theorems

In this section we are going to demonstrate that the state models that we have constructed realize $\mathscr{S}$, and that they are uniquely determined by this property. (We shall keep here the notations of the previous section.)

Let R be a minimal kernel representation of $\mathscr{S}$, i.e., a full row rank polynomial matrix such that $\mathscr{S} = \operatorname{Ker} R(\partial)$. Recall that

$$\operatorname{Ker} R(\partial) = \{w \in \mathscr{U}^q \mid Rw \in \mathscr{L}(sk[s]^p)\}$$

(see [5, Lemma 2]).

Lemma 5. $\overline{\mathscr{S}} = \{w \in \mathscr{U}^q \mid Rw \in \mathscr{L}(k[s]^p)\}$.

Proof. If $w \in \mathscr{S}$, then $Rw \in \mathscr{L}(sk[s]^p)$, and hence $Rtw \in t\mathscr{L}(sk[s]^p) = \mathscr{L}(k[s]^p)$. If $w = \mathscr{L}(a)$, where $a \in k^q$, then $Ra \in k[s]^p$, and hence $Rw \in \mathscr{L}(k[s]^p)$.

Conversely, assume that $w \in \mathscr{U}^q$ and $Rw \in \mathscr{L}(k[s]^p)$. Then $Rw' = R(sw - s\mathscr{L}w(0)) \in \mathscr{L}(sk[s]^p)$. This means that w' is a trajectory in $\mathscr{S}$. It follows that $w = \int w' + \mathscr{L}w(0)$ belongs to $\overline{\mathscr{S}}$. □

In view of the previous lemma, we have a canonical linear map $\mathscr{S} \to k[s]^p$. We remind (see the proof of Theorem 1 in [5]) that the transfer function T can be computed via $R : T = \{a \in O^q \mid Ra = 0\}$. So, we have an exact sequence

$$0 \to T \to O^q \xrightarrow{R} k(s)^p.$$

Tensoring this by $\mathscr{U}$, we get an exact sequence

$$0 \to T \otimes \mathscr{U} \to \mathscr{U}^q \xrightarrow{R} \mathscr{M}^p.$$

The image of $T \otimes \mathscr{U} \to \mathscr{U}^q$ is just $T\mathscr{U}$, and thus we have

$$T\mathscr{U} = \{w \in \mathscr{U}^q \mid Rw = 0\}.$$

There is therefore a well-defined linear map $Z \to k[s]^p$. The latter, in turn, gives rise to a canonical homomorphism

$$\rho : Z[s] \to k[s]^p.$$

We also have a canonical homomorphism

$$Ks - L : X[s] \to Z[s].$$

Note that if x is a state and if ξ is its representative, then

$$\rho(Ks - L)(x) = \rho((Kx)s - Lx) = \mathscr{L}^{-1}(R(t\xi)s - R\xi) = 0.$$

This implies that the sequence

$$0 \to X[s] \to Z[s] \to k[s]^p \to 0 \tag{3}$$

is a complex.

Lemma 6. *The complex* (3) *is exact.*

Proof. *Exactness at* $X[s]$: Obvious, because $K : X \to Z$ is injective.

Exactness at $Z[s]$: Assume that an element $z_0 + z_1 s + \cdots + z_l s^l \in Z[s]$ goes to zero under ρ, and let $\zeta_0, \zeta_1, \ldots, \zeta_l \in \overline{\mathscr{S}}$ be any representatives of $z_0, z_1, \ldots, z_l$, respectively. We then have, by the assumption, that $\mathscr{L}^{-1}(R\zeta_0) + \mathscr{L}^{-1}(R\zeta_1)s + \cdots + \mathscr{L}^{-1}(R\zeta_l)s^l = 0$. Set

$$\xi_0 = -\zeta_0, \xi_1 = -(t\zeta_0 + \zeta_1), \ldots, \xi_{l-1} = -(t^{l-1}\zeta_0 + \cdots + \zeta_{l-1}).$$

All these elements are trajectories of $\mathscr{S}$.

Let $x_0, x_1, \ldots, x_{l-1}$ be the initial states of these trajectories. One can check easily that

$$-Lx_0 = z_0, Kx_0 - Lx_1 = z_1, \ldots, Kx_{l-2} - Lx_{l-1} = z_{l-1}, Kx_{l-1} = z_l.$$

It follows that

$$(sK - L)(x_0 + x_1 s + \cdots + x_{l-1}s^{l-1}) = z_0 + z_1 s + \cdots + z_l s^l.$$

Exactness at $k[s]^p$: Take any $v \in k[s]^p$. Because R has full row rank (and because $k(s) = O + sk[s]$), there exist $g \in O^q$ and $a_1, \ldots, a_l \in k^q$ such that $R(g + a_1 s + \cdots + a_l s^l) = v$. Clearly $Rg \in k[s]^p$, and therefore $\mathscr{L}(g)$ lies in $\overline{\mathscr{S}}$. Letting $z_0 = \mathscr{L}(g), z_1 = M(a_1), \ldots, z_l = M(a_l)$, we see that

$$\rho(z_0 + z_1 s + \cdots + z_l s^l) = v.$$

The lemma is proved. □

Theorem 2. *The state models* (X, Y, F, G, H) *and* (X, Z, K, L, M) *are observable, and they realize* $\mathscr{S}$.

Proof. In view of Proposition 3, it suffices to consider, say, the case of right state model.

The exact sequence (3) splits. It follows from this that (X, Z, K, L, M) is observable. Consider $\mathscr{U}$ as a $k[s]$-module in which s acts on $\mathscr{U}$ as ∂. Then tensoring (3) by this module, we obtain a (split) exact sequence

$$0 \to X \otimes \mathscr{U} \to Z \otimes \mathscr{U} \to \mathscr{U}^p \to 0. \tag{4}$$

(The arrows here are given by $x \mapsto Kx' - Lx$ and $z \mapsto \rho(z)$, respectively.)

Further, the polynomial matrix R determines an obvious linear map $k^q \to k[s]^p$, which can be viewed as the composition of the linear maps $M : k^q \to Z$ and $Z \to k[s]^p$. Consequently, we have a commutative diagram

$$\begin{array}{ccc} k[s]^q & & \\ \downarrow & \searrow & \\ Z[s] & \to & k[s]^p, \end{array}$$

which gives rise to the commutative diagram

$$\begin{array}{ccc} \mathcal{U}^q & & \\ \downarrow & \searrow & \\ Z \otimes \mathcal{U} & \to & \mathcal{U}^p. \end{array} \tag{5}$$

(The downward arrow here is defined by $w \mapsto Mw$ and the south-east arrow is $R(\partial)$.)

Using the exact sequence (4) and the commutative diagram (5), we can see that the behavior of our state model is equal to the kernel of $R(\partial) : \mathcal{U}^q \to \mathcal{U}^p$. Indeed, let $w \in \operatorname{Ker} R(\partial)$. Then, by (5), Mw goes to zero under the map $Z \otimes \mathcal{U} \to \mathcal{U}^p$. Consequently, in view of (4), there exists $x \in X \otimes \mathcal{U}$ such that $Mw = Lx - Kx'$. Conversely, assume that $w \in \mathcal{U}^q$ is such that $Mw = Lx - Kx'$ for some $x \in X \otimes \mathcal{U}$. Then, by (4), the image of Mw under the map $Z \otimes \mathcal{U} \to \mathcal{U}^p$ is zero. Using again (5), we can see that $R(\partial)w = 0$. □

Theorem 3. *Suppose* $(X_1, Y_1, F_1, G_1, H_1)$ *and* $(X_1, Z_1, K_1, L_1, M_1)$ *are observable (left and right, respectively) state models that realize* $\mathcal{S}$. *Then they are equivalent to the canonical representations of* $\mathcal{S}$.

Proof. Using again Proposition 3, we may restrict ourselves by considering the case of left state model. Further, in view of Lemma 2, we may assume without loss of generality that

$$Y_1 = X_1 \oplus k^m, \quad F_1 = [A \;\; B], G_1 = [I \;\; 0] \quad \text{and} \quad H_1 = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}.$$

In other words, we may assume that $\mathcal{S}$ is determined via the classical equation

$$\begin{cases} x' = Ax + Bu, \\ v = Cx + Du, \end{cases}$$

where $x \in X_1 \otimes \mathcal{U}$. As remarked already in Example 1, the pair (A, C) is observable in the classical sense.

Define T, $w_{x,u}$ and $\overline{T}$ as in the proof of Proposition 2. Every trajectory of $\mathcal{S}$ is uniquely represented in the form $w_{x,u}$, and $\overline{T}O^m$ is the transfer function of $\mathcal{S}$.

We have canonical linear maps

$$\alpha : X_1 \to X \quad \text{and} \quad \beta : Y_1 \to Y$$

defined respectively by

$$\alpha(x) = w_{x,0} \bmod \overline{T}\mathcal{U}^m \quad \text{and} \quad \beta(x, u) = w_{x,\mathcal{L}(u)} \bmod t\overline{T}\mathcal{U}^m.$$

If $x \in X_1$ and $u \in \mathcal{U}^m$, then clearly

$$w_{x,u} \equiv w_{x,0} \bmod \overline{T}\mathcal{U}^m \quad \text{and} \quad w_{x,u} \equiv w_{x,\mathcal{L}u(0)} \bmod t\overline{T}\mathcal{U}^m.$$

This means that the linear maps above are surjective. They are injective as well; the injectivity follows immediately from the well-known fact that if a pair (A, C) is observable, then $C(I - tA)^{-1}x = 0$ if and only if $x = 0$.

One can check easily that, for each $x \in X_1$ and $u \in k^q$,

$$(w_{x,u})' = w_{Ax+Bu,0} \quad \text{and} \quad w_{x,u}(0) = \begin{pmatrix} u \\ Cx + Du \end{pmatrix}.$$

From these formulas and the first congruence above, we conclude that

$$F\beta = \alpha[A \;\; B], \quad G\beta = \alpha[I \;\; 0], \quad H\beta = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}.$$

The proof is complete. □

Appendix: Connection with Fuhrmann's realization theory

Let $\mathscr{S}$ be a linear system given by the equation $R(\partial)w = 0$, where R is a full row rank polynomial matrix, say, of size $p \times q$. According to Theorem 1 in [5], the transfer function of $\mathscr{S}$ is equal to the kernel of the homomorphism $R : O^q \to k(s)^p$. Choosing a generator matrix G of this transfer function, we get an exact sequence

$$0 \to O^m \overset{G}{\to} O^q \overset{R}{\to} k(s)^p,$$

where $m = q - p$. Since G is left invertible proper rational matrix, the k-linear map $G(\infty) : k^m \to k^q$ is injective. It follows that there exist rows $g_{i_1}, \dots, g_{i_m}$ of G such that

$$\det \begin{bmatrix} g_{i_1}(\infty) \\ \vdots \\ g_{i_m}(\infty) \end{bmatrix} \neq 0.$$

Let G_0 denote the submatrix of G with rows $g_{i_1}, \dots, g_{i_m}$. Because $\det G_0(\infty) \neq 0$, G_0 is a biproper rational matrix. Therefore GG_0^{-1} also is a generator of the transfer function. We reorder (if necessary) the components in k^q in such a way that GG_0^{-1} becomes $\begin{bmatrix} I \\ T \end{bmatrix}$, where ($I$ is the unit matrix of size $m \times m$ and) T is a proper rational matrix of size $p \times m$. Letting $[-Q \ P]$ be the matrix R after this reordering, we have an exact sequence

$$0 \to O^m \overset{\left[\begin{smallmatrix} I \\ T \end{smallmatrix}\right]}{\to} O^m \oplus O^p \overset{[-Q \ \ P]}{\to} k(s)^p.$$

We see that $-Q + PT = 0$. Next, because $[-Q \ P]$ is of full row rank, we have an exact sequence

$$0 \to k(s)^m \overset{\left[\begin{smallmatrix} I \\ T \end{smallmatrix}\right]}{\to} k(s)^m \oplus k(s)^p \overset{[-Q \ \ P]}{\to} k(s)^p \to 0.$$

This implies that $P : k(s)^p \to k(s)^p$ is bijective, i.e., $\det P \neq 0$. (Compare with Theorem 3.3.22 in [6].)

Thus, we may assume that our linear system $\mathscr{S} \subseteq \mathscr{U}^q (= \mathscr{U}^m \oplus \mathscr{U}^p)$ is given by the equation

$$P(\partial)v = Q(\partial)u \quad (u \in \mathscr{U}^m, v \in \mathscr{U}^p),$$

where P and Q are polynomial matrices respectively of sizes $p \times p$ and $p \times m$ such that P is nonsingular and $T = P^{-1}Q$ is proper.

Following Fuhrmann, we set

$$X = sk[s]^p \cap PO^p.$$

(*Warning*: According to Fuhrmann's definition the state space, in fact, is $k[s]^p \cap tPO^p$. The reason for giving a slightly different definition is that our Laplace transform $\mathscr{L}$ is the conventional one followed by differentiation.) For each $x \in X$ and $u \in k^m$, define

$$Ax = P(P^{-1}x)^\sigma, \quad Bu = PT^\sigma u, \quad Cx = (P^{-1}x)(\infty) \quad \text{and} \quad Du = T(\infty)u.$$

We claim that Ax and Bu belong to X. Indeed, clearly they are contained in PO^p. Next,

$$Ax = sx - s(P^{-1}x)(\infty) \quad \text{and} \quad Bu = sQu - sPT(\infty)u$$

and so they are contained in $sk[s]^p$ as well. Further, it is obvious that Cx and Du belong to k^p. We thus have four canonical linear maps

$$A : X \to X, \quad B : k^m \to X, \quad C : X \to k^p, \quad \text{and} \quad D : k^m \to k^p.$$

The quintuple (X, A, B, C, D) is called the Fuhrmann realization. Our aim here is to express in terms of this construction the state models we have defined. Let $\overline{T}$ be the matrix $\begin{bmatrix} I \\ T \end{bmatrix}$ (as in the proofs of Proposition 2 and Theorem 3).

(1) *Representation via left state models*

The differential equation above can be written in an "operational" form

$$Pv = \mathscr{L}(x) + Qu \quad (x \in X, u \in \mathscr{U}^m, v \in \mathscr{U}^p).$$

For each $x \in X$ and $u \in \mathscr{U}^m$, set

$$w_{x,u} = \begin{pmatrix} u \\ P^{-1}\mathscr{L}x + Tu \end{pmatrix}.$$

It is clear that $(x, u) \mapsto w_{x,u}$ is a bijective map of $X \oplus \mathscr{U}^m$ onto $\mathscr{S}$.

We need to compute the derivative of $w_{x,u}$. We have

$$\begin{aligned}(P^{-1}\mathscr{L}x)' &= sP^{-1}\mathscr{L}x - s(P^{-1}\mathscr{L}x)(0) \\ &= \mathscr{L}(sP^{-1}x - s(P^{-1}x)(\infty)) = \mathscr{L}(P^{-1}x)^{\sigma}.\end{aligned}$$

Also we have

$$\begin{aligned}(Tu)' &= sTu - s(Tu)(0) = sTu - sTu(0) + sTu(0) - s(Tu)(0) \\ &= T(su - su(0)) + s(Tu(0) - T(\infty)u(0)) = Tu' + T^{\sigma}u(0).\end{aligned}$$

We therefore obtain that $(w_{x,u})'$ is equal to

$$\begin{aligned}\begin{pmatrix} u' \\ (P^{-1}\mathscr{L}x)' + (Tu)' \end{pmatrix} &= \begin{pmatrix} u' \\ \mathscr{L}(P^{-1}x)^{\sigma} + T^{\sigma}u(0)) + Tu' \end{pmatrix} \\ &= \begin{pmatrix} u' \\ (P^{-1}\mathscr{L}(Ax + Bu(0)) + Tu' \end{pmatrix}.\end{aligned}$$

Thus, we have

$$(w_{x,u})' = w_{Ax+Bu(0),u'}.$$

Further, we have

$$w_{x,u}(0) = \begin{pmatrix} u(0) \\ (P^{-1}x)(\infty) + T(\infty)u(0) \end{pmatrix} = \begin{pmatrix} u(0) \\ Cx + Du(0) \end{pmatrix}.$$

These two formulas (together with a trivial one) yield the following commutative diagrams:

$$\begin{array}{ccc} \mathscr{S} & \stackrel{\partial}{\to} & \mathscr{S} \\ \downarrow & & \downarrow, \\ X \oplus k^m & \stackrel{[A\ B]}{\to} & X \end{array} \qquad \begin{array}{ccc} \mathscr{S} & \stackrel{id}{\to} & \mathscr{S} \\ \downarrow & & \downarrow, \\ X \oplus k^m & \stackrel{[I\ 0]}{\to} & X \end{array} \qquad \begin{array}{ccc} \mathscr{S} & \stackrel{ev}{\to} & k^q \\ \downarrow & & \| \\ X \oplus k^m & \stackrel{\begin{bmatrix} 0 & I \\ C & D \end{bmatrix}}{\to} & k^q \end{array}.$$

(The top arrow in the third diagram is the "evaluation at 0".) Next, the map $w_{x,u} \mapsto x$ induces an isomorphism $\mathscr{S}/\overline{T}\mathscr{U}^m \simeq X$ and the map $w_{x,u} \mapsto (x, u(0))$ induces an isomorphism $\mathscr{S}/t\overline{T}\mathscr{U}^m \simeq X \oplus k^m$. We can see that the left canonical state model of $\mathscr{S}$ is equivalent to

$$\left(X, X \oplus k^m, [A\ \ B], [I\ \ 0], \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}\right).$$

(2) *Representation via right state models*

Set $Z = k[s]^p \cap PO^p$. Notice that Z contains X and tX. It is easily seen that

$$Z = tX + Pk^p \quad \text{and} \quad tX \cap Pk^p = 0;$$

i.e., every $z \in Z$ can be uniquely represented as

$$z = tx + Pv \quad \text{with } x \in X,\ v \in k^p. \tag{6}$$

Remark. One can observe easily that if $z \in X$, then in the representation above $x = Az$ and $v = Cz$.

We extend the definition of $w_{x,u}$ to all $z \in Z$ by setting

$$w_{z,u} = \begin{pmatrix} u \\ P^{-1}\mathscr{L}z + Tu \end{pmatrix}.$$

By Lemma 5, $(z, u) \mapsto w_{z,u}$ is a bijective map of $Z \oplus \mathscr{U}^m$ onto $\overline{\mathscr{S}}$.

Sending $w_{z,u}$ to $\binom{x}{v}$, where x and v are defined via (6), we obtain a canonical map from $\overline{\mathscr{S}}$ to $X \oplus k^p$.

Take any $x \in X$ and $u \in \mathscr{U}^m$. It is easily seen that $tw_{x,u} = w_{tx,tu}$, and so it goes to $\binom{x}{0}$ under the map $\overline{\mathscr{S}} \to X \oplus k^p$. In view of the remark above, the trajectory $w_{x,u}$ itself goes to $\binom{Ax}{Cx}$. Now take $u \in k^m$ and $v \in k^p$, and put $z = Pv - Qu$. The latter belongs to Z, and it is easily seen that $\mathscr{L}\binom{u}{v} = w_{z,\mathscr{L}(u)}$. We have

$$z = (PT(\infty)u - Qu) + (Pv - PT(\infty)u) = -tBu + P(v - Du).$$

Therefore $\mathscr{L}\binom{u}{v}$ goes to $\binom{-Bu}{v-Du}$ under $\overline{\mathscr{S}} \to X \oplus k^p$.

Thus, there are commutative diagrams

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{\int} & \overline{\mathscr{S}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\begin{bmatrix} I \\ 0 \end{bmatrix}} & X \oplus k^p \end{array}, \qquad \begin{array}{ccc} \mathscr{S} & \xrightarrow{id} & \overline{\mathscr{S}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\begin{bmatrix} A \\ C \end{bmatrix}} & X \oplus k^p \end{array}, \qquad \begin{array}{ccc} k^q & \xrightarrow{\mathscr{L}} & \overline{\mathscr{S}} \\ || & & \downarrow \\ k^q & \xrightarrow{\begin{bmatrix} -B & 0 \\ -D & I \end{bmatrix}} & X \oplus k^p \end{array}.$$

Next, the map $\mathscr{S} \to X$ induces (as we already know) an isomorphism $\mathscr{S}/\overline{T}\mathscr{U}^m \to X$ and the map $\overline{\mathscr{S}} \to X \oplus k^p$ induces an isomorphism $\overline{\mathscr{S}}/\overline{T}\mathscr{U}^m \to X \oplus k^p$. We can see that the right canonical state model of $\mathscr{S}$ is equivalent to

$$\left(X, X \oplus k^m, \begin{bmatrix} I \\ 0 \end{bmatrix}, \begin{bmatrix} A \\ C \end{bmatrix}, \begin{bmatrix} -B & 0 \\ -D & I \end{bmatrix} \right).$$

Acknowledgment

I am much indebted to Eva Zerz and Referee 1 who have made several suggestions and corrections.

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