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Rational differential operators and their kernels

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ABSTRACT

A linear differential/integral operator is associated to a rational matrix in a natural way, and the behavior is defined to be the kernel of this operator. Conditions under which two rational matrices have the same behavior are derived.

0. Introduction

An important class of dynamical systems is formed by linear differential systems, which are defined via polynomial differential operators (see Polderman and Willems [9]). The idea of this article is to enlarge the class of linear differential systems by considering rational differential operators (that is, operators that are rational in the differentiation operator). It was inspired by the works Willems and Yamamoto [13,14], where rational differential operators have been proposed as new representations for linear differential systems.

Let $\mathbb{F}$ be the field of real or complex numbers and s an indeterminate, and let $\mathbb{T}$ be an arbitrary interval. Without loss of generality, we certainly may assume that $\mathbb{T}$ contains 0. Let us denote by $\mathcal{U}$ the

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function space $C^\infty(\mathbb{T}, \mathbb{F})$ and by ∂ the differentiation operator of this space. Besides of the differentiation operator there is another important operator acting on $\mathcal{U}$, namely, the integration operator

$$\int : \mathcal{U} \to \mathcal{U},$$

which is defined by the formula

$$(\int f)(x) = \int_0^x f(\xi)d\xi.$$

The relation $\partial \circ \int = id$ suggests to define ∂^{-1} by setting

$$\partial^{-1} = \int .$$

This, in turn, leads to the following evident definition

$$\partial^{-n} = (\partial^{-1})^n$$

for every $n \geqslant 0$.

Let now G be a rational matrix, say, of size $p \times q$, and let

$$G = G_{-n}s^n + \cdots + G_{-1}s + G_0 + G_1 s^{-1} + G_2 s^{-2} + \cdots$$

be its expansion at infinity. We define the operator

$$G(\partial) : \mathcal{U}^q \to \mathcal{U}^p$$

as

$$G_{-n}\partial^n + \cdots + G_{-1}\partial + G_0 I + G_1\partial^{-1} + G_2\partial^{-2} + \cdots .$$

Thus, in our understanding, a rational differential operator is a usual map (not a "point-to-set" map as in Willems and Yamamoto [13,14]). This, in fact, is a linear differential/integral operator with constant coefficients.

We define the behavior $Bh(G)$ of G to be the kernel of the operator $G(\partial)$, i.e., the solution set of the equation

$$G(\partial)w = 0.$$

For convenience of the reader, we recall that according to the above mentioned works of Willems and Yamamoto, the behavior of G is the solution set of the differential equation

$$Q(\partial)w = 0,$$

where Q is the numerator in a left coprime factorization of G.

The following simple examples illustrate the difference between the two approaches.

Example 1. Consider the rational function $s^{-1}(s^2 - 1) = s - s^{-1}$. Its behavior is the solution set of the differential/integral equation

$$y' - \int y = 0,$$

which is

$$\{C(e^x + e^{-x}) \mid C \in \mathbb{R}\}. \tag{1}$$

The behavior of $s^{-1}(s^2 - 1)$ in the sense of Willems and Yamamoto is the solution set of the differential equation

$$y'' - y = 0,$$

which is

$$\{C_1(e^x + e^{-x}) + C_2(e^x - e^{-x}) \mid C_1, C_2 \in \mathbb{R}\}. \tag{2}$$

Notice that $(e^x + e^{-x})' = e^x - e^{-x}$, and therefore (2) is the differential closure of (1).

Example 2. Consider the rational function $(s-1)^{-1}s^2 = s + 1 + s^{-1} + s^{-2} + \cdots$. Its behavior is the solution set of the differential/integral equation

$$y' + y + \int y + \int^2 y + \int^3 y + \cdots = 0,$$

which is

$$\{C - Cx \mid C \in \mathbb{R}\}. \tag{3}$$

The behavior of $s^2/(s-1)$ in the sense of Willems and Yamamoto is the solution set of the differential equation

$$y'' = 0,$$

which is

$$\{C_1 + C_2x \mid C_1, C_2 \in \mathbb{R}\}. \tag{4}$$

Clearly, (4) is the differential closure of (3).

Example 3. Let A be a proper rational matrix, and let $A = D^{-1}N$ be its left coprime factorization. The behavior of $G = \begin{bmatrix} I & -A \end{bmatrix}$ is the set

$$\left\{ \begin{pmatrix} u \\ y \end{pmatrix} \mid y = Au \right\}. \tag{5}$$

It is clear that $G = D^{-1}\begin{bmatrix} D & -N \end{bmatrix}$ is a left coprime factorization; hence, the behavior of G in the sense of Willems and Yamamoto is the set

$$\left\{ \begin{pmatrix} u \\ y \end{pmatrix} \mid D(\partial)y = N(\partial)u \right\}. \tag{6}$$

As one knows from the classical linear system theory, (5) is the set of zero initial condition trajectories in (6). Again, the differential closure of (5) is equal to (6).

Remark. In view of the above examples, it is tempting to think that, given a rational polynomial matrix G with left coprime factorization $G = P^{-1}Q$, the differential closure of $Bh(G)$ always coincides with $Bh(Q)$. This is indeed so in most cases, but not in general. This is so, for example, when the Wiener–Hopf indices of a rational matrix are positive (see [7]). A simple example showing that the statement is not true is as follows: The behavior of the rational function $g = s/(s-1)$ is $\{0\}$, while the behavior of its numerator s is the set of constant functions.

We shall show how the solution sets of rational differential equations can be constructed explicitly. It is interesting to note that the standard technique of partial fraction expansion works here well.

Our main goal, however, is to study the following question: When two rational matrices determine the same linear system? The question is basic from the point of view of the "behavioral" philosophy (see Willems [12]).

Recall that if R_1 and R_2 are polynomial matrices, then the solution sets of the equations

$$R_1(\partial)w = 0 \quad \text{and} \quad R_2(\partial)w = 0$$

are equal to each other if and only if there exist polynomial matrices A and B such that $R_2 = AR_1$ and $R_1 = BR_2$. Different proofs of this fundamental theorem can be found in Polderman [8], Polderman and Willems [9], Schumacher [11]. (A proof is given also in [4].)

The equivalence theorem that we shall present is a natural generalization of this result.

The reader is referred to Gottimukkala et al. [3] and Trentelman [10], where the same question is studied in the context of Willems and Yamamoto [13,14].

Throughout, $\mathbb{F}$, s and $\mathcal{U}$ will be as above. To avoid confusions, we shall use the symbol $\hbar$ to denote the function that is identically 1 on the interval. We let O be the ring of proper rational functions, and put $t = s^{-1}$. There is exactly one (continuous) action of O on $\mathcal{U}$ for which

$$tw = \int w.$$

An explicit definition is as follows. If $g \in O$ and $w \in \mathcal{U}$, then the product gw is defined by the formula

$$gw = b_0 w + b_1 \int w + b_2 \int^2 w + \cdots + b_n \int^n w + \cdots ,$$

where $b_0, b_1, b_2, \ldots, b_n, \ldots$ are the coefficients in the expansion of g at infinity (The reader can easily prove that the series above converges uniformly on every compact neighborhood of 0.) This action makes $\mathcal{U}$ into a module over O. This module is without torsion, and it is natural therefore to consider its fraction space. We denote it by $\mathcal{M}$ and refer to its elements as Mikusinski functions. Thus, by definition, a Mikusinski function is a ratio w/g, where $w \in \mathcal{U}$ and $g \in O, \neq 0$. Two functions w_1/g_1 and w_2/g_2 are equal if $g_2 w_1 = g_1 w_2$. We identify $\mathcal{U}$ with a subset in $\mathcal{M}$ via the canonical embedding $w \mapsto w/1$. Every Mikusinski function can be represented as $s^n w$, where $w \in \mathcal{U}$ and $n \geqslant 0$. The function $s\hbar$ is an analog of the Dirac's delta; we shall denote it by δ. For every $m \geqslant 0$, $s^m\delta$ should be interpreted as the m-th derivative of δ. Finite linear combinations of derivatives of δ are called purely impulsive functions. Let Δ denote the set of all purely impulsive functions. This is a module over $\mathbb{F}[s]$, and clearly we have $\Delta = \mathbb{F}[s]\delta$.

Given a proper rational function g, we define the L-transform $L(g)$ by the formula

$$L(g) = g\hbar.$$

By definition, the L-transform converts a proper rational function into a C^∞-function. One can view it as a kind of the inverse Laplace transform followed by the derivative.

The Newton–Leibniz formula

$$w = \int w' + w(0)\hbar,$$

where $w \in \mathcal{U}$, can be rewritten as

$$sw = w' + sw(0)\hbar.$$

This, by induction, yields a more general formula

$$s^n w = w^{(n)} + (s^n w(0) + \cdots + sw^{(n-1)}(0))\hbar. \tag{7}$$

Using this, one easily obtains the decomposition

$$\mathcal{M} = \mathcal{U} \oplus \Delta,$$

which gives rise to the projection map $\Pi : \mathcal{M} \to \mathcal{U}$.

(The interested reader is referred to [6] for a more detailed account of the above version of Mikusinski's calculus.)

We shall need the convolution operation. The convolution of two functions u and v, written as $u * v$, is defined by the formula

$$(u * v)(x) = \int_0^x u(x - \xi)v(\xi)d\xi.$$

For later use, we recall the definition of cohomologies of nonsingular rational matrices. If D is a nonsingular rational matrix, we define (see [4,6]) its cohomology spaces to be

$$H^0(D) = \mathbb{F}[s]^p \cap tDO^p \text{ and } H^1(D) = \mathbb{F}(s)^p/(\mathbb{F}[s]^p + tDO^p),$$

where p is the size of D. The reader can notice that H^0 coincides with Fuhrmann's polynomial model construction (see Fuhrmann [1,2]). Letting $D^* = (D^{-1})^{\text{tr}}$, there holds

$$\dim H^0(D) = \dim H^1(D^*). \tag{8}$$

(This is a consequence of Lemma 5 in [6].)

Given an integer sequence $\mu = (m_1, \ldots, m_p)$, we shall write s^μ (resp., ∂^μ) to denote the diagonal matrix with s^{m_i} (resp., ∂^{m_i}) on the diagonal. We let

$$\mathbb{F}[t]^p_\mu = \bigoplus_i \mathbb{F}[t]_{<m_i},$$

where $\mathbb{F}[t]_{<m_i}$ denotes the space of polynomials (in t) of degree $< m_i$.

Concluding the introduction, we note that one can develop the discrete-time theory that is completely parallel to the continuous one. If one wants, one can use the axiomatic framework of [5] in order to treat simultaneously both discrete- and continuous-time cases.

1. Linear differential/integral systems

Let G be a rational matrix of size $p \times q$. Notice that if G is a proper rational matrix, then $G(\partial)$ is just the operator

$$\mathcal{U}^q \to \mathcal{U}^p, \quad w \mapsto Gw.$$

In general, we have the following:

Lemma 1. *$G(\partial)$ is the composition*

$$\mathcal{U}^q \xrightarrow{G} \mathcal{M}^p \xrightarrow{\Pi} \mathcal{U}^p.$$

Proof. By linearity, it suffices to show that, for every integer n, the operator ∂^n is the same as the composition

$$\mathcal{U} \xrightarrow{s^n} \mathcal{M} \xrightarrow{\Pi} \mathcal{U}.$$

In the case when $n \geqslant 0$, this is immediate from (7); when $n \leqslant 0$, this is obvious. □

As a very useful consequence we have the following:

Corollary 1. *There holds*

$$Bh(G) = \{w \in \mathcal{U}^q \mid Gw \in \Delta^p\}.$$

Remark that if $G = P^{-1}Q$ is a factorization, where P is a square nonsingular polynomial matrix and Q an arbitrary rational (not necessarily polynomial) matrix, then

$$Bh(G) = \{w \in \mathcal{U}^q \mid Qw \in P\Delta^p\} \text{ and } Bh(Q) = \{w \in \mathcal{U}^q \mid Qw \in \Delta^p\}.$$

Because $P\Delta^p \subseteq \Delta^p$, we obviously have $Bh(G) \subseteq Bh(Q)$.

It should be pointed out that the equality

$$(G_1G_2)(\partial) = G_1(\partial)G_2(\partial)$$

does not hold, in general. However, we have the following important

Lemma 2. *Let P be a polynomial matrix and G an arbitrary rational matrix (such that the column number of P is equal to the row number of G). Then*

$$(PG)(\partial) = P(\partial) \circ G(\partial).$$

Proof. Let $p \times l$ and $l \times q$ be the sizes of P and G, respectively. We have to show that the two compositions

$$\mathcal{U}^q \xrightarrow{G} \mathcal{M}^l \xrightarrow{P} \mathcal{M}^p \xrightarrow{\Pi} \mathcal{U}^p \text{ and } \mathcal{U}^q \xrightarrow{G} \mathcal{M}^l \xrightarrow{\Pi} \mathcal{U}^l \xrightarrow{P} \mathcal{M}^p \xrightarrow{\Pi} \mathcal{U}^p$$

are equal to each other. For this, it suffices to show that

$$\mathcal{M}^l \xrightarrow{P} \mathcal{M}^p \xrightarrow{\Pi} \mathcal{U}^p \text{ and } \mathcal{M}^l \xrightarrow{\Pi} \mathcal{U}^l \xrightarrow{P} \mathcal{M}^p \xrightarrow{\Pi} \mathcal{U}^p$$

are equal.

Take any $x + y \in \mathcal{M}^l$ with $x \in \mathcal{U}^l$ and $y \in \Delta^l$. We then have

$$\Pi P \Pi(x + y) = \Pi(P(x)) \text{ and } \Pi P(x + y) = \Pi(P(x) + P(y)) = \Pi(P(x)).$$

The lemma is proved. □

The proof of the following corollary is very easy.

Corollary 2. *Let G_1 and G_2 be rational matrices. Suppose that there exists a polynomial matrix P such that $G_2 = PG_1$. Then*

$$Bh(G_1) \subseteq Bh(G_2).$$

Let G_1 and G_2 be two rational matrices with the same column number q. We say that G_1 and G_2 are strongly equivalent if there exist polynomial matrices A and B such that $G_2 = AG_1$ and $G_1 = BG_2$. It immediately follows from the above corollary that two strongly equivalent rational matrices have the same behavior.

If G is a rational matrix, say, of size $p \times q$, we define its associated module to be

$$Ass(G) = G^{\mathrm{tr}}\mathbb{F}[s]^p.$$

Notice that this is a finitely generated $\mathbb{F}[s]$-submodule of $\mathbb{F}(s)^q$.

It is clear that two rational matrices are strongly equivalent if and only if their associated modules are equal. From this (and from the fact that the associated module is a finitely generated torsion free module), one easily obtains the following useful

Lemma 3. *Every rational matrix is strongly equivalent to a one with full row rank.*

Spaces $\mathcal{U}^q$, where $q \geqslant 1$, are called universums (see Willems [12]). A subset $\mathcal{B}$ of an universum will be called a linear differential/integral system if there exists a rational matrix G such that

$$\mathcal{B} = Bh(G).$$

Any such matrix will be called a representation of $\mathcal{B}$.

In what follows, linear differential/integral systems will be referred to simply as linear systems.

2. Solving rational differential equations

In this section we describe a general procedure for solving rational differential equations. We shall try to follow closely Chapter 3 in Polderman and Willems [9].

Before proceeding, we first want to discuss the issue of computing the L-transforms $L(g)$ and the products gu.

It is clear that $L(1) = \hbar$ and $L(t^n) = x^n/n!$ for $n \geqslant 1$. More generally, we have

Lemma 4. *For $n \geqslant 1$ and $\lambda \in \mathbb{F}$,*

$$L\left(\frac{s}{(s-\lambda)^n}\right) = \frac{x^{n-1}}{(n-1)!}e^{\lambda x}.$$

Proof. One knows well that

$$\frac{x^n}{n!} * u = \int^{n+1} u.$$

This can be rewritten as

$$L(t^n) = t^{n+1}u.$$

In particular, for every proper rational function g, we have

$$L(t^n) * L(g) = t^{n+1}L(g).$$

Since $t^{n+1}L(g) = L(t^{n+1}g)$, we get

$$L(t^n) * L(g) = L(tt^n g).$$

From this, one easily gets the following general formula

$$L(f) * L(g) = L(tfg).$$

The proof of the lemma goes now as follows. For $m = 1$, the lemma is obvious: $L(s/(s-\lambda)) = L((1-\lambda t)^{-1}) = e^{\lambda x}$. Using induction and the above formula, we have

$$L\left(\frac{s}{(s-\lambda)^{m+1}}\right) = L\left(t\frac{s}{s-\lambda}\frac{s}{(s-\lambda)^m}\right) = e^{\lambda x} * \frac{x^{m-1}}{(m-1)!}e^{\lambda x}$$
$$= \int_0^x \frac{u^{m-1}}{(m-1)!}e^{\lambda u}e^{\lambda(x-u)}du = e^{\lambda x}\int_0^x \frac{u^{m-1}}{(m-1)!}du = \frac{x^m}{m!}e^{\lambda x}.$$

The proof is complete. □

The lemma above permits us to compute all L-transforms in the complex case. Indeed, let g be an arbitrary proper rational function, and let

$$\frac{g}{s} = \Sigma\frac{a}{(s-\lambda)^n}$$

be the partial fraction expansion of g/s (which is a strictly proper rational function). Then

$$g = \Sigma\frac{as}{(s-\lambda)^n},$$

and hence

$$L(g) = \Sigma\frac{ax^{n-1}}{(n-1)!}e^{\lambda x}.$$

We leave to the reader to consider the real case. One can see that the L-transforms of proper rational functions are precisely Bohl functions (see Polderman and Willems [9]).

Given a proper rational function g, we shall mean by $g(\infty)$ the free coefficient of g in its expansion at infinity and by g^σ the shift of g, that is, $g^\sigma = s(g - g(\infty))$.

Lemma 5. *Let $g \in O$ and $u \in \mathcal{U}$. Then*

$$gu = g(\infty)u + L(g^\sigma) * u.$$

Proof. For $n \geqslant 1$, we have

$$(t^n u)(x) = (\int^n u)(x) = \int_0^x \frac{(x-\xi)^{n-1}}{(n-1)!}u(\xi)d\xi = (L(t^{n-1}) * u)(x), \quad x \in \mathbb{T}.$$

Hence,

$$t^n u = L(t^{n-1}) * u = L((t^n)^\sigma) * u.$$

Thus, the statement is true in the case when $g = t^n$ with $n \geqslant 1$. It is obvious when $g = 1$. These two spacial cases yield the general case when $g = b_0 + \Sigma_{n\geqslant 1}b_n t^n$. □

Corollary 3. *For* $n \geqslant 1$ *and* $\lambda \in \mathbb{F}$,

$$\frac{1}{(s-\lambda)^n}u = \frac{x^{n-1}}{(n-1)!}e^{\lambda x} * u.$$

Proof. This is immediate by the lemma because

$$\frac{1}{(s-\lambda)^n}(\infty) = 0 \quad \text{and} \quad \left(\frac{1}{(s-\lambda)^n}\right)^\sigma = \frac{s}{(s-\lambda)^n}. \quad \square$$

This corollary together with the partial fraction expansion theorem permits us to compute in the complex case all products gu with $g \in O$ and $u \in \mathcal{U}$. (Again, the real case is left to the reader.)

Let now

$$G(\partial)w = 0. \tag{9}$$

be a differential equation, where G is a rational matrix, say, of size $p \times q$. We shall assume that G has full row rank. (In view of Lemma 3, there is no loss of generality in such assumption.)

The cases $p = q$ and $p < q$ are possible.

Case $p = q$**:** Let $G = Us^\mu B$ be a Wiener–Hopf factorization (that is, a factorization, where U is a unimodular polynomial matrix, μ a sequence of integers and B is a biproper rational matrix). The matrix G is strongly equivalent to the matrix $s^\mu B$. Hence, (9) is equivalent to the equation

$$\partial^\mu Bw = 0, \quad w \in \mathcal{U}^p.$$

This is easy to solve. Indeed, w is a solution if and only if Bw is a solution of the equation

$$\partial^\mu \xi = 0, \quad \xi \in \mathcal{U}^p.$$

This latter is a trivial equation; its solution set is equal to $\mathbb{F}[t]^p_\mu \hbar$. Thus, we have the following:

Theorem 1. *The mapping*

$$x \mapsto L(B^{-1}x)$$

establishes a bijection of $\mathbb{F}[t]^p_\mu$ *onto the solution set of (9).*

Case $p < q$**:** To treat this case, we need the following:

Lemma 6. *Up to order of the columns, the matrix G has the form (the so-called input/output form)*

$$G = \begin{bmatrix} -G_2 & G_1 \end{bmatrix},$$

where G_1 *is a square nonsingular rational matrix and* G_2 *a rational matrix satisfying the conditions:*

- $\det(G_1) \neq 0$;
- $G_1^{-1}G_2$ *is a proper rational matrix.*

Proof. We can find a proper rational matrix H such that the sequence

$$0 \to O^m \xrightarrow{H} O^q \xrightarrow{G} \mathbb{F}(s)^p,$$

where $m = q - p$, is exact. This must be a left invertible proper rational matrix. Hence, it contains a square biproper rational submatrix H_1 of size p. Reorder (if necessary) the components in $\mathbb{F}^q$, so that

$$HH_1^{-1} = \begin{bmatrix} I \\ A \end{bmatrix},$$

where I is the unit matrix of size $m \times m$ and A is a proper rational matrix of size $p \times m$. Our matrix G can be written then as

$$G = \begin{bmatrix} -G_2 & G_1 \end{bmatrix},$$

where G_1 is a square rational matrix of size p and G_2 is a rational matrix of size $p \times m$. In view of the exact sequence

$$0 \to O^m \overset{\begin{bmatrix} I \\ A \end{bmatrix}}{\to} O^m \oplus O^p \overset{\begin{bmatrix} -G_2 & G_1 \end{bmatrix}}{\to} \mathbb{F}(s)^p,$$

we have

$$G_1 A - G_2 = 0.$$

Further, because $\begin{bmatrix} -G_2 & G_1 \end{bmatrix}$ is of full row rank, the sequence

$$0 \to \mathbb{F}(s)^m \overset{\begin{bmatrix} I \\ A \end{bmatrix}}{\to} \mathbb{F}(s)^m \oplus \mathbb{F}(s)^p \overset{\begin{bmatrix} -G_2 & G_1 \end{bmatrix}}{\to} \mathbb{F}(s)^p \to 0.$$

is exact. From this, we see that the $\mathbb{F}(s)$-linear map $G_1 : \mathbb{F}(s)^p \to \mathbb{F}(s)^p$ is bijective; hence, $\det(G_1) \neq 0$. From the equality above, we therefore have: $G^{-1}G_2 = A$.

The lemma is proved. $\square$

Remark. We could write $G = d^{-1}R$, where d is a nonzero polynomial and R a polynomial matrix, and derive the lemma from Theorem 3.3.22 in Polderman and Willems [6].

Thus, we may assume that our matrix G has the special form given in the above lemma.

Rewrite our equation as

$$G_1(\partial)y = G_2(\partial)u.$$

It is clear that every solution of this equation can be represented in a unique way as the sum of a particular solution and a solution of the equation $G_1(\partial)y = 0$.

Put $A = G_1^{-1}G_2$. Using Corollary 1, it is easy to see that (u, Au), where $u \in \mathcal{U}^m$, is a particular solution. Indeed,

$$\begin{bmatrix} -G_2 & G_1 \end{bmatrix} \begin{bmatrix} u \\ Au \end{bmatrix} = 0 \in \Delta^p.$$

We already know how to treat the equation $G_1(\partial)y = 0$. Letting $G_1 = Us^{\mu}B$ be a Wiener–Hopf factorization of G_1, we thus have the following:

Theorem 2. *The mapping*

$$\begin{pmatrix} u \\ x \end{pmatrix} \mapsto \begin{pmatrix} u \\ Au + L(B^{-1}x) \end{pmatrix}$$

establishes a bijection of $\mathcal{U}^m \oplus \mathbb{F}[t]^p_{\mu}$ *onto the solution set of (9).*

Example 4. Consider the equation

$$\frac{\partial^3 - 2\partial^2 + \partial}{-2\partial^2 + 7\partial - 4} y = \frac{\partial^2 - 1}{\partial} u,$$

which comes from the rational matrix $\begin{bmatrix} -g_2 & g_1 \end{bmatrix}$, where

$$g_1 = \frac{s^3 - 2s^2 + s}{-2s^2 + 7s - 4} \text{ and } g_2 = \frac{s^2 - 1}{s}.$$

We have

$$g_1 = s \cdot \left(\frac{2s}{s-1} + \frac{s}{(s-1)^2} - 4 \right)^{-1} \text{ and } g_1^{-1} g_2 = -2 + \frac{1}{s} + \frac{4}{s^2} + \frac{2}{s-1}.$$

We conclude that

$$C(2e^x + xe^x - 4), \quad C \in \mathbb{F}$$

are the "homogeneous" solutions and the "input/output" relation between y and u is given by the formula

$$y_{i/o}(x) = -2u(x) + \int_0^x u(\xi)d\xi + 4\int_0^x (x - \xi)u(\xi)d\xi + 2\int_0^x e^{x-\xi}u(\xi)d\xi.$$

3. Transfer function, initial conditions

In this section, we assign to a rational matrix two important invariants, called the transfer function and the initial condition space. The importance of these invariants is due to Theorem 3, which is the main result of the section.

A transfer function with signal number q is a submodule $T \subseteq O^q$ such that O^q/T is free (see [4]). It can be defined also as a submodule of the form $T = E \cap O^q$, where E is an $\mathbb{F}(s)$-linear subspace of $\mathbb{F}(s)^q$.

For any submodule $T \subseteq O^q$, we let $T\mathcal{U}$ denote the set of all finite sums of trajectories gu with $g \in T$ and $u \in \mathcal{U}$.

Let G be an arbitrary rational matrix of size $p \times q$, say, and let $\mathcal{B}$ denote the behavior of G.

We define the transfer function of G to be

$$T = \{g \in O^q \mid Gg = 0\}.$$

In view of the exact sequence

$$0 \to T \to O^q \xrightarrow{G} \mathbb{F}(s)^p,$$

this indeed is a transfer function.

As we have remarked in Introduction, $\mathcal{U}$ is a torsion free module. One knows well that, for modules over a principal ideal domain, the property of torsion freeness is equivalent to the property of flatness. Therefore, tensoring the above sequence by $\mathcal{U}$, we obtain an exact sequence

$$0 \to T \otimes \mathcal{U} \to \mathcal{U}^q \xrightarrow{G} \mathcal{M}^p.$$

We see that the kernel of

$$\mathcal{U}^q \xrightarrow{G} \mathcal{M}^p$$

is $T\mathcal{U}$. As already remarked, $\mathcal{B} = \{w \in \mathcal{U}^q \mid Gw \in \Delta^p\}$; whence,

$$T\mathcal{U} \subseteq \mathcal{B}.$$

Trajectories of $\mathcal{B}$ that lie in $T\mathcal{U}$ are called transfer trajectories of G.

The initial condition space of G is defined to be

$$X = \mathbb{F}[s]^p \cap tGO^q.$$

Clearly, this is a finite-dimensional linear space (over $\mathbb{F}$); its dimension is called the McMillan degree of G.

Lemma 7. *The image of $\mathcal{B}$ under the operator $\mathcal{U}^q \xrightarrow{G} \mathcal{M}^p$ is equal to $X\delta$.*

Proof. By definition, the image is equal to $\Delta^p \cap G\mathcal{U}^q$.

Let r denote the rank of G, and choose a full column rank rational matrix D such that $GO^q = DO^r$. We then have

$$\Delta^p \cap G\mathcal{U}^q = \Delta^p \cap D\mathcal{U}^r = \Delta^p \cap \mathbb{F}(s)^p\hbar \cap D\mathcal{U}^r.$$

We claim that $\mathbb{F}(s)^p\hbar \cap D\mathcal{U}^r = DO^r\hbar$. To show this, take any C such that $CD = I$. If $w \in \mathcal{U}^r$ is such that $Dw \in \mathbb{F}(s)^p\hbar$, then $w = CDw \in \mathbb{F}(s)^r\hbar$. Because $\mathcal{U}^r \cap \mathbb{F}(s)^r\hbar = O^r\hbar$, it follows that $w \in O^r\hbar$. The claim is proved, and thus our image is equal to $\Delta^p \cap DO^r\hbar$.

Further, we have

$$\Delta^p \cap DO^r\hbar = (s\mathbb{F}[s]^p \cap DO^r)\hbar = (\mathbb{F}[s]^p \cap tGO^q)\delta = X\delta.$$

The proof is complete. □

Thus, the operator $\mathcal{U}^q \xrightarrow{G} \mathcal{M}^p$ induces a surjective map of $Bh(G)$ onto $X\delta$. Composing this with the evident bijective map $X\delta \to X$, we obtain a canonical $\mathbb{F}$-linear surjective map

$$\mathcal{B} \to X.$$

If w is a trajectory of G, then its image under this map is called the initial condition of w.

We have proved the following:

Theorem 3. *There is a canonical exact sequence*

$$0 \to T\mathcal{U} \to \mathcal{B} \to X \to 0.$$

Notice that transfer trajectories are precisely trajectories having zero initial condition.

The frequency response of G is defined to be

$$\Phi = \{f \in O^q \mid Gf \in s\mathbb{F}[s]^p\}.$$

It is clear that

$$\mathcal{B} \cap L(O^q) = L(\Phi).$$

Proposition 1. $\mathcal{B} = T\mathcal{U} + L(\Phi)$.

Proof. Let $w \in \mathcal{B}$, and let x be its initial condition. There is $f \in \Phi$ such that $G(f) = sx$. Then $L(f)$ is a trajectory with the same initial condition x. Hence, $w - L(f) \in T\mathcal{U}$.

This completes the proof. □

Remark. If we know the frequency response, we certainly know the transfer function. The proposition says therefore that knowledge of the frequency response (or, what is equivalent, the set of Bohl trajectories) implies knowledge of the whole behavior.

The following corollary will be very helpful for us.

Corollary 4. *Let G_1 and G_2 be two rational matrices (with the same column number), and let Φ_1 and Φ_2 be their frequency responses. Then*

$$Bh(G_1) \subseteq Bh(G_2) \;\Leftrightarrow\; \Phi_1 \subseteq \Phi_2.$$

4. Duality theorem

Let G be a rational matrix of size $p \times q$.

Consider the bilinear form

$$\mathbb{F}[s]^q \times \mathcal{U}^q \;\rightarrow\; \mathbb{F}$$

defined by the formula

$$\langle f, w\rangle = (f^{\text{tr}}(\partial)w)(0).$$

Notice that this is the composition of the well-known pairing

$$\mathbb{F}[s]^q \times \mathcal{U}^q \;\rightarrow\; \mathcal{U}, \quad (f, w) \mapsto f^{\text{tr}}(\partial)w$$

and the canonical map

$$\mathcal{U} \rightarrow \mathbb{F}, \quad u \mapsto u(0).$$

In this section we are interested in computing the orthogonal of $Bh(G)$ with respect to this bilinear form.

For a subset $X \subseteq \mathbb{F}(s)^q$, let us write X_- to denote the set of the polynomial parts of all elements in X. We need the following technical:

Lemma 8. *Let M be a finitely generated $\mathbb{F}[s]$-submodule of $\mathbb{F}(s)^q$ and E an $\mathbb{F}(s)$-linear subspace of $\mathbb{F}(s)^q$ such that $M \subseteq E$. The following conditions are equivalent:*

(a) E is the fraction space of M;
(b) E_-/M_- has finite dimension (over $\mathbb{F}$).

Proof. Consider the canonical epimorphism $E \to E_-/M_-$. We claim that its kernel is equal to $M + (E \cap tO^q$. Indeed, assume that $x + ty \in E$, where $x \in \mathbb{F}[s]^q$ and $y \in O^q$, goes to zero. Then $x + tz \in M$ for some $z \in O^q$. Because $M \subseteq E$, we must have $y - z \in E$. Hence, $x + ty = (x + tz) + t(y - z) \in M + (E \cap O^q)$. The claim is proved, and thus we have a canonical isomorphism

$$E/(M + (E \cap tO^q)) \simeq E_-/M_-.$$

The assertion follows now from Lemma 3 in [4]. □

Theorem 4 (Duality Theorem)**.** *There holds*

$$Bh(G)^\perp = Ass(G)_-.$$

Proof. We can easily reduce to the full row rank case. So, we shall assume that G has full row rank.

Choose D so that $D^{-1}G$ is right biproper rational matrix. Note that the initial condition space of G is equal to

$$\mathbb{F}[s]^p \cap tGO^q = \mathbb{F}[s]^p \cap tDO^p = H^0(D).$$

(Initial conditions are 0-dimensional cohomologies!)

Put

$$\mathcal{B} = KerG(\partial) \quad \text{and} \quad M = Ass(G).$$

We have to prove that

$$\mathcal{B}^\perp = M_-.$$

The inclusion "$\supseteq$" is easily verified. Indeed, take any $x \in M_-$. Then $x + ty = G^{tr}f$ for some $y \in O^q$ and $f \in \mathbb{F}[s]^p$. For each trajectory w of $\mathcal{B}$, we have

$$\langle x, w\rangle = (G^{tr}f - ty)^{tr}(\partial)w(0) = (f^{tr}(\partial)G(\partial)w)(0) - (ty^{tr}w)(0) = ((f^{tr}(\partial)0)(0) - 0 = 0.$$

Thus, $\mathcal{B}^\perp \supseteq M_-$.

Further, we have $T\mathcal{U} \subseteq \mathcal{B}$, and consequently $\mathcal{B}^\perp \subseteq (T\mathcal{U})^\perp$. We claim that

$$(T\mathcal{U})^\perp = E_-,$$

where $E = G^{tr}\mathbb{F}(s)^p$. Indeed, we have $T = Ker(G) \cap O^q$. (Here, in this proof, by $Ker(G)$ we mean the set $\{u \in \mathbb{F}(s)^q \mid Gu = 0\}$.) Let $Ker(G)^\circ$ denote the orthogonal of $Ker(G)$ with respect to the standard bilinear form

$$\mathbb{F}(s)^q \times \mathbb{F}(s)^q \to \mathbb{F}(s), \quad (x, y) \mapsto x^{tr}y.$$

For every $u \in Ker(G)$ and every $v \in \mathbb{F}(s)^p$, we have

$$u^{tr}G^{tr}v = (Gu)^{tr}v = 0.$$

Hence,

$$E \subseteq Ker(G)^\circ.$$

Both these spaces have the same dimension over $\mathbb{F}(s)$, which is p. Therefore, we have

$$E = Ker(G)^\circ.$$

The claim follows now from Lemma 11 and Lemma 8 in [4].

Thus, there is a tower

$$M_- \subseteq \mathcal{B}^\perp \subseteq E_-.$$

We want to prove that

$$\dim(E_-/\mathcal{B}^\perp) = \dim(E_-/M_-).$$

If we can prove this, it will follow that $\mathcal{B}^\perp = M_-$.

Consider the canonical bilinear form

$$(T\mathcal{U})^\perp/\mathcal{B}^\perp \times \mathcal{B}/T\mathcal{U} \to \mathbb{F}.$$

This clearly is nondegenerate from the left. To see that it is nondegenerate from the right as well, take an arbitrary $\xi \in \mathcal{B}$ such that $\langle f, \xi \rangle = 0$ for each $f \in E_-$. Write $\xi = \xi_0 + y\hbar$, where $\xi_0 \in T\mathcal{U}$ and $y \in O^q$. By Lemma 11 in [4], $\langle f, \xi_0 \rangle = 0$ for each $f \in E_-$. It follows that

$$\forall f \in E_-, \quad \langle f, y\hbar \rangle = 0.$$

We see that $y\hbar$ is orthogonal to E_-. In view of Lemma 8 in [4], $y \in T$. Thus, our bilinear form is nondegenerate, and we conclude that the dimension of $E_-/\mathcal{B}^\perp$ is equal to the dimension of $\mathcal{B}/T\mathcal{U}$. The latter is equal to $\dim H^0(D)$.

Further, $G^{\mathrm{tr}}D^*$ is a left invertible proper rational matrix. (This is because $D^{-1}G$ is a right invertible proper rational matrix.) Therefore

$$G^{\mathrm{tr}}D^*O^p = G^{\mathrm{tr}}\mathbb{F}(s)^p \cap O^q = E \cap O^q.$$

It is easily seen that $G^{\mathrm{tr}} : \mathbb{F}(s)^p \to \mathbb{F}(s)^q$ induces an isomorphism

$$\frac{\mathbb{F}(s)^p}{\mathbb{F}[s]^p + tD^*O^p} \simeq \frac{E}{M + (E \cap tO^q)}.$$

The left side is none other than $H^1(D^*)$; the right side, as we saw in the proof of Lemma 8, is canonically isomorphic to E_-/M_-. Thus, $\dim(E_-/M_-) = \dim H^1(D^*)$.

Using the formula (8), we complete the proof. □

5. Equivalence theorem

As already remarked, two strongly equivalent rational matrices determine the same behavior. However, the converse is not true.

Example 5. The rational functions 1 and t have the same behavior (which is $\{0\}$), but they are not strongly equivalent.

To handle the equivalence problem, we have to introduce a somewhat weaker equivalence relation.

Let G_1 and G_2 be two rational matrices. Say that G_1 is more powerful than G_2 (and write $G_1 \succeq G_2$) if, for every nonnegative integer n, there exists a polynomial matrix P such that

$$s^n G_2 - PG_1$$

is a strictly proper rational matrix.

This concept naturally generalizes the one introduced in Willems [12]. Indeed, one can easily verify that, in the case when both G_1 and G_2 are polynomial matrices, G_1 is more powerful than G_2 if and only if there exists a polynomial matrix P such that $G_2 = PG_1$.

Lemma 9. *Let G_1 and G_2 be two rational matrices with the same column number q. Then*

$$G_1 \succeq G_2 \quad \Leftrightarrow \quad Ass(G_2)_- \subseteq Ass(G_1)_-.$$

Proof. Let r_1 be the row number of G_1 and r_2 the row number of G_2. Saying that $Ass(G_2)_- \subseteq Ass(G_1)_-$ is equivalent to saying that

$$G_2^{\mathrm{tr}}\mathbb{F}[s]^{p_2} \subseteq G_1^{\mathrm{tr}}\mathbb{F}[s]^{p_1} + tO^q.$$

"$\Rightarrow$" Let (e_i) be the canonical basis of $\mathbb{F}^{1\times r_2}$. For each $n \geqslant 0$, choose a polynomial matrix P_n such that

$$s^n G_2 - P_n G_1 \in tO^{r_2\times q}.$$

Denote by $P_{n,i}$ the i-th row of P_n. Then

$$s^n e_i G_2 - P_{n,i} G_1 \in tO^{1\times q},$$

and consequently

$$G_2^{\mathrm{tr}} s^n e_i^{\mathrm{tr}} - G_1^{\mathrm{tr}} P_{n,i}^{\mathrm{tr}} \in tO^q.$$

The assertion follows because the columns $s^n e_i^{\mathrm{tr}}$ form a basis of $\mathbb{F}[s]^{r_2}$.

"$\Leftarrow$" Left to the reader.

The lemma is proved. □

The lemma implies, in particular, that the "$\succeq$" is a partial order.

Lemma 10 (Inclusion Lemma)**.** *Let G_1 and G_2 be two rational matrices (with the same column number). Then*

$$Bh(G_1) \subseteq Bh(G_2) \quad \Leftrightarrow \quad G_1 \succeq G_2.$$

Proof. "$\Rightarrow$" This is immediate from Duality Theorem (and Lemma 9).

"$\Leftarrow$" In view of Corollary 4, it suffices to show that Bohl trajectories of G_1 are trajectories of G_2. Let w be an arbitrary Bohl trajectory of G_1. Then $G_2(\partial)w$ is a Bohl function, and to show that it is zero it suffices to show that all the coefficients in its Taylor expansion are zero.

Take i to be an arbitrary nonnegative integer, and choose a polynomial matrix P so that $s^i G_2 - PG_1$ is strictly proper. We then have

$$(\partial^i G_2(\partial)w)(0) = (P(\partial)G_1(\partial)w)(0) = (P(\partial)0)(0) = 0.$$

The proof is complete. □

Two rational matrices will be said to be equivalent if each of them is more powerful than the other. Two strongly equivalent rational matrices are equivalent, of course; but not conversely.

Example 6. The rational functions 1 and t are not strongly equivalent; but they are equivalent.

For polynomial matrices, it is clear that "equivalence" = "strong equivalence".

As an immediate consequence of Inclusion Lemma, we have the following:

Theorem 5 (Equivalence Theorem)**.** *Two rational matrices determine the same linear system if and only if they are equivalent.*

It is interesting to compare this result with the main result of Gottimukkala et al. [3]. Let G_1 and G_2 be two rational matrices with the same column number q. By the theorem above, G_1 and G_2 have the same behavior if and only if

$$Ass(G_1)_- = Ass(G_2)_-.$$

Theorem 6.3 in [3] says that G_1 and G_2 have the same behavior in the sense of Willems and Yamamoto if and only if

$$Ass(G_1) \cap \mathbb{F}[s]^q = Ass(G_2) \cap \mathbb{F}[s]^q.$$

6. Linear differential and integral systems

In the class of all rational matrices polynomial and proper rational matrices form two extreme subclasses. Linear systems determined by polynomial matrices are called, as is well-known, linear differential systems (see Polderman and Willems [9]). In an analogy, let us call linear systems that are determined by proper rational matrices linear integral systems.

A natural question to ask is: What is special with linear differential systems and what is special with linear integral systems?

To proceed, we need to recall the main result of [4].

Let $\mathcal{S}$ be an $\mathbb{F}$-linear subspace of $\mathcal{U}^q$. Letting T denote the submodule

$$\{g \in O^q \mid g\mathcal{U} \subseteq \mathcal{S}\} \subseteq O^q,$$

we define the relative dimension of $\mathcal{S}$ to be the dimension of $\mathcal{S}/T\mathcal{U}$ (over $\mathbb{F}$). Theorem 3 in [5] states that $\mathcal{S}$ is the behavior of a polynomial matrix if and only if it satisfies the following two conditions:

(1) $\mathcal{S}$ is differentiation-invariant;
(2) $\mathcal{S}$ has finite relative dimension.

Lemma 11. *The relative dimension of a linear system is finite; it is equal to the McMillan degree of any its representation.*

Proof. Let $\mathcal{B}$ a linear system, and let G be its representation. Let $p \times q$ be the size of G and T the transfer function. We claim that

$$T = \{g \in O^q \mid g\mathcal{U} \subseteq \mathcal{B}\}.$$

The inclusion "$\subseteq$" is obvious. Indeed, if $g \in T$, then

$$\forall\, u \in \mathcal{U}, \quad G(gu) = (Gg)u = 0u = 0.$$

To show the inclusion "$\supseteq$", take any $g \in O^q$ that does not belong to T, i.e., $Gg \neq 0$. Choose a sufficiently large integer n so that $t^n Gg \in O^p$. We then have

$$G(gt^n\hbar) \in O^p\hbar.$$

Hence, the trajectory $gt^n\hbar$ does not lie in $\mathcal{B}$. Consequently, g is not an element of the right side above.

The claim is proved.

It remains now to apply Theorem 3. □

From this lemma and from the result that we have recalled, we obtain the following:

Proposition 2. *A linear system is differential if and only if it is differentiation-invariant.*

The case of linear integral systems is easier.

First, prove the following proposition that gives an intrinsic characterization of the transfer trajectories of a linear system.

Proposition 3. *Let G be a rational matrix, and let $\mathcal{B}$ be its behavior. Then*

$$T\mathcal{U} = \{w \in \mathcal{B} \mid t^n w \in \mathcal{B} \ \forall n \geqslant 0\};$$

in other words, $w \in \mathcal{B}$ is a transfer trajectory if and only if all its n-fold integrals also are trajectories of $\mathcal{B}$.

Proof. "$\subseteq$" is obvious. To prove "$\supseteq$", take any trajectory w of G and assume that its initial condition x is not zero. Let $x = a_0 s^n + \cdots + a_n$ with $a_0 \neq 0$. Then

$$G(t^{n+1}w) = t^{n+1}x\delta = (a_0 + \cdots + a_n t^n)\hbar.$$

We see that $t^{n+1}w$ is not a trajectory of G. □

The proposition says that, if $\mathcal{B}$ is a linear system with transfer function T, then the biggest integration-invariant subset in $\mathcal{B}$ is $T\mathcal{U}$. It follows that linear systems that are integration-invariant have the form $T\mathcal{U}$.

Proposition 4. *A linear system is integral if and only if it is integration-invariant.*

Proof. "If" Let G be a proper rational matrix. Then saying that Gw is purely impulsive is the same as saying that Gw is zero. Consequently, if T is the transfer function of G, then $KerG(\partial) = T\mathcal{U}$.

"Only if" Let $\mathcal{B}$ be a linear integral system. As we said above, $\mathcal{B} = T\mathcal{U}$, where T is a transfer function. The module O^q/T is free. Therefore, letting p denote its rank, we can find a proper rational matrix G of size $p \times q$ such that the sequence

$$0 \to T \to O^q \stackrel{G}{\to} O^p \to 0$$

is exact. Tensoring this by $\mathcal{U}$, we get an exact sequence

$$0 \to T \otimes \mathcal{U} \to \mathcal{U}^q \stackrel{G}{\to} \mathcal{U}^p \to 0$$

As already remarked, $G(\partial)$ is the same as $\mathcal{U}^q \stackrel{G}{\to} \mathcal{U}^p$. Hence, G represents $\mathcal{B} = T\mathcal{U}$.

The proof is complete. □

Corollary 5. *The mapping*

$$T \mapsto T\mathcal{U}$$

establishes a one-to-one correspondence between transfer functions and linear integral systems.

Proof. The mapping is injective due to the equality $T\mathcal{U} \cap L(O^q) = L(T)$, which was shown in Section 5 of [4]. The surjectivity follows from the above proposition.

The proof is complete. □

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