Linear Algebra and its Applications 434 (2011) 1027–1057

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

First order representations of Fliess models

Vakhtang Lomadze [a,b,*], M. Khurram Zafar [a]

[a] Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan
[b] A. Razmadze Mathematical Institute, Tbilisi 0193, Georgia

ARTICLE INFO

Article history:
Received 13 May 2009
Accepted 23 October 2010
Available online 16 November 2010

Submitted by P. Fuhrmann

AMS classification:
93B25
93B27
93C05

Keywords:
Fliess model
Delay differential equation
State model
Behavior
Exact sequence

ABSTRACT

Fuhrmann's state-space construction (in its generalized form) is used to obtain a general theory of first order representations of Fliess models defined over an arbitrary noetherian commutative ring. The case of arbitrary linear delay differential equations is involved.

Introduction

Let

$$\begin{cases} sx = Fx + Gu \\ y = Hx + Ju \end{cases}$$

be a classical Kalman system. (Here s is an indeterminate, and multiplication by it means the differentiation operator or the backward shift operator.) We then have canonical homomorphisms

$$\begin{bmatrix} sI - F^t \\ -G^t \end{bmatrix} : \mathbb{R}[s]^n \to \mathbb{R}[s]^n \oplus \mathbb{R}[s]^m \quad \text{and} \quad \begin{bmatrix} H^t \\ J^t \end{bmatrix} : \mathbb{R}[s]^p \to \mathbb{R}[s]^n \oplus \mathbb{R}[s]^m$$

* Corresponding author at: A. Razmadze Mathematical Institute, Tbilisi 0193, Georgia.
E-mail addresses: vakholoma@yahoo.com (V. Lomadze), m_khurram_zafar@hotmail.com (M.K. Zafar).

0024-3795/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2010.10.017

where n is the state dimension, m and p are the input and output numbers (and "t" stands for "transpose"). Put

$$M = \frac{\mathbb{R}[s]^n \oplus \mathbb{R}[s]^m}{\begin{bmatrix} sI - F^t \\ -G^t \end{bmatrix} \mathbb{R}[s]^n},$$

and let φ and θ denote respectively the compositions

$$\mathbb{R}[s]^m \xrightarrow{\begin{bmatrix} 0 \\ I \end{bmatrix}} \mathbb{R}[s]^n \oplus \mathbb{R}[s]^m \to M \text{ and } \mathbb{R}[s]^p \xrightarrow{\begin{bmatrix} H^t \\ J^t \end{bmatrix}} \mathbb{R}[s]^n \oplus \mathbb{R}[s]^m \to M.$$

We thus arrive at the triple (M, φ, θ), where M is a finitely generated $\mathbb{R}[s]$-module, $\varphi : \mathbb{R}[s]^m \to M$ is an injective $\mathbb{R}[s]$-homomorphism with torsion cokernel and $\theta : \mathbb{R}[s]^p \to M$ is an $\mathbb{R}[s]$-homomorphism such that $\varphi^{-1}\theta$ is a proper rational matrix. (φ induces a bijective linear map $\mathbb{R}(s)^m \to M \otimes \mathbb{R}(s)$ and θ induces a linear map $\mathbb{R}(s)^p \to M \otimes \mathbb{R}(s)$, and by $\varphi^{-1}\theta$ we mean the composition

$$\mathbb{R}(s)^p \to M \otimes \mathbb{R}(s) \to \mathbb{R}(s)^m,$$

which can be viewed as a rational matrix.) We call such triples causal i/o Fliess models; they were introduced by Fliess [6]. The main theorem in [6] claims: Any causal i/o Fliess model possesses a Kalman realization; moreover, any two Kalman realizations are similar.

The purpose of this paper is to extend this result of Fliess to the ring case. This will be done using a generalized version of Fuhrmann's classical construction. (See Fuhrmann [8].)

Throughout, D is an arbitrary noetherian commutative ring, s an indeterminate, q a fixed positive integer, and $\mathcal{U}$ is an arbitrary fixed module over $D[s]$. (The latter should be thought of as a function space, which is needed to define trajectories.)

By a Fliess model with signal number q, we shall understand any pair (M, μ), where M is a finitely generated $D[s]$-module and $\mu : D[s]^q \to M$ is a "generically" surjective homomorphism. This definition is slightly different from that given in Fliess and Mounier [7], and we believe that it is a very natural starting point for the theory of linear systems over a ring. Following Polderman and Willems [26] and Willems [34,35], the input/output structure is not postulated in the definition. It should be emphasized that postulating such a structure would be a strong restriction. The point is that a "componentwise partition into inputs and outputs" for Fliess models defined over a ring does not always exist. In contrast to the field case, existence of an input/output structure (not necessarily causal) is rather an exception than a rule (see the discussion at the end of Section 2).

A Fliess model (M, μ) is said to be observable if μ is surjective. Observable Fliess models are of particular interest as they include linear delay differential equations (LDDEs), which have attracted much attention in recent years. We recall that in the case of LDDEs

$$D = \mathbb{R}[\delta] \text{ and } \mathcal{U} = C^\infty(\mathbb{R}, \mathbb{R}).$$

(Here $\delta = (\delta_1, \ldots, \delta_r)$ with indeterminates $\delta_1, \ldots, \delta_r$ acting on $\mathcal{U}$ as delay operators; the indeterminate s acts as the differentiation operator.) An LDDE (with q unknowns) is an equation of the form

$$Rw = 0 \quad (w \in \mathcal{U}^q),$$

where $R \in \mathbb{R}[\delta, s]^{\bullet \times q}$. This can be regarded as an observable Fliess model. Indeed, if p is the row number of R, then the module

$$M = \mathbb{R}[\delta, s]^q / R^t \mathbb{R}[\delta, s]^p$$

together with the canonical epimorphism $\mathbb{R}[\delta, s]^q \to M$ is an observable Fliess model.

Remark. The idea of regarding an LDDE as a linear system over a ring is due to Kamen [16]. This point of view was adopted then by many authors (see, e.g., Byrnes [3], Khargonekar [17], Morse [23], Sontag [31]).

LDDEs of the form

$$Py = Qu \quad (u \in \mathcal{U}^m,\ y \in \mathcal{U}^p)$$

with $P \in \mathbb{R}[\delta, s]^{p\times p}$ and $Q \in \mathbb{R}[\delta, s]^{p\times m}$ such that $\det(P)$ is "monic" and $P^{-1}Q$ is "proper" were considered by Gluesing-Luerssen [12]. They correspond to (observable) causal i/o Fliess models. However, as emphasized above, they constitute a small class of systems. The realization theory in this special case is quite easy; Fuhrmann's construction can be straightforwardly generalized to it. The reason (why the theory is easy) is that state modules that appear in this case are projective. Far from it, in general.

Perhaps the reader may find strange that no specific conditions are imposed on the module $\mathcal{U}$. Conditions should be imposed when one deals with the following question: What is a necessary and sufficient condition for two Fliess models (or two state models) to have the same behavior? But this important question will not be addressed here. For interesting results in this direction (for the case of LDDEs) the reader is referred to Gluesing-Luerssen [11], Gluesing-Luerssen et al. [13] and Habets [14].

State-space realization theory is among the first important topics studied in systems theory. Numerous papers were written for the field case. Recently the concept of states has been studied in the Willems behavioral setting (see Fuhrmann [9], Fuhrmann et al. [10], Rapisarda and Willems [28], and also Lomadze [21].) The realization theory of transfer functions and input/output maps defined over a ring has been developed in many papers (see, e.g., Eilenberg [5], Brewer et al. [4], Khargonekar [17], Rouchaleau and Sontag [29], Rouchaleau et al. [30], Sontag [31]).

The content of the paper is as follows:

1. Preliminaries

Here we recall a few definitions and facts from algebra.

One knows well that the field $\mathbb{F}(s)$ of rational functions and the ring $\mathbb{F}(s)_{pr}$ of proper rational functions are indispensable in the theory of linear systems over a field $\mathbb{F}$. We shall need things like them.

A polynomial $g \in D[s]$ is called monic if its leading coefficient is an invertible element of D. The set of monic polynomials is a multiplicative subset in $D[s]$, and the corresponding localization will be denoted by $D(s)$. Elements of $D(s)$ will be called rational functions. Thus, by definition, a rational function is a ratio f/g, where f is an arbitrary polynomial and g is a monic polynomial. Certainly, monic polynomials are not zero-divisors. Therefore, the canonical homomorphism

$$D[s] \to D(s), \quad f \mapsto f/1$$

is an embedding. We shall identify $D[s]$ with its image under this embedding.

A rational function f/g is called proper (resp., strictly proper) if $deg(f) \leqslant deg(g)$ (resp., $deg(f) < deg(g)$). The ring $D(s)_{pr}$ of proper rational functions will be denoted by O. Strictly proper rational functions form an ideal of O. This ideal is principal and is generated by s^{-1}. Notice that $O/s^{-1}O = D$.

By the Euclidean division, any rational function can be written in a unique way as a sum of a polynomial and a strictly proper rational function. In other words, we have:

$$D(s) = D[s] \oplus s^{-1}O.$$

We remark that all the elements in $1 + s^{-1}O$ are invertible in O, and consequently the ideal $s^{-1}O$ is contained in the Jacobson radical of O (see Matsumura [22]).

Remark. The rings $D(s)$ and $D(s)_{pr}$ are taken from the celebrated paper Quillen [27], where they have played a very important role. (See also Lam [19, Ch. IV, Sect. 1].) In many papers on linear systems over D similar rings, namely, the rings $D((s^{-1}))$ (the ring of formal Laurent series in s^{-1}) and $D[[s^{-1}]]$ (the ring of formal series in s^{-1}) are employed.

A homomorphism $M_1 \to M_2$ of modules over $D[s]$ will be said to be generically surjective (resp., bijective) if

$$M_1 \otimes_{D[s]} D(s) \to M_2 \otimes_{D[s]} D(s)$$

is surjective (resp., bijective).

There are two important functors

$$Hom_D(-, \mathcal{U}) \quad \text{and} \quad - \otimes_D \mathcal{U}.$$

We shall consider them on the category of finitely generated D-modules. (It will turn out that the first one is more relevant.)

One has

$$Hom_D(D^n, \mathcal{U}) = \mathcal{U}^n \quad \text{and} \quad D^n \otimes_D \mathcal{U} \simeq \mathcal{U}^n.$$

Remark. In the classical linear systems theory the ground ring is a field $\mathbb{F}$, and one deals (without loss of generality) with finite-dimensional linear spaces of the form $\mathbb{F}^n$. For this reason, the functors above do not occur explicitly.

The following fact is well-known. (For convenience of the reader, we shall provide its proof.)

Lemma 1. *Let X be a finitely generated projective D-module. Then, there is a canonical isomorphism*

$$X \otimes_D \mathcal{U} \simeq Hom_D(X^*, \mathcal{U}).$$

Proof. The isomorphism is established by taking $x \otimes w$ to $\phi : X^* \to \mathcal{U}$ defined by the formula $\phi(y) = y(x)w \ \ (y \in X^*)$. To see that this indeed is an isomorphism, consider an isomorphism $X \oplus X_1 \simeq D^n$, where X_1 is a module and n is an integer. (Such an isomorphism exists because X is projective.) We have:

$$(X \oplus X_1) \otimes_D \mathcal{U} = X \otimes_D \mathcal{U} \oplus X_1 \otimes_D \mathcal{U} \ \text{ and } \ Hom_D((X \oplus X_1)^*, \mathcal{U}) = Hom_D(X^*, \mathcal{U}) \oplus Hom_D(X_1^*, \mathcal{U}).$$

In view of $Hom_D(D^n, \mathcal{U}) = \mathcal{U}^n = D^n \otimes_D \mathcal{U}$, these relations give an isomorphism

$$X \otimes_D \mathcal{U} \oplus X_1 \otimes_D \mathcal{U} \simeq Hom_D(X^*, \mathcal{U}) \oplus Hom_D(X_1^*, \mathcal{U}).$$

This implies what we want. □

Given a D-homomorphism $A : X \to Y$, we let $A^\vee$ denote the homomorphism

$$Hom_D(A, \mathcal{U}) : Hom_D(Y, \mathcal{U}) \to Hom_D(X, \mathcal{U});$$

for simplicity, we shall write A for the canonical homomorphism

$$A \otimes_D \mathcal{U} : X \otimes_D \mathcal{U} \to Y \otimes_D \mathcal{U}.$$

If X is a D-module, then one denotes by $X[s]$ the module $X \otimes_D D[s]$. This is a module over $D[s]$. Every element of $X[s]$ is uniquely represented as $x_0 \otimes 1 + \cdots + x_l \otimes s^l$. (Modules of the type $X[s]$ are called extended modules in Quillen [27].)

For every D-module X, one may identify $Hom_{D[s]}(X[s], \mathcal{U})$ with $Hom_D(X, \mathcal{U})$. (If $u : X \to \mathcal{U}$ is a D-homomorphism, then the map $\tilde{u}$ defined by the formula

$$\tilde{u}(x_0 \otimes 1 + \cdots + x_l \otimes s^l) = u(x_0) + su(x_1) + \cdots + s^l u(x_l)$$

is a $D[s]$-homomorphism from $X[s]$ into $\mathcal{U}$. It is easily seen that $u \mapsto \tilde{u}$ establishes an isomorphism.) Likewise, one may identify $X[s] \otimes_{D[s]} \mathcal{U}$ with $X \otimes_D \mathcal{U}$.

2. Fliess models and AR-models

A Fliess model (with signal number q) is a pair (M, μ) consisting of a finitely generated $D[s]$-module M and a generically surjective homomorphism $\mu : D[s]^q \to M$.

A morphism from one Fliess model (M_1, μ_1) to another Fliess model (M_2, μ_2) is a homomorphism $\phi : M_1 \to M_2$ such that

$$\mu_2 = \phi \circ \mu_1.$$

Clearly, Fliess models form a category.

Let (M, μ) be a Fliess model. The homomorphism μ gives rise in an obvious way to a homomorphism $Hom_{D[s]}(M, \mathcal{U}) \to Hom_{D[s]}(D[s]^q, \mathcal{U})$. Certainly $Hom_{D[s]}(D[s]^q, \mathcal{U}) = \mathcal{U}^q$, and thus we have a canonical homomorphism

$$Hom_{D[s]}(M, \mathcal{U}) \to \mathcal{U}^q.$$

The module $\mathcal{B}_f = Hom_{D[s]}(M, \mathcal{U})$ is called the full (or internal) behavior of the model and the homomorphism itself the manifestation map. The image of this map is called the external (or manifest) behavior.

Remark. The functor $Hom_{D[s]}(-, \mathcal{U})$ has been introduced by Malgrange. First, its importance for linear systems theory was recognized by Oberst [25].

In order to explain how the notions above are related with the classical ones, let us consider the situation at the beginning of Introduction. We have a commutative diagram

$$\begin{array}{ccccccc} \mathbb{R}[s]^n & \xrightarrow{\begin{bmatrix} sI - F^t \\ -G^t \end{bmatrix}} & \mathbb{R}[s]^n \oplus \mathbb{R}[s]^m & \to & M & \to & 0 \\ & & \begin{bmatrix} 0 & H^t \\ I & J^t \end{bmatrix} \uparrow & & \uparrow & & \\ & & \mathbb{R}[s]^m \oplus \mathbb{R}[s]^p & = & \mathbb{R}[s]^m \oplus \mathbb{R}[s]^p & & \end{array}$$

with exact top row. Applying to this diagram the functor $Hom_{\mathbb{R}[s]}(-, C^\infty(I))$, where I is a time interval, we get the commutative diagram

$$\begin{array}{ccccccc} 0 & \to & Hom_{\mathbb{R}[s]}(M, C^\infty(I)) & \to & C^\infty(I)^n \oplus C^\infty(I)^m & \xrightarrow{\begin{bmatrix} sI - F & -G \end{bmatrix}} & C^\infty(I)^n \\ & & \downarrow & & \downarrow \begin{bmatrix} 0 & I \\ H & J \end{bmatrix} & & \\ & & C^\infty(I)^n \oplus C^\infty(I)^p & = & C^\infty(I)^n \oplus C^\infty(I)^p & & \end{array},$$

in which the top row is exact. It follows that the internal behavior $Hom_{\mathbb{R}[s]}(M, C^\infty(I))$ is canonically isomorphic to the solution set of the differential equation

$$sx = Fx + Gu$$

and the manifestation map is given by

$$\begin{pmatrix} x \\ u \end{pmatrix} \mapsto \begin{pmatrix} u \\ Hx + Ju \end{pmatrix}.$$

We say that a Fliess model (M, μ) is controllable if M is projective (see Fliess and Mounier [7]).

Remark. In fact, there are several different kinds of controllability, and the controllability above is strong controllability. There is a vast literature on this fundamental concept for LDDEs (see, e.g., Fliess and Mounier [7], Gluesing-Luerssen [11], Gluesing-Luerssen et al. [13], Mounier [24], Rocha and Willems [28], Vettori and Zampieri [32,33]).

The model is observable if μ is surjective. If this is the case, then the manifestation map is injective, and therefore the internal behavior can be identified with the external one.

An AR-model (with signal number q) is just a polynomial matrix $R \in D[s]^{\bullet \times q}$. Two AR-models R_1 and R_2 are said to be equivalent if $R_2 = FR_1$ and $R_1 = GR_2$ for some polynomial matrices F and G. The behavior of an AR-model R is defined to be the solution set of the equation

$$Rw = 0, \quad w \in \mathcal{U}^q.$$

(Needless to say that an LDDE is a special case of AR-model.)

One associates a Fliess model to an AR-model in a very natural way. Indeed, if R is an AR-model, then the pair consisting of the cokernel $Coker(R^t)$ and the canonical epimorphism of $D[s]^q$ onto this cokernel is a Fliess model. A Fliess model obtained this way is observable, of course. Conversely, if (M, μ) is observable, one can always find a polynomial matrix $R \in D[s]^{\bullet \times q}$ such that the sequence

$$D[s]^p \xrightarrow{R^t} D[s]^q \xrightarrow{\mu} M \to 0,$$

where p is the row number of R, is exact.

AR-models and observable Fliess models are equivalent objects. More precisely, there is a one-to-one correspondence between equivalence classes of AR-models and isomorphism classes of observable Fliess models.

Applying the functor $Hom_{D[s]}(-, \mathcal{U})$ to the sequence above, we get an exact sequence

$$0 \to Hom_{D[s]}(M, \mathcal{U}) \to \mathcal{U}^q \xrightarrow{R} \mathcal{U}^p.$$

This tells us that the behavior of an AR-model and the (external) behavior of the corresponding Fliess model coincide.

Let (M, μ) be a Fliess model. We say that (M, μ) admits an input/output structure if there are an integer m, a permutation matrix Π, a generically bijective homomorphism $\varphi : D[s]^m \to M$ and a homomorphism $\theta : D[s]^{q-m} \to M$ such that

$$(M, \mu \circ \Pi) = (M, [\varphi \;\; \theta]).$$

We see that for (M, μ) to admit an input/otput structure it is necessary that $M \otimes D(s)$ be a free $D(s)$-module. (If D is a field, then so is $D(s)$, and the condition is fulfilled automatically. But this is not the case when D is not a field.) At this point, we should perhaps mention a relation with Quillen's theorem (see Theorem 3 in Quillen [27]) stating that if M is a finitely generated projective module such that $M \otimes D(s)$ is free over $D(s)$, then M is free. (We remind that the famous Quillen–Suslin theorem is an easy consequence of this theorem.) It follows that if (M, μ) is controllable, then it admits an input/output structure if and only if it has a representation of the form $(D[s]^m, [R_1 \;\; R_2])$, where R_1 is a nonsingular $m \times m$ matrix and R_2 is an arbitrary $m \times (q - m)$ matrix. (A square matrix is called nonsingular if its determinant is an invertible element of $D(s)$.) This fact was observed by Khargonekar [17] in a somewhat different context.

3. Left and right state models

From the point of view that does not make distinction between inputs and outputs, there are two kinds of state models (see, e.g., Fuhrmann et al. [10], Kuijper [18], Lomadze [20,21], Polderman and Willems [26], Rapisarda and Willems [28], Willems [34,35]).

A left state model is a quintuple (X, Y, A, B, C), where X, Y are finitely generated D-modules and $A, B : Y \to X, C : Y \to D^q$ are linear maps such that B is surjective and $\begin{bmatrix} B \\ C \end{bmatrix}$ is injective. The module X is called the state module and Y the (left) internal variable module.

A right state model is a quintuple (X, Z, E, F, G), where X, Z are finitely generated D-modules and $E, F : X \to Z, G : D^q \to Z$ are linear maps such that E is injective and $\begin{bmatrix} E & G \end{bmatrix}$ is surjective. The module X is called the state module and Z the (right) internal variable module.

Remark. In Kuijper [18] right and left state models are referred to as **P** and **DP** representations, respectively.

Example 1. Let (X, F, G, H, J) be a Kalman model over D with m inputs and p outputs, in other words, a quintuple, where X is a finitely generated *projective* D-module and

$$F : X \to X,\ G : D^m \to X,\ H : X \to D^p,\ J : D^m \to D^p$$

are D-linear maps.

Associated with this system there is the following left state model

$$\left(X^*, X^* \oplus D^p, \begin{bmatrix} I & 0 \end{bmatrix}, \begin{bmatrix} F^t & H^t \end{bmatrix}, \begin{bmatrix} -G^t & -J^t \\ 0 & I \end{bmatrix}\right)$$

and the following right state model

$$\left(X^*, X^* \oplus D^m, \begin{bmatrix} I \\ 0 \end{bmatrix}, \begin{bmatrix} F^t \\ G^t \end{bmatrix}, \begin{bmatrix} 0 & H^t \\ I & J^t \end{bmatrix}\right).$$

Remark. Example 6 (in Section 7) explains why we use above the dualizing functor *.

Here are concrete examples of state models.

Example 2. Let $D = \mathbb{R}[\delta]$.

(a) The quintuple (X, Y, A, B, C) with $X = D^3$, $Y = D^4$ and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} -\delta & 0 & 0 & 0 \end{bmatrix}$$

is a left state model.

(b) Let $X = D^3$, and let Z be the module generated by the symbols a_1, a_2, a_3 and a subject to the following relation

$$\delta a = a_1.$$

Define E, F, G by the following formulas

$$E(e_1) = a_1,\ E(e_2) = a_2,\ E(e_3) = a_3; \quad F(e_1) = a_2,\ F(e_2) = a_3,\ F(e_3) = 0; \quad G(1) = a.$$

(Here e_1, e_2, e_3 is the standard basis of D^3.) One can easily check that (X, Z, E, F, G) is a right state model.

Example 3. (a) Let $D = \mathbb{R}[\delta_1, \delta_2]$. The quintuple (X, Y, A, B, C) with $X = D$, $Y = D^3$ and

$$A = \begin{bmatrix} 0 & \delta_2 & -\delta_1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_1 \end{bmatrix}$$

is a left state model.

(b) Let $X = D$, and let Z be the module generated by the symbols a, b, c_0 and c_1 subject to the following relations

$$\delta_1 a = c_1 \quad \text{and} \quad \delta_1 b + \delta_2 c_0 = 0.$$

Define E, F, G by the following formulas

$$E(1) = c_0; \quad F(1) = c_1; \quad G(e_1) = a,\ G(e_2) = b,\ G(e_3) = c_0.$$

(Here again e_1, e_2, e_3 is the standard basis of D^3.) One can easily check that (X, Z, E, F, G) is a right state model.

Assume that (X, Y, A, B, C) and (X, Z, E, F, G) are left and right state models, respectively. We shall say that they form an exact couple if the sequence

$$0 \to Y \overset{\begin{bmatrix} A \\ B \\ C \end{bmatrix}}{\to} X \oplus X \oplus D^q \overset{\begin{bmatrix} E & -F & G \end{bmatrix}}{\to} Z \to 0 \qquad (1)$$

is exact. We then have, in particular, that $EA + GC = FB$.

Example 4. The two state models associated with the classical linear system in Example 1 form an exact couple.

Example 5. The two state models in Example 2 as well as in Example 3 form an exact couple.

Lemma 2. *Let X, Y and Z be finitely generated A-modules and $A, B : Y \to X, C : Y \to D^q, E, F : X \to Z, G : D^q \to Z$ linear maps such that the sequence (1) is exact. If one of the quintuples (X, Y, A, B, C) and (X, Z, E, F, G) is a state model, then so is the other.*

Proof. We have a commutative diagram

$$\begin{array}{ccccccc} 0 \to & Y & \to & X \oplus X \oplus D^q & \to & Z & \to 0 \\ & B \downarrow & & \downarrow & & \downarrow & \\ 0 \to & X & = & X & \to & 0 & \end{array}.$$

Applying the snake lemma (see Proposition 2.10 in Atiyah and Macdonald [1]), we obtain the following exact sequence

$$X \oplus D^q \overset{\begin{bmatrix} E & G \end{bmatrix}}{\to} Z \to Coker(B) \to 0;$$

whence

$$\text{“}B \text{ is surjective"} \Leftrightarrow \text{“} \begin{bmatrix} E & G \end{bmatrix} \text{ is surjective"}.$$

Likewise, from the commutative diagram

$$\begin{array}{ccccccc} & 0 & \to & X & = & X & \to 0 \\ & \downarrow & & \downarrow & & \downarrow E & \\ 0 \to & Y & \to & X \oplus X \oplus D^q & \to & Z & \to 0 \end{array},$$

we get the exact sequence

$$0 \to Ker(E) \to Y \overset{\begin{bmatrix} B \\ C \end{bmatrix}}{\to} X \oplus D^q.$$

It follows from this that

$$\text{“} \begin{bmatrix} B \\ C \end{bmatrix} \text{ is injective"} \Leftrightarrow \text{“}E \text{ is injective"}.$$

The proof is complete. □

A morphism from one left state model $(X_1, Y_1, A_1, B_1, C_1)$ to another left state model $(X_2, Y_2, A_2, B_2, C_2)$ is a pair consisting of D-linear maps $\alpha : X_1 \to X_2$ and $\beta : Y_1 \to Y_2$ such that

$$\alpha A_1 = A_2\beta, \ \ \alpha B_1 = B_2\beta \ \text{ and } \ C_1 = C_2\beta.$$

Similarly, a morphism from a right state model $(X_1, Z_1, E_1, F_1, G_1)$ to a right state model $(X_2, Z_2, E_2, F_2, G_2)$ is a pair consisting of D-linear maps $\alpha : X_1 \to X_2$ and $\beta : Z_1 \to Z_2$ such that

$$\beta E_1 = E_2\alpha, \ \ \beta F_1 = F_2\alpha \ \text{ and } \ G_2 = \beta G_1.$$

Using the previous lemma, we can define in an obvious way functors from state models of one kind to state models of the other kind. It is clear that these functors are inverse to each other, and thus we have

Proposition 1. *The two categories of state models are canonically equivalent.*

4. From Fliess models to (right) state models

Suppose that we are given a Fliess model $\Phi = (M, \mu)$. Let V be the module of fractions of M defined by monic polynomials, and let $i : M \to V$ be the canonical map given by $i(x) = x/1$. We have a canonical $D(s)$-linear map $D(s)^q \to V$, which is onto by definition; let N denote the image of O^q under this map.

Define

$$X = \{x \in M |\, i(x) \in s^{-1}N\} \ \text{ and } \ Z = \{z \in M |\, i(z) \in N\}.$$

Next, define two canonical linear maps $E, F : X \to Z$ by the following formulas

$$E(x) = x \ \text{ and } \ F(x) = sx.$$

For each $a \in D^q$, clearly $\mu(a)$ belongs to Z. Hence, we also have a canonical linear map $D^q \to Z$, which will be denoted by G.

We thus have a quintuple (X, Z, E, F, G).

Remark. The reader can notice that the construction above naturally generalizes the construction given in the classical paper Fuhrmann [8] (see also Fuhrmann [9], Fuhrmann et al. [10].)

Theorem 1. (X, Z, E, F, G) *is a right state model.*

Proof. First of all, we need to show that X and Z are finitely generated D-modules. This is easy to do once we have at our disposal Finiteness Theorem (see Appendix A). (A direct proof seems to be hard.) Indeed, let $\mathcal{F}$ denote the quintuple (V, M, N, i, j), where j is the canonical inclusion map $N \to V$. This is a sheaf. Moreover, this certainly is a coherent sheaf. It is clear that

$$X \simeq H^0\mathcal{F}(-1) \ \text{ and } \ Z \simeq H^0\mathcal{F}.$$

By Finiteness Theorem, we get that X and Z are finitely generated modules.

Further, it is clear that E is injective. To see that the linear map $[E \quad G] : X \oplus D^q \to Z$ is surjective, take any element z in Z. Then $i(z) \in N$. We can find $a \in D^q$ and $g \in O^q$ such that $\mu(a + s^{-1}g) = i(z)$. Obviously $x = z - \mu(a)$ belongs to X, and we have $z = E(x) + G(a)$.

The proof is complete. $\square$

The right state model that we have constructed will be denoted by $Sigma(\Phi)$. Clearly, the construction is functorial. In other words, given a morphism $\Phi_1 \to \Phi_2$, there is a morphism between the state representations

$$Sigma(\Phi_1) \to Sigma(\Phi_2).$$

Once the right state representation of a Fliess model is defined, the left state representation can be defined using Lemma 2.

Now, we want to show that a Fliess model can be reconstructed from its right state representation. Let (M, μ) be a Fliess model, and let (X, Z, E, F, G) be its right state representation.

The canonical linear maps E, F determine the $D[s]$-homomorphism

$$sE - F : X[s] \to Z[s],$$

and the canonical map $Z \to M$ gives rise in an obvious way to a $D[s]$-homomorphism

$$Z[s] \to M.$$

For every $x \in X$, the element $sE(x) - F(x)$ goes to $sx - sx = 0$ under the latter homomorphism. It follows that the sequence

$$X[s] \to Z[s] \to M$$

is a complex.

The following proposition implies what we want.

Proposition 2. (*a*) *The complex*

$$0 \to X[s] \to Z[s] \to M \to 0$$

is exact.

(*b*) *The diagram*

$$\begin{array}{ccc} D[s]^q & = & D[s]^q \\ \downarrow & & \downarrow \\ Z[s] & \to & M \end{array}$$

is commutative.

Proof. (a) Exactness at $X[s]$:

This is obvious, because $E : X \to Z$ is injective.

Exactness at $Z[s]$:

Assume that an element $z_0 \otimes 1 + z_1 \otimes s + \cdots + z_l \otimes s^l \in Z[s]$ goes to zero. Then

$$z_0 + sz_1 + \cdots + s^l z_l = 0.$$

Set

$$x_0 = -s^{-1}z_0,\ x_1 = -(s^{-2}z_0 + s^{-1}z_1),\ \ldots,\ x_{l-1} = -(s^{-l}z_0 + \cdots + s^{-1}z_{l-1}).$$

Using the relation above, one can easily see that all these elements are states. One can check easily that

$$-Fx_0 = z_0,\ Ex_0 - Fx_1 = z_1,\ \ldots, Ex_{l-2} - Fx_{l-1} = z_{l-1},\ Ex_{l-1} = z_l.$$

It follows that

$$(sE - F)(x_0 \otimes 1 + x_1 \otimes s + \cdots + x_{l-1} \otimes s^{l-1}) = z_0 \otimes 1 + z_1 \otimes s + \cdots + z_l \otimes s^l.$$

Exactness at M:

Take any $m \in M$. Because $\mu : D(s)^q \to V$ is surjective (and because $D(s) = sD[s] + O$),

$$i(m) = n + i(\mu(a_1 s + \cdots + a_l s^l))$$

for some $n \in N$ and $a_1, \ldots, a_l \in D^q$. From this it follows that $z = m - \mu(a_1 s + \cdots a_l s^l) \in Z$. We can see that

$$z \otimes 1 + \mu(a_1) \otimes s + \cdots + \mu(a_l) \otimes s^l$$

goes to m.

(b) Obvious.

The proposition is proved. □

5. From state models to Fliess models

We begin with the following two lemmas.

Lemma 3. *Let (X, Y, A, B, C) be a left state model. Then, the homomorphism*

$$\begin{bmatrix} sB - A \\ C \end{bmatrix} : Y[s] \to X[s] \oplus D[s]^q \tag{2}$$

is injective.

Proof. Let $y_0, \ldots, y_l \in Y$. We have

$$\begin{bmatrix} sB - A \\ C \end{bmatrix} (y_0 \otimes 1 + y_1 \otimes s + \cdots + y_l \otimes s^l)$$
$$= \begin{bmatrix} -Ay_0 \\ Cy_0 \end{bmatrix} \otimes 1 + \begin{bmatrix} By_0 - Ay_1 \\ Cy_1 \end{bmatrix} \otimes s + \begin{bmatrix} By_{l-1} - Ay_l \\ Cy_l \end{bmatrix} \otimes s^l + \begin{bmatrix} By_l \\ 0 \end{bmatrix} \otimes s^{l+1}.$$

Assuming that the right hand side is zero, we obtain that

$$By_l = 0 \text{ and } Cy_l = 0, \;\; By_{l-1} - Ay_l = 0 \text{ and } Cy_{l-1} = 0, \; \ldots, \; By_0 - Ay_1 = 0 \text{ and } Cy_0 = 0.$$

Because $\begin{bmatrix} B \\ C \end{bmatrix}$ is injective, it follows that all $y_l, \ldots, y_1, y_0$ are zero.

The proof is complete. □

Lemma 4. *Let (X, Z, E, F, G) be a right state model. Then, the homomorphism*

$$sE - F : X[s] \to Z[s] \tag{3}$$

is injective.

Proof. Let $x_0, \ldots, x_l \in X$. We have

$$(sE - F)(x_0 \otimes 1 + x_1 \otimes s + \cdots + x_l \otimes s^l)$$
$$= -F(x_0) \otimes 1 + (Ex_0 - Fx_1) \otimes s + \cdots + (Ex_{l-1} - Fx_l) \otimes s^l + Ex_l \otimes s^{l+1}.$$

Assuming that the right hand side is zero, we obtain that

$$Ex_l = 0, \; Ex_{l-1} = Fx_l, \; \ldots, \; Ex_0 = Fx_1.$$

Because E is injective, it follows that all $x_l, \ldots, x_1, x_0$ are zero.

The proof is complete. □

Theorem 2. *Let (X, Y, A, B, C) be a left state model. Define L to be the cokernel of (2), and define λ to be the composition*

$$D[s]^q \to X[s] \oplus D[s]^q \to L.$$

Then, (L, λ) is a Fliess model.

Proof. Consider the homomorphism of O-modules

$$B - s^{-1}A : Y \otimes O \to X \otimes O.$$

We claim that this is surjective. Indeed,

$$(B - s^{-1}A) \otimes O/s^{-1}O : (Y \otimes O) \otimes O/s^{-1}O \to (X \otimes O) \otimes O/s^{-1}O$$

is the same as $B : Y \to X$, and hence is surjective. By Nakayama's lemma (see Matsumura [22]), our homomorphism must be surjective. It follows that $B - s^{-1}A : Y(s) \to X(s)$ is surjective. Multiplying this by s, we get that

$$sB - A : Y(s) \to X(s)$$

is surjective.

Now, consider the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & 0 & \to & D(s)^q & = & D(s)^q & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & Y(s) & \to & X(s) \oplus D(s)^q & \to & L \otimes D(s) & \to 0 \end{array}.$$

Applying the snake lemma, we obtain the following exact sequence

$$Y(s) \to X(s) \to Coker(\lambda \otimes D(s)) \to 0,$$

from which we get that

$$Coker(\lambda \otimes D(s)) = 0.$$

Thus,

$$\lambda \otimes D(s) : D(s)^q \to L \otimes D(s)$$

is an epimorphism.

The proof is complete. □

Theorem 3. *Assume* (X, Z, E, F, G) *is a right state model. Define* M *to be the cokernel of* (3), *and define* μ *to be the composition*

$$D[s]^q \to Z[s] \to M.$$

Then, (M, μ) *is a Fliess model.*

Proof. Consider the homomorphism

$$\begin{bmatrix} E - s^{-1}F & G \end{bmatrix} : X \otimes O \oplus O^q \to Z \otimes O.$$

We claim that this is surjective. Indeed,

$$\begin{bmatrix} E - s^{-1}F & G \end{bmatrix} \otimes O/s^{-1}O : (X \otimes O \oplus O^q) \otimes O/s^{-1}O \to (Z \otimes O) \otimes O/s^{-1}O$$

is the same as $\begin{bmatrix} E & G \end{bmatrix} : X \oplus D^q \to Z$, and hence is surjective. Applying Nakayama's lemma, we see that our homomorphism must be surjective. It follows that $\begin{bmatrix} E - s^{-1}F & G \end{bmatrix} : X(s) \oplus D(s)^q \to Z(s)$ is surjective. Hence,

$$\begin{bmatrix} sE - F & G \end{bmatrix} : X(s) \oplus D(s)^q \to Z(s)$$

is surjective. This, in turn, implies that

$$X(s) \to Coker(D(s)^q \to Z(s))$$

is surjective. Now, consider the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & 0 & \to & D(s)^q & = & D(s)^q & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & X(s) & \to & Z(s) & \to & M \otimes D(s) & \to 0 \end{array}.$$

Applying the snake lemma, we obtain the following exact sequence

$$X(s) \to Coker(D(s)^q \to Z(s)) \to Coker(\mu \otimes D(s)) \to 0,$$

from which we get that

$$Coker(\mu \otimes D(s)) = 0.$$

Thus,

$$\mu \otimes D(s) : D(s)^q \to M \otimes D(s)$$

is an epimorphism.

The proof is complete. □

6. Equivalence theorem

Our aim in this section is to show that Fliess models and state (right or left) models are equivalent objects.

Proposition 3. *Let (X, Y, A, B, C) and (X, Z, E, F, G) be left and right state models, respectively, and suppose that they form an exact couple. Then, their Fliess models are canonically isomorphic.*

Proof. Let (L, λ) and (M, μ) denote the corresponding Fliess models.

Consider the diagram

$$\begin{array}{ccc} Y[s] & \overset{\begin{bmatrix} sB - A \\ C \end{bmatrix}}{\longrightarrow} & X[s] \oplus D[s]^q \\ \downarrow & & \downarrow \\ X[s] & \overset{sE-F}{\longrightarrow} & Z[s] \end{array}$$

with the vertical arrows given by B and $\begin{bmatrix} E & -G \end{bmatrix}$. The diagram commutes:

$$\begin{bmatrix} E & -G \end{bmatrix} \begin{bmatrix} sB - A \\ C \end{bmatrix} = sEB - EA - GC = sEB - FB = (sE - F)B.$$

So, there is a homomorphism $L \to M$ making the diagram

$$\begin{array}{ccccccc} 0 \to & Y[s] & \overset{\begin{bmatrix} sB - A \\ C \end{bmatrix}}{\longrightarrow} & X[s] \oplus D[s]^q & \to & L & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & X[s] & \overset{sE-F}{\longrightarrow} & Z[s] & \to & M & \to 0 \end{array}$$

commutative. Because the left and the middle vertical arrows here are surjective, by the snake lemma, the homomorphism $L \to M$ also is surjective. Further, by the same lemma, we have an exact sequence

$$0 \to Ker(Y[s] \to X[s]) \to Ker(X[s] \oplus D[s]^q \to Z[s]) \to Ker(L \to M) \to 0.$$

Therefore, to prove injectivity of $L \to M$ it suffices to prove surjectivity of

$$Ker(Y[s] \to X[s]) \to Ker(X[s] \oplus D[s]^q \to Z[s]).$$

Let

$$x = x_0 \otimes 1 + x_1 \otimes s + x_2 \otimes s^2 + \cdots + x_l \otimes s^l \in X[s] \text{ and}$$
$$w = w_0 + w_1 s + w_2 s^2 + \cdots + w_k s^l \in D[s]^q$$

be such that

$$Ex - Gw = 0.$$

Using (1), we can find $y_0, \ldots, y_l \in Y$ such that

$$(\forall\ 0 \leqslant i \leqslant l) \quad Ay_i = -x_i, \quad By_i = 0, \quad Cy_i = w_i.$$

Putting

$$y = y_0 \otimes 1 + y_1 \otimes s + y_2 \otimes s^2 + \cdots + y_k \otimes s^l,$$

we have

$$By = 0 \quad \text{and} \quad \begin{bmatrix} sB - A \\ C \end{bmatrix} y = \begin{bmatrix} x \\ w \end{bmatrix}.$$

This proves the surjectivity, and thus the homomorphism $L \to M$ is bijective.

Finally, we see from the diagram above that the square

$$\begin{array}{ccc} D[s]^q & \to & L \\ \downarrow & & \downarrow \\ Z[s] & \to & M \end{array}$$

commutes, and this implies that the isomorphism $L \to M$ determines an isomorphism

$$(L, \lambda) \simeq (M, \mu).$$

The proof is complete. □

In Section 3 we saw that, for every Fliess model Φ, there is a canonical isomorphism

$$Phi(Sigma(\Phi)) \simeq \Phi.$$

We leave to the reader to show that, for every right state model Σ, there is a canonical isomorphism

$$Sigma(Phi(\Sigma)) \simeq \Sigma.$$

It follows that the category of Fliess models is canonically equivalent to that of right state models. The latter, as we already know, is canonically equivalent to the category of left state models.

Thus, we have

Theorem 4. *The are canonical categorical equivalences*:

$$\{\text{Fliess models}\} \sim \{\text{Left state models}\} \sim \{\text{Right state models}\}.$$

7. Behavioral equivalence

We have seen that Fliess models and state models are equivalent from the purely mathematical point of view. The aim of this section is to show that they are equivalent from the behavioral point of view as well.

Let $\Sigma = (X, Y, A, B, C)$ be a left state model, and let $\Phi = (M, \mu)$ be the corresponding Fliess model. The state model determines the diagram

$$D[s]^q \stackrel{C}{\leftarrow} Y[s] \stackrel{sB-A}{\rightarrow} X[s].$$

Applying to this diagram the functor $Hom_D(-, \mathcal{U})$, we get

$$Hom_D(X, \mathcal{U}) \stackrel{sB^\vee - A^\vee}{\rightarrow} Hom_D(Y, \mathcal{U}) \stackrel{C^\vee}{\leftarrow} \mathcal{U}^q.$$

Associated with this there is an equation

$$B^\vee sx - A^\vee x + C^\vee w = 0.$$

The solution set of this equation, that is, the set

$$\{(x, w) \in Hom_D(X, \mathcal{U}) \oplus \mathcal{U}^q \mid B^\vee sx - A^\vee x + C^\vee w = 0\}$$

is called the full (or internal) behavior of Σ and is denoted by $\mathcal{B}_f(\Sigma)$. The canonical map

$$\mathcal{B}_f(\Sigma) \to \mathcal{U}^q,$$

induced by the projection $Hom_D(X, \mathcal{U}) \oplus \mathcal{U}^q \to \mathcal{U}^q$, is called the manifestation map. Its image is called the manifest (or external) behavior. (These definitions are based on Willems [35].) By the very definition, we have an exact sequence

$$0 \to \mathcal{B}_f(\Sigma) \to Hom_D(X, \mathcal{U}) \oplus \mathcal{U}^q \to Hom_D(Y, \mathcal{U}).$$

Apply now $Hom_{D[s]}(-, \mathcal{U})$ to the exact sequence

$$Y[s] \to X[s] \oplus D[s]^q \to M \to 0.$$

We then get an exact sequence

$$0 \to Hom_{D[s]}(M, \mathcal{U}) \to Hom_D(X, \mathcal{U}) \oplus \mathcal{U}^q \to Hom_D(Y, \mathcal{U}).$$

We can see that the diagram

$$\begin{array}{ccc} Hom_{D[s]}(M, \mathcal{U}) & \simeq & \mathcal{B}_f(\Sigma) \\ \downarrow & & \downarrow \\ \mathcal{U}^q & = & \mathcal{U}^q \end{array}$$

commutes, which means that Σ and Φ generate the same behavior.

Now, let $\Sigma = (X, Z, E, F, G)$ be a right state model, and let $\Phi = (M, \mu)$ be the corresponding Fliess model. The state model determines the diagram

$$X[s] \overset{sE-F}{\to} Z[s] \overset{G}{\leftarrow} D[s]^q.$$

Applying to this the functor $Hom_D(-, \mathcal{U})$, we get the diagram

$$\mathcal{U}^q \overset{G^\vee}{\leftarrow} Hom_D(Z, \mathcal{U}) \overset{sE^\vee - F^\vee}{\to} gfHom_D(X, \mathcal{U}).$$

Associated with this there is an equation

$$\begin{cases} sE^\vee z & = F^\vee z \\ w & = G^\vee z \end{cases}.$$

The solution set of $E^\vee sz = F^\vee z$ is called the full (or internal) behavior of Σ and is denoted by $\mathcal{B}_f(\Sigma)$. The canonical map

$$\mathcal{B}_f(\Sigma) \to \mathcal{U}^q,$$

induced by $G^\vee$, is called the manifestation map. Its image is called the manifest (or external) behavior.

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X[s] & \to & Z[s] & \to & M & \to & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & D[s]^q & = & D[s]^q & & \end{array},$$

in which the top row is exact. Applying to this diagram the functor $Hom_{D[s]}(-, \mathcal{U})$, we get the following commutative diagram

$$\begin{array}{ccccccc} 0 & \to & Hom_{D[s]}(M, \mathcal{U}) & \to & Hom_D(Z, \mathcal{U}) & \to & Hom_D(X, \mathcal{U}) \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{U}^q & = & \mathcal{U}^q & & \end{array}.$$

The top row in this diagram is exact, and so we have a commutative diagram

$$\begin{array}{ccc} Hom_{D[s]}(M, \mathcal{U}) & \simeq & \mathcal{B}_f(\Sigma) \\ \downarrow & & \downarrow \\ \mathcal{U}^q & = & \mathcal{U}^q \end{array}.$$

Thus, as above, Σ and Φ generate the same behavior.

We have proved the following

Theorem 5. *A state (left or right) model has the same behavior as the corresponding Fliess model.*

As we already know, if two state models form an exact couple, then their Fliess models are canonically isomorphic. As a consequence of Theorem 5 we have

Corollary 1. *Two state models are behaviorally equivalent if they form an exact couple.*

In the following examples we want to illustrate this corollary.

Example 6. Let (X, F, G, H, J) be as in Example 1. In view of Lemma 1, we may identify $Hom_D(X^*, \mathcal{U})$ with $X \otimes_D \mathcal{U}$. The behavioral equation of the associated left state model is

$$\begin{bmatrix} I \\ 0 \end{bmatrix} sx - \begin{bmatrix} F \\ H \end{bmatrix} x + \begin{bmatrix} -G & 0 \\ -J & I \end{bmatrix} \begin{pmatrix} u \\ y \end{pmatrix} = 0;$$

similarly, the behavioral equation of the associated right state model is

$$\begin{cases} [I \quad 0]\, s \begin{pmatrix} x \\ u \end{pmatrix} & = & [F \quad G] \begin{pmatrix} x \\ u \end{pmatrix} \\ \begin{pmatrix} u \\ y \end{pmatrix} & = & \begin{bmatrix} 0 & I \\ H & J \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \end{cases}.$$

Both these equations can be rewritten as

$$\begin{cases} sx = Fx + Gu \\ y = Hx + Ju \end{cases}.$$

The state models in the following two examples are observable. Remark that the manifestation maps of observable models are injective, and consequently their full behaviors may be identified with the manifest behaviors.

Example 7. (a) Consider the left state model of Example 2. Its behavioral equation is

$$s \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} \delta w \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Eliminating the state variables, we obtain the manifest behavior

$$\{w \in \mathcal{U} \mid \delta s^3 w = 0\}.$$

(b) Consider the right state model of Example 2. Since Z is generated by the symbols a_1, a_2, a_3, a satisfying the relation $\delta a = a_1$, we have

$$Hom_D(Z, \mathcal{U}) = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z \end{pmatrix} \in \mathcal{U}^4 \mid \delta z = z_1 \right\}.$$

The full behavior is equal to

$$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z \end{pmatrix} \in Hom_D(Z, \mathcal{U}) \mid sz_1 = z_2,\ sz_2 = z_3,\ sz_3 = 0 \right\}.$$

This, in turn, is equal to

$$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z \end{pmatrix} \in \mathcal{U}^4 \mid \delta z = z_1,\ sz_1 = z_2,\ sz_2 = z_3,\ sz_3 = 0 \right\}.$$

The manifest behavior is exactly the same as in (a).

Example 8. (a) Consider the left state model of Example 3. Its behavioral equation is

$$\begin{bmatrix} -I \\ 0 \\ 0 \end{bmatrix} sx + \begin{bmatrix} 0 \\ \delta_2 \\ -\delta_1 \end{bmatrix} x + \begin{pmatrix} u \\ y_1 \\ y_2 \end{pmatrix} = 0.$$

Eliminating x, we obtain the manifest behavior

$$\left\{ \begin{pmatrix} u \\ y_1 \\ y_2 \end{pmatrix} \in \mathcal{U}^3 \mid \delta_1 u = sy_2,\ \ \delta_1 y_1 + \delta_2 y_2 = 0 \right\}.$$

(b) Consider the right state model of Example 3. Since Z is generated by the symbols a, b, c_0, c_1 satisfying the relations $\delta_1 a = c_1$ and $\delta_1 b + \delta_2 c_0 = 0$, we have

$$Hom_D(Z, \mathcal{U}) = \left\{ \begin{pmatrix} x \\ y \\ z_0 \\ z_1 \end{pmatrix} \in \mathcal{U}^4 \mid \delta_1 x = z_1,\ \ \delta_1 y + \delta_2 z_0 = 0 \right\}.$$

The full behavior is equal to

$$\left\{ \begin{pmatrix} x \\ y \\ z_0 \\ z_1 \end{pmatrix} \in Hom_D(Z, \mathcal{U}) \mid sz_0 = z_1 \right\} = \left\{ \begin{pmatrix} x \\ y \\ z_0 \\ z_1 \end{pmatrix} \in \mathcal{U}^4 \mid \delta_1 x = z_1,\ \delta_1 y + \delta_2 z_0 = 0,\ sz_0 = z_1 \right\}.$$

Again, the manifest behavior is exactly the same as in (a).

8. State models of classical type

Our definition of the behavior of a state model uses the functor $Hom_D(-, \mathcal{U})$ and is motivated by Theorem 5. There is an alternative way for defining the behavior of a state model, which is based on applying the functor $- \otimes_D \mathcal{U}$: The behavior of a left state model (X, Y, A, B, C) can be defined via the equation

$$\begin{cases} sBy = \ Ay \\ w \ \ = \ Cy \end{cases},$$

where $y \in Y \otimes_D \mathcal{U}$ and $w \in \mathcal{U}^q$; the behavior of a right state model (X, Z, E, F, G) can be defined via the equation

$$sEx - Fx + Gw = 0,$$

where $x \in X \otimes_D \mathcal{U}$ and $w \in \mathcal{U}^q$.

Given a state model Σ, let us denote by $BE^{(I)}(\Sigma)$ the behavioral equation of Σ as defined in the previous section and by $BE^{(II)}(\Sigma)$ the one as defined above.

In general, there is no relation between the two methods in defining the behavior of a state model. (It is worth noting that the first method is “contravariant" and the second one is “covariant".) We shall now introduce a class of state models for which these methods are closely related; one is expressed by the other.

Say that a left state model (X, Y, A, B, C) is of classical type if the modules X, Y are projective and the homomorphism $\begin{bmatrix} B \\ C \end{bmatrix}$ is left invertible. Likewise, say that a right state model (X, Z, E, F, G) is of classical type if the modules X, Z are projective and the homomorphism E is left invertible.

Notice that the state models associated with a Kalman model (see Example 1) are of classical type.

It is clear that if $\Sigma = (X, Y, A, B, C)$ is a left state model of classical type, then its dual

$$\Sigma^* = (X^*, Y^*, B^t, A^t, C^t)$$

is a right state model of classical type. Conversely, if $\Sigma = (X, Z, E, F, G)$ is a right state model of classical type, then its dual

$$\Sigma^* = (X^*, Z^*, F^t, E^t, G^t)$$

is a left state model of classical type. For every state model Σ of classical type, we clearly have

$$(\Sigma^*)^* = \Sigma.$$

Using Lemma 1, the reader can easily see that, for every state model Σ of classical type,

$$BE^{(II)}(\Sigma) = BE^{(I)}(\Sigma^*) \text{ and } BE^{(I)}(\Sigma) = BE^{(II)}(\Sigma^*).$$

Remark. In case when D is a field $\mathbb{F}$, all state models are of classical type, of course. Therefore, in view of the relations above, the two methods to define the behaviors are equivalent. This explains the absence of the first method in the linear systems theory over a field. One prefers to use the second method (the "covariant" one), which is simpler and more direct.

9. State models corresponding to AR-models

As already remarked, AR-models are of particular interest. In this section we want to characterize state models that correspond to them.

A left state model (X, Y, A, B, C) is called observable if

$$sB - A : Y[s] \to X[s]$$

is surjective. A right state model (X, Z, E, F, G) is called observable if

$$\begin{bmatrix} sE - F & G \end{bmatrix} : X[s] \oplus D[s]^q \to Z[s]$$

is surjective.

The following justifies the above definitions.

Proposition 4. *Let (M, μ) be a Fliess model, and let (X, Y, A, B, C) and (X, Z, E, F, G) be the corresponding left and right state models. The following conditions are equivalent*:

(*a*) *(M, μ) is observable*;
(*b*) *(X, Y, A, B, C) is observable*;
(*c*) *(X, Z, E, F, G) is observable*.

Proof. (a) $\Leftrightarrow$ (b) Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & Y[s] & \to & X[s] \oplus D[s]^q & \to & M & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & X[s] & = & X[s] & \to & 0 & \end{array}.$$

Applying the snake lemma, we get that the sequence

$$D[s]^q \to M \to Coker(Y[s] \to X[s]) \to 0$$

is exact; whence the assertion.

(a) $\Leftrightarrow$ (c) Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & X[s] & \to & X[s] \oplus D[s]^q & \to & D[s]^q & \to 0 \\ & || & & \downarrow & & \downarrow & \\ 0 \to & X[s] & \to & Z[s] & \to & M & \to 0 \end{array}.$$

Applying the snake lemma, we get that the sequence

$$0 \to Coker(X[s] \oplus D[s]^q \to Z[s]) \to Coker(D[s]^q \to M) \to 0$$

is exact; whence the assertion. $\square$

An immediate consequence of the proposition is that state models corresponding to AR-models are observable ones.

Now, we shall give some characterization of observable state models in terms of linear algebra (over D).

Proposition 5. *A left state model (X, Y, A, B, C) is observable if and only if, for all sufficiently large k, the homomorphism*

$$\begin{bmatrix} -A & & & \\ B & & & \\ & \ddots & \ddots & \\ & & & -A \\ & & & B \end{bmatrix} : Y^k \to X^{k+1}$$

is surjective.

Proof. From the proof of Theorem 2, we know that the homomorphism $B - s^{-1}A : Y \otimes O \to X \otimes O$ is always surjective. Therefore, to say that $sB - A : Y[s] \to X[s]$ is surjective is the same as to say that the sheaf homomorphism

$$\mathcal{O} \otimes Y \to \mathcal{O}(1) \otimes X \tag{4}$$

is surjective. Notice that our linear map can be identified with

$$H^0\mathcal{O}(k) \otimes Y \to H^0\mathcal{O}(k+1) \otimes X. \tag{5}$$

Assume that (4) is surjective. Letting $\mathcal{E}$ denote the kernel, we then have an exact sequence

$$0 \to \mathcal{E} \to \mathcal{O} \otimes Y \to \mathcal{O}(1) \otimes X \to 0.$$

"Twisting" this sequence by k and passing to cohomology, we get the exact sequence

$$H^0\mathcal{O}(k) \otimes Y \to H^0\mathcal{O}(k+1) \otimes X \to H^1\mathcal{E}(k).$$

For all sufficiently large k, $H^1\mathcal{E}(k) = 0$ (see Theorem 7 in Appendix A). Hence, (5) is surjective for all $k \gg 0$.

Conversely, assume that (5) is surjective for all $k \gg 0$. Letting $\mathcal{E}$ denote the image of (4) and $\mathcal{F}$ the cokernel, we have an exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}(1) \otimes X \to \mathcal{F} \to 0.$$

Using the same argument as above, we find that

$$0 \to H^0\mathcal{E}(k) \to H^0\mathcal{O}(k+1) \otimes X \to H^0\mathcal{F}(k) \to 0$$

is an exact sequence for all $k \gg 0$. Noticing that (5) is the composition

$$H^0\mathcal{O}(k) \otimes Y \to H^0\mathcal{E}(k) \to H^0\mathcal{O}(k+1) \otimes X,$$

we find that $H^0\mathcal{E}(k) \to H^0\mathcal{O}(k+1) \otimes X$ is surjective for all $k \gg 0$. In view of the previous cohomological exact sequence, this implies that $H^0\mathcal{F}(k) = 0$ for all sufficiently large k. By Theorem 7 (see Appendix A), $\mathcal{F} = 0$.

The proposition is proved. □

Proposition 6. *A right state model (X, Z, E, F, G) is observable if and only if, for all sufficiently large k, the homomorphism*

$$\begin{bmatrix} -F & & & G & & \\ E & & & & & \\ & \ddots & \ddots & & \ddots & \\ & & -F & & & \\ & & E & & & G \end{bmatrix} : X^{k-1} \oplus D^{qk} \to Z^k$$

is surjective.

Proof. From the proof of Theorem 3, we know that the homomorphism $[E - s^{-1}F\ G] : X \otimes O \oplus O^q \to Z \otimes O$ is always surjective. Therefore, to say that $[sE - F\ G] : X[s] \oplus D[s]^q \to Z[s]$ is surjective is the same as to say that

$$\mathcal{O}(-1) \otimes X \oplus \mathcal{O}^q \to \mathcal{O} \otimes Z$$

is surjective. Notice that our linear map can be identified with

$$H^0\mathcal{O}(k-1) \otimes X \oplus D^{qk} \to H^0\mathcal{O}(k+1) \otimes X.$$

We can repeat the arguments in the proof of the previous proposition to complete the proof. $\square$

10. Two examples

In this section we want to consider two very simple examples of LDDEs, just to illustrate the construction in Section 4. We begin by the following evident lemma, which somewhat facilitates the computations of the "X" and the "Z".

Let $R \in D[s]^{p\times q}$, and let (M, μ) be the corresponding Fliess model. By the very definition, we then have an exact sequence

$$D[s]^p \xrightarrow{R^t} D[s]^q \xrightarrow{\mu} M \to 0.$$

Let $D((s^{-1}))$ be the ring of formal Laurent series, and define $\overline{V}$ by the exact sequence

$$D((s^{-1}))^p \xrightarrow{R^t} D((s^{-1}))^q \to \overline{V} \to 0.$$

Next, define $\bar{i}$ to be the canonical map $M \to \overline{V}$ and $\overline{N}$ the image of $D[[s^{-1}]]^q$ under the canonical map $D((s^{-1}))^q \to \overline{V}$.

Lemma 5. *We can compute the modules X and Z by the following formulas*

$$X = \{x \in M|\ \bar{i}(x) \in s^{-1}\overline{N}\} \text{ and } Z = \{z \in M|\ \bar{i}(z) \in \overline{N}\}.$$

The point of the lemma is that it is easier to compute $\overline{V}$ and $\overline{N}$ than the "V" and the "N" in Section 4.

Example 9. Let $D = \mathbb{R}[\delta]$, and consider the LDDE determined by δs^3. The module M, as a linear space over $\mathbb{R}$, is equal to

$$\delta D \oplus \delta Ds \oplus \delta Ds^2 \oplus \mathbb{R}[s].$$

Put

$$a_1 = \delta,\ \ a_2 = \delta s,\ \ a_3 = \delta s^2,\ \ a = 1.$$

These are generators of M with the following defining relations

$$\delta a = a_1,\ \ sa_1 = a_2,\ \ sa_2 = a_3,\ \ sa_3 = 0.$$

One easily finds

$$\overline{V} = \mathbb{R}((s^{-1})) \text{ and } \overline{N} = \mathbb{R}[[s^{-1}]].$$

We have

$$X = \delta D + \delta Ds + \delta Ds^2 \simeq D^3 \text{ and } Z = \delta D + \delta Ds + \delta Ds^2 + \mathbb{R} \simeq D^3 \oplus \mathbb{R}.$$

The maps $E, F : D^3 \to D^3 \oplus \mathbb{R}$ are given respectively by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ 0 \end{pmatrix};$$

the map $G : D \to Z$ is given by

$$x \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ x \bmod \delta \end{pmatrix}.$$

The right state model that we obtain coincides with the one given in Example 2b).

Example 10. Let $D = \mathbb{R}[\delta_1, \delta_2]$, and consider the LDDE determined by

$$R = \begin{bmatrix} \delta_1 & 0 & -s \\ 0 & \delta_1 & \delta_2 \end{bmatrix}.$$

(The matrix is taken from Gluesing-Luerssen [12] (see Example 5.2.5).)

The submodule $R^t D[s]^2 \subseteq D[s]^3$ consists of elements of the form

$$\begin{pmatrix} \delta_1 f \\ \delta_1 g \\ g\delta_2 - sf \end{pmatrix},$$

where $f, g \in D[s]$. It is clear that every element in $M = D[s]^3 / R^t D[s]^2$ has a representative of the form

$$\begin{pmatrix} h_1 \\ h_2 \\ h \end{pmatrix},$$

where $h_1, h_2 \in \mathbb{R}[\delta_2][s]$ and $h \in D[s]$. It is clear also that two such columns are congruent modulo $R^t D[s]^2$ if and only if they are equal. Thus, the module M, as a linear space over $\mathbb{R}$, is equal to

$$\mathbb{R}[\delta_2][s] \oplus \mathbb{R}[\delta_2][s] \oplus D[s].$$

Let e_1, e_2, e_3 be the canonical basis of D^3, and let a, b, c denote their classes in M. Certainly, these are generators of M. We have

$$\delta_1 e_1 \equiv s e_3 \bmod R^t D[s]^2 \quad \text{and} \quad \delta_1 e_2 + \delta_2 c \equiv 0 \bmod R^t D[s]^2.$$

Hence, $a,\ b,\ c$ satisfy the following relations

$$\delta_1 a = sc \quad \text{and} \quad \delta_1 + \delta_2 c = 0.$$

And these are defining relations.

Likewise, one can easily finds that

$$\overline{V} = \mathbb{R}[\delta_2]((s^{-1})) \oplus \mathbb{R}[\delta_2]((s^{-1})) \oplus D((s^{-1}))$$

and the canonical homomorphism $D((s^{-1}))^3 \to \overline{V}$ is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x(0, \delta_2, s^{-1}) \\ y(0, \delta_2, s^{-1}) \\ z - \delta_2 g + sf \end{pmatrix},$$

where $f = \delta_1^{-1}(x - x(0, \delta_2, s^{-1}))$ and $g = \delta_1^{-1}(y - y(0, \delta_2, s^{-1}))$. (Notice the entrance of the "s"!) Further,

$$\overline{N} = \mathbb{R}[\delta_2][[s^{-1}]] \oplus \mathbb{R}[\delta_2][[s^{-1}]] \oplus sD[[s^{-1}]].$$

We get

$$X = D \quad \text{and} \quad Z = \mathbb{R}[\delta_2] \oplus \mathbb{R}[\delta_2] \oplus D \oplus sD.$$

Let a and b be as above, and put $c_0 = c$ and $c_1 = sc$. These elements generate the module Z over D, and satisfy the following relations

$$\delta_1 a = c_1 \quad \text{and} \quad \delta_1 b + \delta_2 c_0 = 0.$$

(There are no other relations.) Further, one easily calculates $E,\ F,\ G$ and obtains precisely the right state model that we gave in Example 3b).

Acknowledgment

The authors thank the referee whose remarks and suggestions have led to a more readable and improved version of the paper.

Appendix A. "Coherent sheaves" over $(D(s), D[s], O)$

We regard the triple $(D(s), D[s], O)$ as an algebraic analog of the projective line over D. (Intuitively, the projective line consists of the "finite" domain and the "infinite" domain intersecting along the "generic" domain. And one may think of rational functions as functions on the "generic" domain, of polynomials as functions on the "finite" domain and of proper rational functions as functions on the "infinite" domain.)

A sheaf (over $(D(s), D[s], O)$) is a quintuple (V, M, N, i, j), where V is a module over $D(s)$, M and N are respectively modules over $D[s]$ and O, and $i : M \to V$ and $j : N \to V$ are respectively $D[s]$- and O-homomorphisms such that the $D(s)$-homomorphisms

$$M \otimes_{D[s]} D(s) \to V \quad \text{and} \quad N \otimes_O D(s) \to V$$

are isomorphisms.

Remark. Traditionally, one obtains the projective line $\mathbb{P}^1_D$ by "gluing" in a certain way two copies of the affine line $\mathbb{A}^1_D$. One can find a formal definition in Lam [19], for example. A standard definition of a sheaf over $\mathbb{P}^1_D$ would be: A sheaf is a quintuple (V, M, N, i, j), where V is a module over $D[s, s^{-1}]$, M and N are respectively modules over $D[s]$ and $D[s^{-1}]$, and $i : M \to V$ and $j : N \to V$ are respectively $D[s]$- and $D[s^{-1}]$-homomorphisms such that the $D[s, s^{-1}]$-homomorphisms

$$M \otimes_{D[s]} D[s, s^{-1}] \to V \quad \text{and} \quad N \otimes_{D[s^{-1}]} D[s, s^{-1}] \to V$$

are isomorphisms.

In our opinion, the definition that we offer is more appropriate for purposes of linear systems theory.

A sheaf is said to be coherent if its modules are finitely generated.

Example 11. $\mathcal{O} = (D(s), D[s], O, i, j)$, where i, j are the canonical inclusion maps, is a coherent sheaf.

Let $\mathcal{F} = (V, M, N, i, j)$ be a sheaf. Elements in V are called generic sections, elements in M finite sections and elements in N infinite sections. A global section is a pair (x, y), where $x \in M$ and $y \in N$ are such that $i(x) = j(y)$. The set of global sections is denoted by $\Gamma(\mathcal{F})$. This is a module over D.

Given a sheaf $\mathcal{F}$ and a D-module X, one defines in an obvious way $\mathcal{F} \otimes X$.

If $\mathcal{F} = (V, M, N, i, j)$ is a sheaf and k an integer, one defines a new sheaf

$$\mathcal{F}(k) = (V, M, N, i, s^k j).$$

(By s^k, we mean here the automorphism $V \to V$ given by multiplication by s^k.) It should be pointed out that this extremely simple operation is very much important.

Let $\mathcal{F} = (V, M, N, i, j)$ be a sheaf. We define the cohomology modules $H^0\mathcal{F}$ and $H^1\mathcal{F}$ respectively as the kernel and cokernel of the D-linear map

$$L \oplus M \to V, \quad (x, y) \mapsto i(x) - j(y).$$

It should be emphasized that cohomology modules are modules over D. Notice that

$$H^0(\mathcal{F}) = \Gamma(\mathcal{F}).$$

Example 12. For $n \geqslant 0$, we have

$$H^0\mathcal{O}(n) \simeq D^{n+1} \text{ and } H^1\mathcal{O}(-n-2) \simeq D^{n+1}.$$

For $n < 0$, we have

$$H^0\mathcal{O}(n) = 0 \text{ and } H^1\mathcal{O}(-n-2) = 0.$$

Let $(V_1, M_1, N_1, i_1, j_1)$ and $(V_2, M_2, N_2, i_2, j_2)$ be sheaves. A homomorphism between them is a triple (ϕ, f, g) consisting of homomorphisms $\phi : V_1 \to V_2, f : M_1 \to M_2$ and $g : N_1 \to N_2$ such that the following diagrams

$$\begin{array}{ccc} M_1 \otimes D(s) & \overset{f\otimes D(s)}{\to} & M_2 \otimes D(s) \\ \downarrow & & \downarrow \\ V_1 & \overset{\phi}{\to} & V_2 \end{array} \quad \text{and} \quad \begin{array}{ccc} N_1 \otimes D(s) & \overset{g\otimes D(s)}{\to} & N_2 \otimes D(s) \\ \downarrow & & \downarrow \\ V_1 & \overset{\phi}{\to} & V_2 \end{array}$$

commute.

It is worth noting that

$$Hom(\mathcal{O}, \mathcal{F}) = \Gamma(\mathcal{F}).$$

One defines in an obvious way the kernels, images and cokernels of homomorphisms of sheaves. Consequently, we have the notion of exact sequences of sheaves.

Proposition 7. *A short exact sequence of sheaves*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

yields a long exact sequence of cohomologies

$$0 \to H^0\mathcal{F}_1 \to H^0\mathcal{F} \to H^0\mathcal{F}_2 \to H^1\mathcal{F}_1 \to H^1\mathcal{F} \to H^1\mathcal{F}_2 \to 0.$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & M_1 \oplus N_1 & \to & M \oplus N & \to & M_2 \oplus N_2 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & V_1 & \to & V & \to & V_2 & \to 0 \end{array}$$

The rows are exact, and so we can apply the snake lemma. □

Lemma 6. *Let $\mathcal{F} = (V, M, N, i, j)$ be a sheaf.*

(a) *Given $x \in M$, there exists an integer $k_0 \geqslant 0$ such that, for every $k \geqslant k_0$, x extends to a global section of $\mathcal{F}(k)$.*
(b) *Given $y \in N$, there exist an integer $k_0 \geqslant 0$ and an invertible element $g \in O$ such that, for every $k \geqslant k_0$, gy extends to a global section of $\mathcal{F}(k)$.*

Proof. (a) We have

$$i(x) = \frac{j(y)}{s^{-k_0}}$$

for some $k_0 \geqslant 0$ and $y \in N$. If $k \geqslant k_0$, then

$$i(x) = s^k j(s^{k_0-k}y).$$

This means that $(x, s^{k_0-k}y)$ is a global section of $\mathcal{F}(k)$.

(b) We have

$$\frac{i(x)}{f} = j(y)$$

for some monic polynomial f and $y \in N$. Let k_0 denote the degree of f. Then $f = s^{k_0}g$, where g is an invertible element of O. For every $k \geqslant k_0$, we then have

$$i(s^{k-k_0}x) = s^k j(gy).$$

This means that $(s^{k-k_0}x, gy)$ is a global section of $\mathcal{F}(k)$. $\square$

Theorem 6. (Serre) *Let $\mathcal{F}$ be a coherent sheaf. There exists $k_0 \geqslant 0$ such that for every $k \geqslant k_0$, $\mathcal{F}(k)$ is generated by global sections.*

Proof. Let $\mathcal{F} = (V, M, N, i, j)$. Select generator sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ of the modules M and N. Using the previous lemma, one can easily find an integer $k_0 \geqslant 0$ and invertible elements $g_1, \ldots, g_n$ such that for every $k \geqslant k_0$ all elements $x_1, \ldots, x_m$ and $g_1y_1, \ldots, g_ny_n$ extend to global sections of $\mathcal{F}(k)$. All these global sections certainly generate the sheaf $\mathcal{F}(k)$. $\square$

The following is an immediate consequence of the theorem.

Lemma 7. *Given a coherent sheaf $\mathcal{F}$, there is an epimorphism*

$$\mathcal{O}^r \to \mathcal{F}(k_0).$$

In the following "lim" stands for "the direct limit". (The reader is referred to Atiyah and Macdonald [1] for the notion of direct limit.) The following says, in particular, that knowledge of $H^0\mathcal{F}(k)$ for all sufficiently large k implies knowledge of $\mathcal{F}$.

Lemma 8. *Let $\mathcal{F} = (V, M, N, i, j)$ be a sheaf. Then,*

$$M = \lim_k H^0\mathcal{F}(k) \quad \text{and} \quad N = \lim_{g \text{ monic}} g^{-1}H^0\mathcal{F}(deg(g)).$$

Proof. The assertion is true for $\mathcal{F} = \mathcal{O}$. Indeed, it is easily seen that

$$D[s] = \cup_k H^0\mathcal{O}(k) \quad \text{and} \quad O = \cup_{g \text{ monic}} g^{-1}H^0\mathcal{O}(deg(g)).$$

To prove the general case consider an epimorphism $\mathcal{O}^r \to \mathcal{F}(k_0)$, which exists for some $r \geqslant 0$ and some $k_0 \geqslant 0$. We have epimorphisms of modules

$$D[s]^r \to M \quad \text{and} \quad O^r \to s^{k_0}N.$$

From these and from the commutative diagrams

$$\begin{array}{ccc} \lim H^0\mathcal{O}^r(k) & \to & \lim H^0\mathcal{F}(k_0+k) \\ \downarrow & & \downarrow \\ D[s]^r & \to & M \end{array} \quad \text{and} \quad \begin{array}{ccc} \lim \frac{1}{g}H^0\mathcal{O}^r(deg(g)) & \to & \lim \frac{1}{g}H^0\mathcal{F}(k_0+deg(g)) \\ \downarrow & & \downarrow \\ O^r & \to & s^{k_0}N \end{array}$$

the assertion follows. (Here k runs over all nonnegative integers, and g over all monic polynomials.) $\square$

Theorem 7. *Let $\mathcal{F}$ be a coherent sheaf.*
(*a*) *If $H^0\mathcal{F}(k) = 0$ for all sufficiently large k, then $\mathcal{F} = 0$.*
(*b*) *For all sufficiently large k, $H^1\mathcal{F}(k) = 0$.*

Proof. (a) Follows immediately from the previous lemma.

(b) Consider an epimorphism $\mathcal{O}^r(-k_0) \to \mathcal{F}$, which exists by Lemma 7. For every k, we have an epimorphism

$$H^1\mathcal{O}^r(k-k_0) \to H^1\mathcal{F}(k).$$

Because $H^1\mathcal{O}(k-k_0) = 0$ for all sufficiently large k (see Example 11), this completes the proof. $\square$

The following theorem, which also is due to Serre, expresses the most fundamental fact about coherent sheaves.

Theorem 8 (Finiteness Theorem)**.** *The cohomology modules of a coherent sheaf are finitely generated*.

Proof. Let $\mathcal{F}$ be a coherent sheaf. In view of Lemma 7, there is an exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}^r(-k_0) \to \mathcal{F} \to 0.$$

Passing to cohomology, we get an exact sequence

$$0 \to H^0\mathcal{E} \to H^0\mathcal{O}^r(-k_0) \to H^0\mathcal{F} \to H^1\mathcal{E} \to H^1\mathcal{O}^r(-k_0) \to H^1\mathcal{F} \to 0.$$

From this and from Example 11 it immediately follows that $H^1\mathcal{F}$ is finitely generated.

The sheaf $\mathcal{E}$ clearly is coherent, and applying the same argument to it, we find that $H^1\mathcal{E}$ also is finitely generated. In view of this and Example 11, our exact sequence yields that $H^0\mathcal{F}$ is finitely generated.

The proof is complete. □

For the general theory of cohomologies of coherent sheaves the interested reader is referred to Hartshorne [15].

Appendix B. Fuhrmann's realization over *D*

In this appendix, which is self-contained, we revisit the main result of Gluesing-Luerssen [12].

Let m and p be integers such that $m + p = q$, and assume we have

$$P \in D[s]^{p\times p},\ Q \in D[s]^{p\times m} \text{ with } \det(P) \text{ being monic and } P^{-1}Q \text{ being proper}.$$

(This is precisely the starting point for the development in Gluesing-Luerssen [12].) Set

$$X = D[s]^p \cap s^{-1}PO^p.$$

Call elements of X states of the equation $Py = Qu$.

Lemma 9. *The module X is finitely generated and projective*.

Proof. For sufficiently large k, we have $PO^p \subseteq s^{k+1}O^p$. Consequently, X is a submodule of $D[s]^p \cap s^kO^p$, which is finitely generated of course.

Notice that X can be viewed as the kernel of the homomorphism

$$D[s]^p \oplus s^{-1}PO^p \to D(s)^p$$

given by $\begin{pmatrix} f \\ g \end{pmatrix} \mapsto g - f$. This homomorphism is surjective because

$$D(s)^p = PD(s)^p = P(D[s]^p + s^{-1}O^p) \subseteq D[s]^p + s^{-1}PO^p.$$

Further, the modules $D[s]^p \oplus s^{-1}PO^p$ and $D(s)^p$ are flat over D. ($D[s]$ is flat because it is free; $D(s)$ is flat over $D[s]$ (as a localization), and therefore also is flat; O can be viewed as a localization of $D[s^{-1}]$ (with respect to $1 + s^{-1}D[s^{-1}]$), and consequently is flat.) Hence, in view of the exact sequence

$$0 \to X \to D[s]^p \oplus s^{-1}PO^p \to D(s)^p \to 0,$$

X must be flat. (See Proposition 5 in Bourbaki [2], Ch. I, Sect. 2.) This proves the lemma because, for finitely generated modules, projectiveness and flatness are equivalent properties. □

Let

$$\pi : PO^p \to D^p$$

denote the canonical map taking $z \in PO^p$ to the free coefficient of $P^{-1}z$ in the expansion at infinity. This clearly is a D-linear map. It is worth noting that this map vanishes on $s^{-1}PO^p$.

Define D-linear maps

$$F : X \to X,\ \ G : D^m \to X,\ \ H : X \to D^p,\ \ J : D^m \to D^p$$

by the following formulas

$$\begin{aligned} Fx &= sx - P\pi(sx) \\ Gu &= Qu - P\pi(Qu) \\ Hx &= \pi(sx) \\ Ju &= \pi(Qu). \end{aligned}$$

Thanks of the inclusion $X \subseteq D[s]^p$, there is a canonical $D[s]$-homomorphism $\chi : X[s] \to D[s]^p$ given by the formula

$$\chi(x_0 \otimes 1 + x_1 \otimes s + \cdots + x_l \otimes s^l) = x_0 + sx_1 + \cdots + s^l x_l.$$

If x is a state, then

$$\begin{bmatrix} \chi & P \end{bmatrix} \begin{bmatrix} sI - F \\ H \end{bmatrix} x = \begin{bmatrix} \chi & P \end{bmatrix} \begin{pmatrix} P\pi(sx) \\ -\pi(sx) \end{pmatrix} = P\pi(sx) - P\pi(sx) = 0.$$

Because $X[s]$ is generated by state elements, it follows from this that the composition

$$\begin{bmatrix} \chi & P \end{bmatrix} \begin{bmatrix} sI - F \\ H \end{bmatrix}$$

is zero.

Proposition 8. (*a*) *The complex*

$$0 \to X[s] \xrightarrow{\begin{bmatrix} sI - F \\ H \end{bmatrix}} X[s] \oplus D[s]^p \xrightarrow{\begin{bmatrix} \chi & P \end{bmatrix}} D[s]^p \to 0$$

is exact.

(*b*) *The square*

$$\begin{array}{ccc} D[s]^m \oplus D[s]^p & = & D[s]^m \oplus D[s]^p \\ \begin{bmatrix} G & 0 \\ -J & I \end{bmatrix} \downarrow & & \downarrow \begin{bmatrix} -Q & P \end{bmatrix} \\ X[s] \oplus D[s]^p & \to & D[s]^p \end{array}$$

is commutative.

Proof. (a) Exactnessat $X[s]$: This is obvious, because $E : X \to Z$ is injective.

Exactnessat $X[s] \oplus D[s]^p$: Suppose that an element

$$\begin{pmatrix} x_0 \otimes 1 + x_1 \otimes s + \cdots + x_l \otimes s^l \\ a_0 \otimes 1 + a_1 \otimes s + \cdots + a_l \otimes s^l \end{pmatrix} \in X[s] \oplus D[s]^p$$

goes to zero. We then have the following relation

$$(x_0 + Pa_0) + (x_1 + Pa_1)s + \cdots + (x_l + Pa_l)s^l = 0.$$

Set

$$\begin{aligned} \bar{x}_0 &= -s^{-1}(x_0 + Pa_0), \\ \bar{x}_1 &= -s^{-2}(x_0 + Pa_0) - s^{-1}(x_0 + Pa_0), \\ &\vdots \\ \bar{x}_{l-1} &= -s^{-l}(x_0 + Pa_0) - \cdots - s^{-1}(x_{l-1} + Pa_{l-1}). \end{aligned}$$

Using the relation above, one can easily see that all these elements are states. We have

$$\begin{array}{llll} & -s\bar{x}_0 & = x_0 & +Pa_0 \\ \bar{x}_0 & -s\bar{x}_1 & = x_1 & +Pa_1 \\ & & \vdots & \\ \bar{x}_{l-2} & -s\bar{x}_{l-1} & = x_{l-1} & +Pa_{l-1} \\ \bar{x}_{l-1} & & = x_l & +Pa_l. \end{array}$$

Remark that if $x \in X$ and $a \in D^p$, then

$$\begin{bmatrix} I - P\pi \\ \pi \end{bmatrix} x = \begin{bmatrix} I \\ 0 \end{bmatrix} x, \quad \begin{bmatrix} I - P\pi \\ \pi \end{bmatrix} sx = \begin{bmatrix} F \\ -H \end{bmatrix} x \text{ and } \begin{bmatrix} I - P\pi \\ \pi \end{bmatrix} (x + Pa) = \begin{pmatrix} x \\ a \end{pmatrix}$$

Applying therefore $\begin{bmatrix} I - P\pi \\ \pi \end{bmatrix}$ to the equalities above, we get

$$\begin{array}{lll} & -\begin{bmatrix} F \\ -H \end{bmatrix}\bar{x}_0 & = \begin{pmatrix} x_0 \\ a_0 \end{pmatrix} \\ \begin{bmatrix} I \\ 0 \end{bmatrix}\bar{x}_0 & -\begin{bmatrix} F \\ -H \end{bmatrix}\bar{x}_1 & = \begin{pmatrix} x_1 \\ a_1 \end{pmatrix} \\ & & \vdots \\ \begin{bmatrix} I \\ 0 \end{bmatrix}\bar{x}_{l-2} & -\begin{bmatrix} F \\ -H \end{bmatrix}\bar{x}_{l-1} & = \begin{pmatrix} x_{l-1} \\ a_{l-1} \end{pmatrix} \\ \begin{bmatrix} I \\ 0 \end{bmatrix}\bar{x}_{l-1} & & = \begin{pmatrix} x_l \\ a_l \end{pmatrix}. \end{array}$$

We can see that our element is equal to

$$\begin{bmatrix} sI - F \\ H \end{bmatrix} (\bar{x}_0 \otimes 1 + \bar{x}_1 \otimes s + \cdots + \bar{x}_{l-1} \otimes s^{l-1}).$$

Exactness at $D[s]^p$: Take any $h \in D[s]^p$. Because $P : D(s)^p \to D(s)^p$ is bijective (and because $D(s) = D[s] + s^{-1}O$, we have

$$h = P(f + s^{-1}g)$$

for some $f \in D[s]^p$ and $g \in O^p$. Put

$$x = s^{-1}Pg = h - Pf.$$

This belongs both to $s^{-1}O^p$ and $D[s]^p$; in other words, this is a state. We see that h is the image of $\begin{pmatrix} x \\ f \end{pmatrix}$.

(b) Take arbitrary $u \in D^m$ and $y \in D^p$. We have

$$\begin{aligned} \begin{bmatrix} \chi & P \end{bmatrix} \begin{bmatrix} G & 0 \\ -J & I \end{bmatrix} \begin{pmatrix} u \\ y \end{pmatrix} &= Gu + Py - PDu = -Qu + P\pi(Qu) + Py - P\pi(Qu) \\ &= -Qu + Py = \begin{bmatrix} -Q & P \end{bmatrix} \begin{pmatrix} u \\ y \end{pmatrix}. \end{aligned}$$

The commutativity follows.

The proposition is proved. □

Let $\mathcal{B}$ denote the solution set of the equation $Py = Qu$.

Theorem 9. *The Kalman model* (X, F, G, H, J) *is a realization of* $Py = Qu$. *In other words, one has*

$$\mathcal{B} = \left\{ \begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{U}^m \oplus \mathcal{U}^p \mid \exists\, x \in X \otimes \mathcal{U} : \ sx = Fx + Gu, \ y = Hx + Ju \right\}.$$

Proof. Tensoring the diagram

$$\begin{array}{ccccccccc} & & & & D[s]^m \oplus D[s]^p & = & D[s]^m \oplus D[s]^p & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & X[s] & \to & X[s] \oplus D[s]^p & \to & D[s]^p & \to & 0 \end{array}$$

(in which the bottom row is exact and the square is commutative) by $\mathcal{U}$, we obtain a diagram

$$\begin{array}{ccccccccc} & & & & \mathcal{U}^m \oplus \mathcal{U}^p & = & \mathcal{U}^m \oplus \mathcal{U}^p & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & X \otimes \mathcal{U} & \to & X \otimes \mathcal{U} \oplus \mathcal{U}^p & \to & \mathcal{U}^p & \to & 0 \end{array}$$

Here also the bottom row is exact and the square is commutative.

This diagram proves the theorem. Indeed, suppose that $\begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{B}$. Due to commutativity of the square, $\begin{pmatrix} Gu \\ y - Ju \end{pmatrix}$ is contained in the kernel of $X \otimes \mathcal{U} \oplus \mathcal{U}^p \to \mathcal{U}^p$. Since the sequence is exact, there exists a (unique) state trajectory $x \in X \otimes \mathcal{U}$ such

$$\begin{pmatrix} Gu \\ y - Ju \end{pmatrix} = \begin{bmatrix} sI - F \\ H \end{bmatrix} x.$$

The implication "$\subseteq$" follows.

Suppose now that $\begin{pmatrix} u \\ y \end{pmatrix}$ is such that $sx = Fx + Gu$ and $y = Hx + Ju$ for some $x \in X \otimes \mathcal{U}$. Then

$$\begin{pmatrix} Gu \\ y - Ju \end{pmatrix} = \begin{bmatrix} sI - F \\ H \end{bmatrix} x.$$

Because the right hand side goes to zero under the map $X \otimes \mathcal{U} \oplus \mathcal{U}^p \to \mathcal{U}^p$, by commutativity of the square, $\begin{pmatrix} u \\ y \end{pmatrix}$ goes to zero under the operator $\begin{bmatrix} -Q & P \end{bmatrix}$. $\square$

Remark. We remind that D is an arbitrary noetherian commutative ring and $\mathcal{U}$ is an arbitrary $D[s]$-module. In Gluesing-Luerssen [12] the ground ring is a polynomial ring (over a field) and the function space is a divisible module.

Appendix C. Connection with Fuhrmann's realization

In this section we clarify connection with Fuhrmann's realization.

We keep the assumptions and notations of the previous appendix.

Consider the sequence

$$0 \to X[s] \oplus D[s]^m \to X[s] \oplus D[s]^m \oplus D[s]^p \to D[s]^p \to 0, \tag{6}$$

where the second and third arrows are given respectively by the matrices

$$\begin{bmatrix} sI - F & -G \\ 0 & I \\ H & J \end{bmatrix} \text{ and } \begin{bmatrix} \chi & -Q & P \end{bmatrix}.$$

If $x \in X$ and $u \in D^m$, then the composition of these two arrows sends $\begin{pmatrix} x \\ u \end{pmatrix}$ to

$$P\pi(sx) + Qu - P\pi(Qu) - Qu - P\pi(sx) + P\pi(Qu) = 0$$

It follows that our sequence is a complex.

Lemma 10. *The complex* (6) *is exact*.

Proof. From Proposition 8, we can see that the homomorphism $\begin{bmatrix} sI-F \\ H \end{bmatrix} : X[s] \to X[s] \oplus D[s]^p$ is left invertible; in other words, $S(sI-F) + TH = I$ for some $S : X[s] \to X[s]$ and $T : D[s]^p \to X[s]$. We then have

$$\begin{bmatrix} S & SG-TJ & T \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} sI-F & -G \\ 0 & I \\ H & J \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

It follows that the second homomorphism is left invertible. Hence, it is injective and its cokernel is a projective module. Let N denote this cokernel.

The third homomorphism is obviously surjective because $\begin{bmatrix} \chi & P \end{bmatrix}$ is surjective (see Proposition 8). It follows that the induced homomorphism

$$N \to D[s]^p$$

is surjective.

To complete the proof, we need to show that this is bijective. But this follows from two facts in Bourbaki [2, Ch. II, Sect. 3] (see Theorem 1 and Corollary of Proposition 6). □

Proposition 9. *There is a canonical homomorphism $X^*[s] \oplus D[s]^m \to M$ such that the sequence*

$$0 \to X^*[s] \xrightarrow{\begin{bmatrix} sI-F^t \\ -G^t \end{bmatrix}} X^*[s] \oplus D[s]^m \to M \to 0$$

is exact and the square

$$\begin{array}{ccc} D[s]^m \oplus D[s]^p & = & D[s]^m \oplus D[s]^p \\ \begin{bmatrix} 0 & H^t \\ I & J^t \end{bmatrix} \downarrow & & \downarrow \\ X^*[s] \oplus D[s]^m & \to & M \end{array}$$

is commutative.

Proof. Dualizing the sequence (6), we get an exact sequence

$$0 \to D[s]^p \to X^*[s] \oplus D[s]^m \oplus D[s]^p \to X^*[s] \oplus D[s]^m \to 0.$$

Consider the diagram

$$\begin{array}{ccccccccc} & & & & X^*[s] & = & X^*[s] & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & D[s]^p & \to & X^*[s] \oplus D[s]^m \oplus D[s]^p & \to & X^*[s] \oplus D[s]^m & \to & 0 \end{array}$$

in which the left and the right vertical arrows respectively are

$$\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} sI-F^t \\ -G^t \end{bmatrix}.$$

This diagram is commutative, and we can apply the snake lemma. Doing this we get and exact sequence

$$0 \to D[s]^p \to D[s]^m \oplus D[s]^p \to \frac{X^*[s] \oplus D[s]^m}{\begin{bmatrix} sI-F^t \\ -G^t \end{bmatrix} X^*[s]} \to 0,$$

which gives an isomorphism

$$M \simeq \frac{X^*[s] \oplus D[s]^m}{\begin{bmatrix} sI-F^t \\ -G^t \end{bmatrix} X^*[s]}.$$

This proves the proposition. □

The proposition tells us that

$$\left(X^*, X^* \oplus D^m, \begin{bmatrix} I \\ 0 \end{bmatrix}, \begin{bmatrix} F^t \\ G^t \end{bmatrix}, \begin{bmatrix} 0 & H^t \\ I & J^t \end{bmatrix}\right).$$

is a right state representation of (M, μ).

According to Example 6, the behavioral equation of this model is

$$\begin{cases} \begin{bmatrix} I & 0 \end{bmatrix} s \begin{bmatrix} x \\ u \end{bmatrix} & = \begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \\ \begin{bmatrix} u \\ y \end{bmatrix} & = \begin{bmatrix} 0 & I \\ H & J \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \end{cases},$$

which, as already noted, can be rewritten as

$$\begin{cases} sx = Fx + Gu \\ y = Hx + Ju \end{cases}.$$

References

[1] M. Atiyah, I.G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley, Reading, MA, 1969.
[2] N. Bourbaki, Commutative Algebra, Springer, Verlag, New York, 1989 (chapters 1–7).
[3] C. Byrnes, On the control of certain deterministic infinite-dimensional systems by algebro-geometric techniques, Amer. J. Math. 100 (1978) 1333–1381.
[4] J.W. Brewer, J.W. Bunce, F.S. Van Vleck, Linear Systems over Commutative Rings, Lecture Notes in Pure and Applied Mathematics, vol. 114, Marcel Dekker, 1986.
[5] S. Eilenberg, Automata, Languages and Machines, Academic Press, New York, 1974.
[6] M. Fliess, Some basic structural properties of generalized linear systems, Systems Control Lett. 15 (1990) 391–396.
[7] M. Fliess, H. Mounier, Controllability and observability of linear delay systems: an algebraic approach, ESAIM Control Optim. Calc. Var. 3 (1998) 301–314.
[8] P.A. Fuhrmann, Algebraic system theory: an analyst's point of view, J. Franklin Inst. 301 (1976) 521–540.
[9] P.A. Fuhrmann, A study of behaviors, Linear Algebra Appl. 351/352 (2002) 303–380.
[10] P.A. Fuhrmann, P. Rapisarda, Y. Yamamoto, On the state of behaviors, Linear Algebra Appl. 424 (2007) 570–614.
[11] H. Gluesing-Luerssen, A behavioral approach to delay-differential systems, SIAM J. Control Optim. 35 (1997) 480–499.
[12] H. Gluesing-Luerssen, The Fuhrmann-realization for multi-operator systems in the behavioral context, Multidimens, Systems Signal Process. 11 (2000) 193–211.
[13] H. Gluesing-Luerssen, P. Vettori, S. Zampieri, The algebraic structure of delay-differential systems: a behavioral perspective, Kybernetika 37 (2001) 397–426.
[14] L.C.G.J.M. Habets, System equivalence for AR-systems over rings – with an application to delay-differential systems, Math. Control Signals Systems 12 (1999) 219–244.
[15] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York, 1977.
[16] E.W. Kamen, On an algebraic theory of systems defined by convolution operators, Math. Systems Theory 9 (1975) 57–74.
[17] P.P. Khargonekar, On matrix fraction representations for linear systems over commutative rings, SIAM J. Control Optim. 20 (1982) 172–197.
[18] M. Kuijper, First Order Representations of Linear Systems, Birkhäuser, Boston, 1994.
[19] T. Lam, Serre's Problem on Projective Modules, Springer-Verlag, Berlin, 2006.
[20] V. Lomadze, Finite-dimensional time-invariant linear dynamical systems: algebraic theory, Acta Appl. Math. 19 (2) (1990) 149–201.
[21] V. Lomadze, State and internal variables for linear systems, Linear Algebra Appl. 425 (2007) 534–547.
[22] H. Matsumura, Commutative Ring Theory, Cambridge University Press, 1986.
[23] A.S. Morse, Ring models for delay-differential systems, Automatica 12 (1976) 529–531.
[24] H. Mounier, Algebraic interpretations of the spectral controllability of a linear delay system, Forum Math. 10 (1998) 3958
[25] U. Oberst, Multidimensioanl constant linear systems, Acta Appl. Math. 29 (1990) 1–175.
[26] J.W. Polderman, J.C. Willems, Introduction to Mathematical Systems Theory, Springer, New York, 1998.
[27] D. Quillen, Projective modules over polynomial rings, Invent. Math. 36 (1976) 167–171.
[28] P. Rapisarda, J.C. Willems, State maps for linear systems, SIAM J. Control Optim. 35 (1997) 1053–1091.
[29] Y. Rouchaleau, E.D. Sontag, On the existence of minimal realizations of linear dynamical systems over noetherian integral domains, J. Comp. Syst. Sci. 18 (1979) 65–75.
[30] Y. Rouchaleau, B. Wyman, R. Kalman, Algebraic structure of linear dynamical systems. III. Realization theory over a commutative ring, Proc. Nat. Acad. Sci. 69 (1972) 3404–3406.
[31] E.D. Sontag, Linear systems over commutative rings: a survey, Ricerche Automat. 7 (1976) 1–34.
[32] P. Vettori, S. Zampieri, Controllability of systems described by convolutional and delay-differential equations, SIAM J. Control Optim. 39 (2000) 728–756.
[33] P. Vettori, S. Zampieri, Module theoretic approach to controllability of convolutional systems, Linear Algebra Appl. 351–352 (2002) 739–759.

[34] J.C. Willems, From time series to linear system – part I. Finite dimensional linear time invariant systems, Automatica 22 (1986) 561–580.
[35] J.C. Willems, Paradigms and puzzles in the theory of dynamical systems, IEEE Trans. Automat. Control 36 (1991) 259–294.