ON A PROGRESS IN THE THEORY OF INTEGRAL OPERATORS IN WEIGHTED BANACH FUNCTION SPACES

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Dedicated to Alois Kufner on the occasion of his 70th birthday

Abstract. The paper presents the recent results concerning boundedness criteria in weighted Banach function spaces with non-standard growth both for classical integral operators and integral transforms defined, generally speaking, on metric spaces with measure.

1. Introduction

It is a great pleasure for me to take this opportunity and pay a tribute to Professor Alois Kufner for his outstanding mathematical abilities and exceptional personal quality. In a series of books and papers Alois Kufner has given comprehensive treatment of the weight theory and its application to harmonic analysis, partial differential equations. He was an inspirer of my long-standing and very fruitful collaboration with the Function Spaces group in Prague. Our contacts gave a profound impact on my subsequent scientific activity.

I would like to thank the organizers of the conference "Function Spaces, Differential Operators and Nonlinear Analysis" FSDONA 2004, and especially Professors Pavel Drábek and Jiří Rákosník for this opportunity, for their warm hospitality and creating friendly atmosphere in the meeting.

The present paper is a survey of the very recent results in the theory of integral operators in weighted Banach functional spaces. For the considerable achievement in this area we refer to [GR], [K1], [OK], [GGKK], [BK], [KP], [GM], [EKM]. The latter book focuses our attention on boundedness

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(compactness) criteria of integral operators arising naturally in boundary value problems for PDE, the spectral theory of differential operators, continuum and quantum mechanics, stochastic processes, etc. This monograph was published in 2002. Various new results in weighted theory were obtained later on. The exposition of the present paper is based on the recent results of the author, his pupils and co-authors. The results collected here are scattered with proofs through various journals, some of them are announced quite recently. A characteristic feature of the statements presented here is that most of them are given in the form of criteria.

The survey deals with the boundedness (compactness) criteria for a large class of classical integral operators (or their generalizations) in Banach function spaces with non-standard growth. The latter have been studied intensively by many mathematicians (see, e.g., [Sh], [S1]–[S7], [P1], [P2], [KR], [ELN], [EN], [EM], [ER1], [ER2], [D1]–[D3], [DR1], [DR2], [FSZ], [F], [H1], [H2], [KS1]–[KS5], [CFN], [CCF], [PR], [CFMP]) and references therein). The study of these spaces has been stimulated by various problems of elasticity, fluid mechanics, calculus of variation and differential equations with non-standard growth conditions (see, e.g., [Z1], [Z2], [Z3], [AM], [Ma], [FZhang], [R], [ADS]). The impulse to the intensive development of the theory of integral operators in variable exponent Lebesgue spaces has been given by L. Diening's [D2] pioneering paper on the boundedness of maximal functions in the $L^{p(\cdot)}$ space on a bounded subset of $\mathbb{R}^n$. Then he has also proved that the boundedness holds in the case of an unbounded set in $\mathbb{R}^n$ if the exponent is constant outside of some ball (see [D2]). Similar results for Riesz potentials and singular integrals have been obtained in [D1] and [DR1]. In the case of unbounded domains for the exponent not necessarily constant at infinity the boundedness of maximal functions was obtained independently by A. Nekvinda [Ne], and D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer [CFN1], [CFN2]. The latter paper proves the boundedness for more general fractional maximal functions and, consequently, for the Riesz potentials in $\mathbb{R}^n$. In all the above-mentioned papers the exponent should satisfy the weak Lipschitz condition, except the paper [Ne] in which the result was obtained with the condition replaced by somewhat general integral condition at infinity.

Recently, Diening discovered the necessary and sufficient condition on the exponent function $p(\cdot)$ for the maximal operator to be bounded in $L^{p(\cdot)}$.

Applying the techniques of the theory of weighted norm inequalities and extrapolation, it has recently been shown in [CFMP] that the boundedness in unweighted variable exponent Lebesgue spaces of a wide class of classical operators (potentials, singular integrals, etc.) defined in the Euclidean space with Lebesgue measure follows from that of a maximal operator on the variable $L^{p(\cdot)}$ space and from the known weighted inequalities in the Lebesgue space with a constant exponent.

Following another approach, the present paper focuses on weighted estimates in variable Lebesgue spaces for integral transforms defined both in the Euclidean space with Lebesgue measure and on the spaces of homogeneous type (quasi-metric space with doubling measure).

The study of integral operators in weighted $L^{p(\cdot)}$ spaces was started by the author jointly with S. Samko. The boundedness criteria for the maximal functions, singular integral operators and Riesz potentials in weighted $L^{p(\cdot)}$ spaces with power-type weight is proved in [KS1]–[KS3]. The application of the above-mentioned results for the singular integrals to the Fredholm criteria for singular operators in $L^{p(\cdot)}$ is given in [KS4] and [Ka]. Quite recently the boundedness of the Cauchy-type singular integral operator in weighted $L^{p(\cdot)}$ spaces has been applied in [KPS] to the solution of the Riemann problem for analytic functions in the class of the Cauchy-type integrals with the densities of the spaces mentioned above.

The comprehensive survey on the progress in the theory of integral operators in the variable exponent Lebesgue spaces is given in [S8]. Lately there appeared a vast number of new results concerning integral operators in weighted Lebesgue spaces for weights, more general than power ones. Note that not long ago the necessary condition for the boundedness of the Hardy-Littlewood operator in Musielak-Orlicz spaces has been established in [D3]. In the special case this condition was found to be sufficient. By this an attempt to generalize the concept of Muckenhoupt class was made. However, it is still not clear to what extent the above-mentioned result covers the case of power weights.

In the present paper, along with the power-type weights we discuss more general cases. In this direction we can point out the weight estimates for Hardy operators, trace inequalities for generalized Riesz potentials and the Helson-Szegö-type results for singular integrals. In addition to what we said above, it should be emphasized that in Sections 3 and 4 we present the results for a much more general variable exponent $p(x)$ than the functions with the weak Lipschitz condition.

The paper is organized as follows. In Section 3 we develop the study of the weight theory for Hardy-Littlewood maximal functions and Riesz potentials in variable exponent Lebesgue spaces. Section 4 deals with maximal functions and fractional integrals defined on the spaces of homogeneous type (SHT). In Section 5, trace inequalities for generalized potentials defined on SHT in the weighted $L^{p(\cdot)}$ spaces are treated under a very general condition for the variable exponent. The necessary conditions and the sufficient boundedness/compactness conditions for the Hardy-type operators are established in Section 6. In Section 7 we concentrate our attention on the conditions which govern the boundedness of singular integral operators in weighted variable exponent Lebesgue spaces. Along with the power-type weights, we can cover more oscillating weights. For the Cauchy singular integrals in variable L^p spaces the extensions of the well-known Helson-Szegö theorem are treated.

2. Preliminaries

Let X be a space with a complete measure μ. Suppose that w is a non-negative locally μ-integrable function on X. Such functions are called weight functions. By $L^p_w(X)$ $(1 \le p < \infty)$ we denote a set of all μ-measurable functions f for which the norm

$$\|f\|_{p,w} = \left(\int_\Omega |f(x)|^p w(x)^p \,\mathrm{d}x \right)^{1/p} < \infty.$$

Given a measurable set in $\Omega \subset X$, and a measurable function $p\colon \Omega \to [1,\infty)$, let $L^{p(\cdot)}(\Omega)$ denote the Banach space of measurable functions f on Ω such that $\rho_p(f/\lambda) < \infty$ for some $\lambda > 0$, where

$$\rho_p(f) = \int_\Omega |f(x)|^{p(x)} \,\mathrm{d}x < \infty,$$

with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_\Omega \rho_p(f/\lambda) \le 1 \right\}. \tag{2.1}$$

These spaces are referred to as Nakano spaces [Na] or variable L^p spaces. They are the special case of the Musielak-Orlicz spaces (cf. [Mu], [MO]) and generalize the classical Lebesgue spaces: when $p(x) = p$ is constant, $L^{p(\cdot)} = L^p(\Omega)$.

The weighted variable exponent Lebesgue space is defined as the set of all μ-measurable functions f, for which

$$\| \cdot \|_{p(\cdot),w} = \|fw\|_{p(\cdot)} < \infty.$$

The set of all measurable functions $p\colon \Omega \to [1,\infty)$, for which

$$\underline{p} = \operatorname*{ess\,inf}_{x\in\Omega} p(x) > 1 \quad \text{and} \quad \overline{p} = \operatorname*{ess\,sup}_{x\in\Omega} p(x) < 1, \tag{2.2}$$

is denoted by $\mathcal{P}$.

Let (X, d, μ) be a measure space with a quasi-metric d and let Ω be an open set in X. The class of $p \in \mathcal{P}$ satisfying the condition

$$|p(x) - p(y)| \le \frac{c}{-\log d(x,y)}, \quad x, y \in \Omega,\ d(x,y) \le 1/2, \tag{2.3}$$

will be denoted by $\mathcal{V}(\Omega)$. The condition (2.3) is sometimes referred to as the weak Lipschitz, w-Lip or the log-Hölder condition.

By $\mathcal{V}_\infty(\Omega)$ we denote the set of functions $p \in \mathcal{P}$ which along with (2.3) satisfy the condition

$$|p(x) - p(y)| \le \frac{c}{\log(e + d(x, x_0))}, \quad x, y \in \Omega, \tag{2.4}$$

where $d(y, x_0) \ge c\, d(x, x_0)$.

Let Γ be a simple, closed, rectifiable curve, $\Gamma = \{t \in \mathbb{C} : t = t(s),\ 0 \le s \le \ell\}$, where $\ell = \nu(\Gamma)$ and ν is an arc-length measure on Γ. Let p be a measurable function on Γ such that $p\colon \Gamma \to [1, \infty)$. By analogy with the above definition the space $L^{p(\cdot)}(\Gamma)$ is defined through the modular

$$I_p(f/\lambda) = \int_0^\ell \left(\frac{|f(t(s))|}{\lambda} \right)^{p(t(s))} \mathrm{d}\nu < \infty$$

by the norm

$$\|f\|_{p(\cdot)} = \inf\{\lambda > 0 : I_p(f/\lambda) \le 1\}.$$

By $L^{p(\cdot)}_w(\Gamma)$ we denote the Banach space of measurable functions $f\colon \Gamma \to \mathbb{C}$ for which

$$\|f\|_{p(\cdot),w} = \|fw\|_{p(\cdot)} < \infty.$$

The classes $\mathcal{V}(\Gamma)$ and $\mathcal{V}([0,\ell])$ are then defined in a natural way. The latter is defined for the functions $s \mapsto p(t(s))$. Since $|t(s_1) - t(s_2)| \le |s_1 - s_2|$, we have $\mathcal{V}(\Gamma) \subset \mathcal{V}([0,\ell])$. The inverse inclusion holds, for instance, if there exists a $\gamma > 0$ such that $|s_1 - s_2| \le c\,|t(s_1) - t(s_2)|^\gamma$ with some $c > 0$ or if Γ satisfies the so called arc-chord condition.

3. Weighted inequalities for maximal functions, potentials and singular integrals in $L^{p(\cdot)}(\mathbb{R}^n)$

This section is devoted to the classical integral transforms in the Euclidean space. We present the necessary and sufficient conditions ensuring the boundedness of the Hardy-Littlewood maximal operator, the Riesz potentials and singular integrals in weighted Lebesgue spaces.

Theorem 3.1 [KS1]. *Let Ω be an open bounded set in $\mathbb{R}^n$. Let $p \in \mathcal{V}(\Omega)$ and let $\alpha\colon \Omega \to \mathbb{R}^1$ satisfy the w-*Lip *condition. Then the Hardy-Littlewood maximal operator M, $Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|\,\mathrm{d}y$, is bounded in $L^{p(\cdot)}_w$ with $w(x) = |x-x_0|^{\alpha(x)}$, $x_0 \in \Omega$ if and only if*

$$-\frac{n}{p(x_0)} < \alpha(x_0) < \frac{n}{q(x_0)}.$$

For unbounded sets the following assertion holds true.

Theorem 3.2 [Kh1]. *Let Ω be an open set in $\mathbb{R}^n$. Let $p \in \mathcal{V}(\Omega)$. If $p(x) = p_\infty$ outside some ball then the operator M is bounded in $L^{p(\cdot)}_w(\Omega)$ with $w(x) = |x-x_0|^\beta$ if and only if*

$$-\frac{n}{p_0} < \beta < \frac{n}{q_0} \quad \textit{and} \quad -\frac{n}{p_\infty} < \beta < \frac{n}{q_\infty}.$$

Given a measurable function $f\colon \mathbb{R}^n \to \mathbb{R}^1$ let us consider the Calderón-Zygmund operator

$$\mathcal{K}f(x) = \int_{\mathbb{R}^n} \frac{K(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y,$$

where $K\colon \mathbb{R}^n \to \mathbb{R}^1$ is homogeneous of degree 0, does not vanish on the sphere S^{n-1}, satisfies the Dini condition on S^{n-1} and

$$\int_{S^{n-1}} K(x')\,\mathrm{d}x' = 0.$$

Theorem 3.3. *Let $p \in \mathcal{P}$ be constant outside some ball and let $w(x) = |x-x_0|^\alpha$, $x_0 \in \mathbb{R}^n$. Then the inequality*

$$\|\mathcal{K}w\|_{p(\cdot),w} \le c\,\|fw\|_{p(\cdot),w}$$

with the constant c independent on f holds if only if conditions (2.3) *and* (2.4) *are fulfilled.*

In particular, this statement holds for the Riesz transforms $\mathcal{R}_j$ which are defined by

$$\mathcal{R}_j f(x) = \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y)\,\mathrm{d}y, \quad x \in \mathbb{R}^n,\ j = 1,2,\dots,n,$$

for measurable functions $f\colon \mathbb{R}^n \to \mathbb{R}^1$ satisfying the condition

$$\int_{\mathbb{R}^n} \frac{f(x)\,\mathrm{d}x}{(1+|x|)^n} < \infty.$$

If the inequalities

$$\|\mathcal{R}_j f w\|_{p(\cdot)} \le c\,\|fw\|_{p(\cdot)}$$

hold for all $j = 1, 2, \ldots, n$ then the conditions for the weight from Theorem 3.2 are satisfied. See [KS2] for the case $n = 1$.

Theorem 3.3 in its sufficiency part holds true for more general Calderón-Zygmund operators.

For the generalized Riesz potential

$$I_{\alpha(x)} f(x) = \int_\Omega \frac{f(y)}{|x-y|^{n-\alpha(x)}}\,\mathrm{d}y, \quad x \in \Omega,$$

over bounded domain Ω in $\mathbb{R}^n$ the following extension of the Stein-Weiss theorem (see [ST]) holds.

Theorem 3.4 [S7]**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ and $x_0 \in \Omega$, let $p \in \mathcal{V}(\Omega)$ and let α satisfy the conditions*

$$\inf_{x\in\Omega} \alpha(x) > 0, \quad \sup_{x\in\Omega} \alpha(x)p(x) < n,$$

$$|\alpha(x) - \alpha(y)| \le \frac{A}{\log \frac{1}{|x-y|}}, \quad |x-y| \le \frac{1}{2},\ x, y \in \Omega.$$

Then the generalized Riesz potential operator $I_{\alpha(\cdot)}$ satisfies the estimate

$$\|I_{\alpha(\cdot)} f\|_{q(\cdot),|x-x_0|^\mu} \le C\,\|f\|_{p(\cdot),|x-x_0|^\gamma}, \tag{3.1}$$

where

$$\frac{1}{q(x)} \equiv \frac{1}{p(x)} - \frac{\alpha(x)}{n}, \tag{3.2}$$

$$\alpha(x_0)p(x_0) - n < \gamma < n[p(x_0) - 1] \tag{3.3}$$

and

$$\mu = \frac{q(x_0)}{p(x_0)}\,\gamma. \tag{3.4}$$

Now we shall deal with $\Omega = \mathbb{R}^n$ and a constant α, $0 < \alpha < n$. Consider the weight fixed to the origin and to infinity:

$$\rho(x) = \rho_{\gamma_0,\gamma_\infty}(x) = |x|^{\gamma_0}(1+|x|)^{\gamma_\infty-\gamma_0}.$$

Theorem 3.5 [SV]. *Let $p \in \mathcal{V}_\infty(\mathbb{R}^n)$. Let $\rho_1 = \rho_{\gamma_0,\gamma_\infty}$ and $\rho_2 = \rho_{\mu_0,\mu_\infty}$, where*

$$\mu_0 = \frac{q(0)}{p(0)}\gamma_0, \quad \mu_\infty = \frac{q(\infty)}{p(\infty)}\gamma_\infty.$$

Then the operator I_α is bounded from $L^{p(\cdot)}_{\rho_1}(\mathbb{R}^n)$ into the space $L^{q(\cdot)}_{\rho_2}(\mathbb{R}^n)$ if

$$\alpha p(0) - n < \gamma_0 < n[p(0) - 1], \quad \alpha p(\infty) - n < \gamma_\infty < n[p(\infty) - 1],$$

and the exponents γ_0 and γ_∞ are related to each other by the equality

$$\frac{q(0)}{p(0)}\gamma_0 + \frac{q(\infty)}{p(\infty)}\gamma_\infty = \frac{q(\infty)}{p(\infty)}\big[(n+\alpha)p(\infty) - 2n\big].$$

In [SV] the spherical potential operators were treated as well. Certain Hardy-type inequality for the Riesz potential was proved in [S6].

4. Maximal functions on $L^{p(\cdot)}_w(X)$

In this section we present results concerning weight estimates in variable exponent Lebesgue spaces for maximal functions and generalized potentials defined in the spaces of homogeneous type (SHT). These are spaces (X, d, μ) defined in Section 2 satisfying in addition the following so called *doubling condition*: There exists a positive constant $c > 0$ such that

$$\mu B(x, 2r) \le c\mu B(x, r)$$

for any $x \in X$ and $r > 0$.

Let M be the maximal operator defined by

$$Mf(x) = \sup_{r>0} \frac{1}{\mu B(x,r)} \int_{B(x,r)} |f(y)| \,\mathrm{d}\mu.$$

Theorem 4.1 [Kh2]. *Let $p \in \mathcal{V}(X)$ and $p(x) = p_\infty$ outside some ball. Then the maximal operator M is bounded in $L^{p(\cdot)}(X)$.*

Theorem 4.2 [KS5]. *Let Ω be an open bounded set in X and let*

$$\mu B(x, r) \sim r^s, \quad s > 0.$$

Let $p \in \mathcal{V}(\Omega)$ and let $w(x) = d(x_0, x)^\alpha$, $x_0 \in X$. Then M is bounded in $L^{p(\cdot)}_w$ if and only if

$$-\frac{s}{p(x_0)} < \alpha < \frac{s}{p'(x_0)}. \tag{4.1}$$

It should be mentioned that for the unweighted $L^{p(\cdot)}$ spaces the boundedness of the maximal operator on a bounded set in a metric space with doubling measure has been independently proved in [HHP]. In the same paper it is shown that the maximal operator may be bounded even though the variable exponent is not weak Lipschitz continuous.

Given $0 < \alpha < 1$, define the fractional maximal operator on SHT by

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r)^{1-\alpha}} \int_{B(x,r)} |f(y)| \, \mathrm{d}y.$$

Theorem 4.3 [K3]**.** *Let* $0 \le \alpha < 1$, $1 < \underline{p} \le \overline{p} < 1/\alpha$ *and let* $p \in \mathcal{V}_\infty(X)$. *Define* $q\colon X \to (1,\infty)$ *by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \alpha, \quad x \in X. \tag{4.2}$$

Then the fractional maximal operator M_α *is bounded from* $L^{p(\cdot)}(X)$ *to* $L^{q(\cdot)}(X)$.

5. Trace inequalities for generalized potentials

In this section we shall deal with the generalized potential operators defined as follows:

$$I_{\alpha(x)} f(x) = \int_X d(x,y)^{\alpha(x)-s} f(y) \, \mathrm{d}\mu(y), \quad 0 < \alpha(x) < s. \tag{5.1}$$

Our goal is to present two statements for this operator. The first one concerns criteria for a two-weights inequality for the operator $I_{\alpha(\cdot)}$ in weighted Lebesgue spaces with constant exponents. The second one establishes the trace inequality for $I_{\alpha(\cdot)}$ in variable exponent Lebesgue spaces.

In the sequel we always assume that $\mu X < \infty$, that α satisfies the w-Lip(X) condition and that there exist positive constants c_0, $s > 0$ such that

$$\mu B(x,r) \le c\, r^s$$

for all $x \in X$ and $r > 0$. We set

$$l := \operatorname{diam} X = \sup\{d(x,y) : x, y \in X\}.$$

Theorem 5.1 [EKM2]. *Let $1 < \gamma < \lambda < \infty$, $0 < \alpha(x) < s$ and let α satisfy the w-Lip(X) condition. Let ρ and w be weights.*

Then the operator $I_{\alpha(\cdot)}$ is bounded from $L^{\gamma}_{w}(X)$ to $L^{\lambda}_{\rho}(X)$ if and only if

$$\sup_{\substack{x\in X\\ 0<r<l}} \rho B(x,Nr)^{1/\lambda} \times \left(\int_{X\setminus B(x,r)} w(y)^{-\gamma'} d(x,y)^{(\alpha(x)-s)\gamma'}\,\mathrm{d}\mu(y)\right)^{1/\gamma'} < \infty \tag{5.2}$$

and

$$\sup_{\substack{x\in X,\\ 0<r<l}} w(x)^{-1} B(x,Nr)^{1/\gamma'} \times \left(\int_{X\setminus B(x,r)} \rho(y)^{\lambda} d(x,y)^{(\alpha(x)-s)\lambda}\,\mathrm{d}\mu(y)\right)^{1/\lambda} < \infty, \tag{5.3}$$

where $N = 2a_1(1+2a_0)$. The constants a_0 and a_1 are from the definition of the quasi-metric (see, e.g., [GGKK, p. 1]*).*

This theorem extends the results of [GGK] in the case of an open bounded set Ω.

Corollary 5.1. *Let $1 < \gamma < \lambda < \infty$. Let α satisfy the w-Lip(X) condition and let $\sup_{x\in X} \alpha(x) < s/\gamma$. Then*

(i) *the operator $I_{\alpha(\cdot)}$ acts boundedly from $L^{\gamma}(X)$ into $L^{\lambda}_{\rho}(X)$ if*

$$\sup_{\substack{x\in X\\ 0<r<l}} r^{\lambda(\alpha(x)-s/r)} \int_{B(x,r)} \rho^{\lambda}(y)\,\mathrm{d}\mu(y) < \infty; \tag{5.4}$$

(ii) *if X is compact and*

$$b_1 r^s \le \mu B(x,r) \le b_2 r^s \tag{5.5}$$

for some positive constants b_1 and b_2, then the condition (5.4) *is also necessary for the boundedness of $I_{\alpha(\cdot)}$ from $L^{\gamma}(X)$ to $L^{\lambda}_{\rho}(X)$.*

Now we return to the variable exponent Lebesgue spaces and give a trace inequality for $I_{\alpha(\cdot)}$. By this we present some extensions of the well-known theorems of Adams and Sobolev type.

Theorem 5.2 [EKM2]. *Let $p(\cdot)$ and $q(\cdot)$ be measurable functions on X with $1 < \underline{p} \le \bar{q} < \infty$, let α satisfy the w-*Lip(X) *condition and suppose that* $\sup_{x\in X} \alpha(x) < s/\underline{p}$. *Let v be a weight. Then the condition*

$$\sup_{\substack{x\in X\\ 0<r<l}} r^{\bar{q}(\alpha(x)-s/\underline{p})} \int_{B(x,r)} v(y)^{q(y)}(y)\,\mathrm{d}\mu(y) < \infty \tag{5.6}$$

implies the boundedness of $I_{\alpha(\cdot)}$ from $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}_v(X)$.

From this theorem a statement of the Sobolev type follows for $I_{\alpha(\cdot)}$.

Corollary 5.2. *Let $p(\cdot)$ and $q(\cdot)$ be measurable functions on X such that $1 < \underline{p} \le \bar{q} < \infty$. Let α satisfy the w-*Lip(X) *condition and suppose that $s(1/\underline{p}-1/\bar{q}) \le \inf_{x\in X}\alpha(x) \le \sup_{x\in X}\alpha(x) < s/\underline{p}$. Then $I_{\alpha(\cdot)}$ acts boundedly from $L^{p(\cdot)}(X)$ into $L^{q(\cdot)}(X)$.*

Theorems 5.1 and 5.2 together with Corollaries 5.1 and 5.2 are new even for Euclidean spaces. From these results several special cases can be indicated, such as potentials on thin sets. Let Γ be a subset of $\mathbb{R}^n$ which is an s-set $(0 < s \le n)$ in the sense that there is a Borel measure μ in $\mathbb{R}^n$ such that $\operatorname{supp}\mu = \Gamma$ and there are positive constants c_1, c_2 such that for all $x \in \Gamma$ and $r \in (0,1)$,

$$c_1 r^s \le \mu(B(x,r)\cap\Gamma) \le c_2 r^s. \tag{5.7}$$

It is known (see [T]) that μ is equivalent to the restriction of the Hausdorff s-measure H_s; we shall thus identify μ with $H_s|_\Gamma$. Given $x \in \Gamma$, put $\Gamma(x,r) = B(x,r)\cap\Gamma$.

Let us indicate some examples of SHT for which the condition (5.7) is satisfied. Let $\Gamma \subset \mathbb{C}$ be a connected rectifiable curve and let ν be an arc-length measure on Γ. By definition, Γ is *Carleson (regular)* if

$$\nu(\Gamma \cap B(z,r)) \le c\,r$$

for every $z \in \mathbb{C}$ and $r > 0$.

For $r < \operatorname{diam}\Gamma$, the reverse inequality

$$\nu(\Gamma \cap B(x,r)) \ge r$$

holds for all $z \in \Gamma$. Equipped with ν and the Euclidean metric, the regular curve becomes an SHT. Now let

$$T_{\alpha(t)}f(t) = \int_\Gamma \frac{f(\tau)}{|t-\tau|^{1-\alpha(t)}}\,\mathrm{d}\tau \tag{5.8}$$

be an integral with weak variable singularities.

The Cantor set in $\mathbb{R}^n$ is an s-set, where

$$s = \frac{\log(3^n - 1)}{\log 3}.$$

Consider the following potential-type integral transform on a bounded Cantor set F:

$$J_{\alpha(x)} f(x) = \int_F \frac{f(y)}{|x-y|^{s-\alpha}} \, \mathrm{d}H_s, \quad 0 < \alpha(x) < s. \tag{5.9}$$

From the previous results we can derive trace inequalities for the operators $T_{\alpha(\cdot)}$ and $J_{\alpha(\cdot)}$. In some cases the statements have the form of criteria.

Let us illustrate it by the case of $J^{\alpha(\cdot)}$.

Theorem 5.3 [EKM2]. *Let $1 < \gamma < \lambda < \infty$, let α satisfy the w-Lip(X) condition and let $\sup_{x \in F} < s/\gamma$. Then the operator $J_{\alpha(\cdot)}$ acts boundedly from $L^\gamma(F)$ into $L^\lambda_\rho(F)$ if and only if*

$$\sup_{\substack{x \in F \\ 0 < r < \operatorname{diam} F}} r^{\lambda(\alpha(x) - s/\gamma)} \int_{\Gamma(x,r)} \rho^\lambda(y) \, \mathrm{d}H_s(y) < \infty.$$

Theorem 5.4 [KS6]. *Let Γ be a closed regular curve with finite length $\ell < \infty$. Let $p(t)$ satisfy conditions* (2.2), (2.3) *on Γ and $\alpha(t)$ satisfy the assumptions*

$$\inf_{t \in \Gamma} \alpha(t) > 0, \quad \sup_{t \in \Gamma} \alpha(t) p(t) < 1, \quad |\alpha(t) - \alpha(\tau)| \le \frac{A}{\log \frac{2\ell}{|t-\tau|}}$$

for all $t, \tau \in \Gamma$ with $A > 0$ independent of t and τ. Then the operator $T_{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(\Gamma)$ into $L^{q(\cdot)}(\Gamma)$ where $1/q(t) = 1/p(t) - \alpha(t)$.

Theorem 5.5 [KS6]. *Let Γ be an arbitrary infinite regular curve in $\mathbb{C}$. Let $p \in \mathcal{V}(\Gamma)$ be constant on Γ outside some circle in $\mathbb{C}$. Then the operator T_α with a constant α is bounded from $L^{p(\cdot)}(\Gamma)$ into $L^{q(\cdot)}(\Gamma)$ with $1/q(t) = 1/p(t) - \alpha$.*

6. Two-weight problem for the Hardy operators in $L^{p(\cdot)}$

During the last 30 years a vast amount of research has been carried out on the boundedness/compactness of the weighted Hardy operator

$$H_{v,w} f(x) = v(x) \int_0^x f(t) w(t) \, \mathrm{d}t.$$

In the present section we expose integral-type necessary conditions and sufficient conditions on measurable almost everywhere positive functions (weights) v and w governing the boundedness/compactness of the operator $H_{v,w}$ from $L^{p(x)}(I)$ to $L^{q(x)}(I)$, where $p(x)$ and $q(x)$ are continuous functions and $I = [0,1]$ or $I = R_+ \equiv [0,\infty)$. We also investigate the corresponding problems for the dual operators

$$\overline{H}_{v,w}f(x) = v(x)\int_x^1 f(t)w(t)\,\mathrm{d}t, \quad x \in (0,1),$$

and

$$\widetilde{H}_{v,w}f(x) = v(x)\int_x^\infty f(t)w(t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+.$$

From the beginning we should emphasize that in the sequel we shall not need the condition

$$\operatorname*{ess\,sup}_{x\in I} p(x) \le \operatorname*{ess\,inf}_{x\in I} q(x).$$

Under this heavy restriction it is possible to get the boundedness of $\widetilde{H}_{v,w}$ from $L^{p(\cdot)}$ to $L^{q(\cdot)}$ using the standard methods of BFS (see, for example, [Be], [K]). It should be noted that all conditions derived in the present section are simultaneously necessary and sufficient in the case of classical Lebesgue spaces. The distance of $H_{v,w}$ from the space of finite rank operators is estimated as well. Finally, we mention that the boundedness of $H_{v,w}$ and $\overline{H}_{v,w}$ in weighted $L^{p(x)}$ spaces for some power weights v and w was established in [KS1].

Let $I \equiv [0,1]$ or $I \equiv \mathbb{R}_+$. Suppose that a is a fixed number in I. We shall need the following notation:

$$\bar{p} := \operatorname*{ess\,sup}_{x\in I} p(x), \quad \underline{p} := \operatorname*{ess\,inf}_{x\in I} p(x),$$

$$p_0(x) := \operatorname*{ess\,inf}_{y\in[0,x]} p(y), \quad p_{1,a}(x) := \operatorname*{ess\,inf}_{y\in[x,a]} p(y),$$

$$p_1(x) := p_{1,a}(x) \quad \text{for } a = 1,$$

$$\widetilde{p}(x) := \begin{cases} p(x) & \text{if } x \in [0,a] \\ p_2 \equiv \text{const} & \text{if } x \in [a,\infty), \end{cases}$$

$$\widetilde{p}_0(x) := \begin{cases} p_0(x) & \text{if } x \in [0,a] \\ p_2 \equiv \text{const} & \text{if } x \in [a,\infty), \end{cases}$$

$$\widetilde{P}(x) := \begin{cases} \bar{p} & \text{if } x \in [0,a] \\ p_2 \equiv \text{const} & \text{if } x \in [a,\infty), \end{cases}$$

$$\widetilde{p}_1(x) := \begin{cases} p_{1,a}(x) & \text{if } x \in [0,a] \\ p_2 \equiv \text{const} & \text{if } x \in [a,\infty), \end{cases}$$

$$\underline{q} := \operatorname*{ess\,inf}_{x\in I} q(x), \quad \bar{q} := \operatorname*{ess\,sup}_{x\in I} q(x),$$
$$g'(x) := \frac{g(x)}{(g(x)-1)}$$

for measurable g with $g(x) \in (1,\infty)$.

We begin with the following statement:

Theorem 6.1. *Let $p(x)$ and $q(x)$ be measurable functions on $I \equiv [0,1]$ with $1 < \underline{p} \le p_0(x) \le q(x) \le \bar{q} < \infty$ for a.a. $x \in I$. If*

$$\begin{aligned} B &\equiv \sup_{0<t<1} B(t) \\ &\equiv \sup_{0<t<1} \int_t^1 v(x)^{q(x)} \left(\int_0^t w^{p_0'(x)}(\tau)\,\mathrm{d}\tau \right)^{q(x)/p_0'(x)} \mathrm{d}x < \infty, \end{aligned} \tag{6.1}$$

then the operator $H_{v,w}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.

Theorem 6.2. *Let $p(x)$ and $q(x)$ be measurable functions on $\mathbb{R}_+$ with $1 < \underline{p} \le p_0(x) \le q(x) \le \bar{q} < \infty$ for a.a. $x \in I$. Then the condition*

$$\begin{aligned} B_\infty \equiv \sup_{t>0} \int_0^\infty & v(x)^{q(x)} \\ & \times \left(\int_0^t w(\tau)^{p_0'(x)} (1+\tau)^{2(p'(x)/p_0'(\tau)-1)}\,\mathrm{d}\tau \right)^{q(x)/p_0'(x)} \mathrm{d}x < \infty \end{aligned}$$

guarantees the boundedness of $H_{v,w}$ from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$.

Under additional assumptions the following rather simpler sufficient condition for boundedness of $H_{v,w}$ can be obtained.

Theorem 6.3. *Let $p(x)$ and $q(x)$ be measurable functions on $I \equiv \mathbb{R}_+$ with $1 < \underline{p} \le p_0(x) \le q(x) \le \bar{q} < \infty$ for a.a. $x \in I$. Suppose that there exists a positive number a such that $p(x) \equiv p_2 \equiv \mathrm{const}$ and $q(x) \equiv q_2 \equiv \mathrm{const}$ when $x > a$. Assume also that $p_2 \le q_2$. If*

$$\begin{aligned} A &\equiv \sup_{t>0} A(t) \\ &\equiv \sup_{t>0} \int_t^\infty v(x)^{q(x)} \left(\int_0^t w(\tau)^{\widetilde{p}_0'(x)}\,\mathrm{d}\tau \right)^{q(x)/\widetilde{p}_0'(x)} \mathrm{d}x < \infty, \end{aligned} \tag{6.2}$$

then the operator $H_{v,w}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.

Theorem 6.4. *Let Let $p(x)$ and $q(x)$ be measurable functions on $I \equiv [0,1]$ with $1 < \underline{p} \le \bar{p} < \infty$, $1 < \underline{q} \le \bar{q} < \infty$. If $H_{v,w}$ is bounded from $L^{p(\cdot)}(0,1)$ to $L^{q(\cdot)}(0,1)$ and $\rho_{p'}(w) < \infty$, then*

$$\begin{aligned}\overline{B} &\equiv \sup_{0<t<1} \overline{B}(t)\\ &\equiv \sup_{0<t<1} \left(\int_t^1 v(x)^{q(x)} \left(\int_0^t w(\tau)^{p'(\tau)}\,\mathrm{d}\tau \right)^{q(x)/\bar{p}'} \mathrm{d}x \right)^{1/\underline{q}} < \infty.\end{aligned} \tag{6.3}$$

Theorem 6.5. *Let $p(x)$ and $q(x)$ be measurable functions on $I \equiv \mathbb{R}_+$ with $1 < \underline{p} \le \bar{p} < \infty$, $1 < \underline{q} \le \bar{q} < \infty$. Suppose that there exists a positive number a such that $p(x) \equiv p_2 \equiv \mathrm{const}$ and $q(x) \equiv q_2 \equiv \mathrm{const}$ when $x > a$. Assume that $\int_0^a w^{p'(\tau)}(\tau)\,\mathrm{d}\tau < \infty$ and $H_{v,w}$ is bounded from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$. Then*

$$\begin{aligned}\overline{A}_\infty &\equiv \sup_{t>0} \overline{A}_\infty(t)\\ &\equiv \sup_{t>0} \left(\int_t^\infty v(x)^{q(x)} \left(\int_0^t w(\tau)^{\widetilde{p}\,'(\tau)}\,\mathrm{d}\tau \right)^{q(x)/\widetilde{P}\,'(x)} \mathrm{d}x \right)^{1/q_2} < \infty.\end{aligned} \tag{6.4}$$

Now we turn to the dual operators

$$\overline{H}_{v,w} f(x) = v(x) \int_x^1 f(t) w(t)\,\mathrm{d}t$$

for measurable $f\colon (0,1) \to \mathbb{R}$.

Theorem 6.6. *Let p and q be measurable functions on $I \equiv [0,1]$ with $1 < \underline{p} \le p_1(x) \le q(x) \le \bar{q} < \infty$ for a.a. $x \in I$. If*

$$\begin{aligned}G &\equiv \sup_{0<t<1} G(t)\\ &\equiv \sup_{0<t<1} \int_0^t v(x)^{q(x)} \left(\int_t^1 w^{p_1'(x)}(\tau)\,\mathrm{d}\tau \right)^{q(x)/p_1'(x)} \mathrm{d}x < \infty,\end{aligned} \tag{6.5}$$

then the operator $\overline{H}_{v,w}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.

For the operator

$$\widetilde{H}_{v,w} f(x) = v(x) \int_x^\infty f(t) w(t)\,\mathrm{d}t, \quad x \in (0,\infty),$$

we have:

Theorem 6.7. *Let p and q be measurable functions on $I \equiv \mathbb{R}_+$. Suppose that there exists a positive number a such that $p(x) \equiv p_2 \equiv \text{const}$, $q(x) \equiv q_2 \equiv \text{const}$ for $x > a$, and $1 < p_2 \le \widetilde{p}_1(x) \le q(x) \le \bar{q} < \infty$ for a.a. $x \in I$. If*

$$\begin{aligned}\overline{G} &\equiv \sup_{t>0} \overline{G}(t) \\ &\equiv \sup_{t>0} \int_0^t v(x)^{q(x)} \left(\int_t^\infty w^{\widetilde{p}_1'(x)}(\tau)\,\mathrm{d}\tau \right)^{q(x)/\widetilde{p}_1'(x)} \mathrm{d}x < \infty,\end{aligned} \tag{6.6}$$

then the operator $\widetilde{H}_{v,w}$ is bounded from $L^{p(\cdot)}(I)$ to $L^{q(\cdot)}(I)$.

Theorem 6.8. *Let p and q be measurable functions on $I \equiv [0,1]$. Suppose that $1 < \underline{p} \le \bar{p} < \infty$, $1 < \underline{q} \le \bar{q} < \infty$. Then from the boundedness of $\overline{H}_{v,w}$ from $L^{p(\cdot)}(0,1)$ to $L^{q(\cdot)}(0,1)$ it follows that*

$$\begin{aligned}F &\equiv \sup_{0<t<1} F(t) \\ &\equiv \sup_{0<t<1} \int_0^t v(x)^{q(x)} \left(\int_t^1 w^{p'(\tau)}(\tau)\,\mathrm{d}\tau \right)^{q(x)/\widetilde{P}'(x)} \mathrm{d}x < \infty,\end{aligned} \tag{6.7}$$

provided that $\rho_{p'}(w) < \infty$.

Theorem 6.9. *Let p and q be measurable functions on $I \equiv \mathbb{R}_+$ with $1 < \underline{p} \le \bar{p} < \infty$, $1 < \underline{q} \le \bar{q} < \infty$. Suppose that there exists a positive number a such that $p(x) \equiv p_2 \equiv \text{const}$ and $q(x) \equiv q_2 \equiv \text{const}$ when $x > a$. Then from the boundedness of $\widetilde{H}_{v,w}$ from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(\mathbb{R}_+)$ it follows that*

$$\begin{aligned}F_\infty &\equiv \sup_{t>0} F_\infty(t) \\ &\equiv \sup_{t>0} \int_0^t v(x)^{q(x)} \left(\int_t^\infty w^{(\widetilde{p})'(\tau)}(\tau)\,\mathrm{d}\tau \right)^{q(x)/\widetilde{P}'(x)} \mathrm{d}x < \infty,\end{aligned} \tag{6.8}$$

provided that $\int_0^a w^{p'(\tau)}(\tau)\,\mathrm{d}\tau < \infty$.

Now we deal with compactness properties of the operators $H_{v,w}$, $\overline{H}_{v,w}$ and $\widetilde{H}_{v,w}$.

Theorem 6.10. *Let p and q satisfy the conditions of Theorem 6.1.*

(i) *If (6.1) holds and $\lim_{t\to 0} B(t) = 0$, then $H_{v,w}$ is compact from $L^{p(\cdot)}(0,1)$ to $L^{q(\cdot)}(0,1)$.*

(ii) *Let $\rho_{p'}(w) < \infty$. If $H_{v,w}\colon L^{p(\cdot)}(0,1) \to L^{q(\cdot)}(0,1)$ is compact then (6.3) holds and $\lim_{t\to\infty} \overline{B}(t) = 0$, where $\overline{B}(t)$ is from (6.3).*

Analogously we have:

Theorem 6.11. *Let p and q satisfy the conditions of Theorem* 6.3. *Then the following statements hold:*

(i) *If* (6.4) *holds and* $\lim\limits_{t\to 0} A(t) = \lim\limits_{t\to\infty} A(t) = 0$, *then $H_{v,w}$ is compact from $L^{p(\cdot)}(\mathbb{R}_+)$ to $L^{q(\cdot)}(R_+)$.*

(ii) *Let* $\int_0^a w^{p'(\tau)}(\tau)d\tau < \infty$. *If* $H_{v,w}\colon L^{p(\cdot)}(\mathbb{R}_+) \to L^{q(\cdot)}(\mathbb{R}_+)$ *is compact, then* (6.4) *holds and* $\lim\limits_{t\to 0} \overline{A}_\infty(t) = \lim\limits_{t\to\infty} \overline{A}_\infty(t) = 0$, *where* $\overline{A}_\infty(t)$ *is from* (6.4).

Theorem 6.12. *Let p and q satisfy the conditions of Theorem* 6.6. *Then the following statements hold:*

(i) *If* (6.5) *holds and* $\lim\limits_{t\to 0} G(t) = 0$, *then the operator* $\overline{H}_{v,w}\colon L^{p(\cdot)}(0,1) \to L^{q(\cdot)}(0,1)$ *is compact.*

(ii) *If* $\rho_{p'}(w) < \infty$ *and if* $H_{v,w}\colon L^{p(\cdot)}(0,1) \to L^{q(\cdot)}(0,1)$ *is compact, then* (6.7) *holds and* $\lim\limits_{t\to 0} F(t) = 0$.

Theorem 6.13. *Let $p(x)$ and $q(x)$ satisfy the conditions of Theorem* 6.7. *Then the following statements hold:*

(i) *If* (6.7) *holds and* $\lim\limits_{t\to 0} G(t) = \lim\limits_{t\to\infty} G(t) = 0$ *then the operator* $\widetilde{H}_{v,w}$ *is compact from* $L^{p(\cdot)}(\mathbb{R}_+)$ *to* $L^{q(\cdot)}(\mathbb{R}_+)$.

(ii) *Let* $\int_0^a w^{p'(\tau)}\,\mathrm{d}\tau < \infty$. *If* $\widetilde{H}_{v,w}\colon L^{p(\cdot)}(\mathbb{R}_+) \to L^{q(\cdot)}(\mathbb{R}_+)$ *is compact then* (6.8) *holds and* $\lim\limits_{t\to 0} F_\infty(t) = \lim\limits_{t\to\infty} F_\infty(t) = 0$.

The next statement gives the estimate of the distance of $H_{v,w}$ from the space of finite rank operators acting from $L^{p(\cdot)}$ to $L^{q(\cdot)}$. We denote this distance by $\alpha(H_{v,w})$.

Theorem 6.14. *Let functions p and q satisfy the conditions of Theorem* 6.1. *Suppose that $cB \le 1$, where B is from* (6.1) *and* $c = 2^{2\bar{q}+\bar{q}/\underline{p}}$. *Then*

$$\lim_{a\to 0} \overline{A}_a^{1/\underline{q}} \le \alpha(H_{v,w}) \le c^{1/\bar{q}} \lim_{a\to 0} B_a^{1/\bar{q}},$$

where

$$B_a \equiv \sup_{0<t<a} \int_t^a v(x)^{q(x)} \left(\int_0^t w^{p_0'(x)}(\tau)\,\mathrm{d}\tau \right)^{q(x)/p_0'(x)} \mathrm{d}x,$$

$$\overline{A}_a \equiv \sup_{0<t<a} \overline{A}_a(t) \equiv \sup_{0<t<a} \int_t^a v(x)^{q(x)} \left(\int_0^t w^{p'(\tau)}(\tau)\,\mathrm{d}\tau \right)^{q(x)/\bar{p}'} \mathrm{d}x.$$

In the lower estimate we assume that $\rho_{p'}(w) \le 1$.

Example 6.1. Let $w(x) = x^\beta$ and $v(x) = x^{\alpha/p(x)}$, where either $\beta \geq 0$ and $-\beta\bar{p}'(\underline{p}-1) - \underline{p} \leq \alpha < -1$ or $-1/\underline{p}' < \beta < 0$ and $-\underline{p} - \beta\underline{p} \leq \alpha < -1$. Then $H_{v,w}$ is bounded in $L^{(p(\cdot)}(0,1)$. If either $\beta \geq 0$ and $-\beta\bar{p}'(\underline{p}-1) - \underline{p} < \alpha < -1$ or $-1/\underline{p}' < \beta < 0$ and $-\underline{p} - \beta p_- < \alpha < -1$, then $H_{v,w}$ is compact in $L^{p(\cdot)}(0,1)$.

Example 6.2. If $w(x) = x^\beta$, $v(x) = x^{\alpha/p(x)}$, where either $\beta < -1/p'_-$ and $\alpha > -1$ or $\beta > 0$ and $\alpha > -1$, then $\overline{H}_{v,w}$ is compact in $L^{p(\cdot)}(0,1)$.

All the results presented in this section are contained in [EKM2].

At last we note that the Hardy inequalities with power weights in $L^{p(\cdot)}$ were treated in [KS1], [S6], [S7], [HHP].

7. Weighted inequalities for the Cauchy singular integral operator

One of the main goal of our investigation is the Cauchy singular integral

$$(S_\Gamma f)(t) = \frac{1}{\pi i}\int_\Gamma \frac{f(\tau)\,\mathrm{d}\tau}{\tau - t}, \quad t \in \Gamma,\ f \in L_1(\Gamma).$$

In the case that the operator $S_\Gamma\colon f \to S_\Gamma f$ is bounded in $L^{p(\cdot)}(\Gamma)$ we denote its norm as $\|S\|_{p(\cdot)}$. Recall that Γ is called the Lyapunov curve if $t' \in \operatorname{Lip}\beta$, $0 < \beta \leq 1$, and it is called the *curve of bounded curvature (Radon curve)* if t' is a function of bounded variation on $[0,l]$. The definition of a regular (Carleson) curve was given in Section 5.

Theorem 7.1 [KS6]**.** *Let Γ be a bounded Carleson curve and let $p \in \mathcal{V}(\Gamma)$. Then S_Γ is bounded in $L^{p(\cdot)}_w(\Gamma)$ with*

$$w(t) = \prod_{k=1}^{n} |t - t_k|^{\alpha_k},$$

where t_k are distinct points of Γ, if and only if

$$-\frac{1}{p(t_k)} < \alpha_k < \frac{1}{q(t_k)}, \quad q(t) = \frac{p(t)}{p(t)-1}.$$

This result can be obtained in a more general form.

Theorem 7.1'. *Let Γ be a bounded Carleson curve and let $p \in \mathcal{V}(\Gamma)$. Suppose that α and β are real-valued functions satisfying the w-*Lip *condition on Γ. Let $w(t) = |t-t_0|^{\alpha(t)}$ and $w_1(t) = |t-t_0|^{\beta(t)}$, $t_0 \in \Gamma$. Then the operator S_Γ is bounded from $L^{p(\cdot)}_w(\Gamma)$ into $L^{p(\cdot)}_{w_1}(\Gamma)$ if and only if*

$$-\frac{1}{p(t_0)} < \alpha(t_0) \le \beta(t_0) < \frac{1}{p'(t_0)}.$$

Via the solution of boundary value problems for analytic functions in the class of the Cauchy-type integrals with a density in $L^{p(\cdot)}$ we are able to obtain the weight results for S_Γ in $L^{p(\cdot)}$-space when the weight is not necessarily of power type. We use the following notation:

$$\mathcal{R}^{p(\cdot)} = \{\Gamma : S_\Gamma \text{ is bounded in } L^{p(\cdot)}(\Gamma)\}$$

and

$$W^{p(\cdot)}(\Gamma) = \{\sigma : \sigma S_\Gamma \tfrac{1}{\sigma} \text{ is bounded in } L^{p(\cdot)}(\Gamma)\}.$$

Theorem 7.2 [KPS]**.** *Let $p \in \mathcal{V}(\Gamma)$, let $\Gamma \in \mathcal{R}^{p(\cdot)}$ and let φ be a real-valued function in $C(\Gamma)$. Then the function σ,*

$$\sigma(t) = \left| \exp\left(\frac{1}{2\pi}\int_\Gamma \frac{\varphi(\tau)\,\mathrm{d}\tau}{\tau - t}\right)\right|,$$

belongs to the class $W^{p(\cdot)}(\Gamma)$.

Theorem 7.3. *Let p and φ satisfy the assumptions of Theorem 7.2. Let $\alpha\colon \Gamma \to \mathbb{R}^1$ satisfy the w-*Lip(Γ) *condition. Then the function σ_1,*

$$\sigma_1(t) = |t-t_0|^{\alpha(t)} \left| \exp\left(\frac{1}{2\pi}\int_\Gamma \frac{\varphi(\tau)\,\mathrm{d}\tau}{\tau - t}\right)\right|, \quad t_0 \in \Gamma,$$

belongs to $W^{p(\cdot)}$ if

$$-\frac{1}{p(t_0)} < \alpha_k(t_0) < \frac{1}{q(t_0)}.$$

Theorem 7.4 [KPS]**.** *Let Γ be a Lyapunov curve or a Radon curve and let $p \in \mathcal{V}(\Gamma)$. Assume that*

$$\operatorname*{ess\,sup}_{\tau\in\Gamma} |\alpha(\tau)| < \min\bigl\{2\operatorname{arcctg} \|S_{\Gamma_0}\|_{p(t(\cdot))}, \pi/\overline{q}\bigr\}.$$

Then the function σ,

$$\sigma(t) = \left| \exp\left(\frac{1}{2\pi} \int_\Gamma \frac{\alpha(\tau)}{\tau - t} \, \mathrm{d}\tau \right) \right|,$$

belongs to $W^{p(\cdot)}(\Gamma)$.

Theorem 7.4 contains the sufficient part of the well-known Helson-Szegö result. The following example given on the basis of Theorem 7.3 is of interest, in our opinion. Let t_k, $k = 1, 2, \ldots$, be arbitrary distinct points on Γ. Then the function

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}$$

belongs to $W^{p(\cdot)}(\Gamma)$ under the conditions

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{q(t_k)}, \quad k = 1, 2, \ldots,$$

and

$$\left| \sum_{k=1}^{n} \beta_k \right| < \infty.$$

Finally, it should be emphasized that the results presented in the survey turned out to be for us an important tool for solving a variety of problems in several areas of analysis and its application. Namely, in the development of the theory of function spaces (Triebel-Lizorkin, Bessel, etc.) generated by the norm of variable exponent Lebesgue spaces, in BVP in general setting for the elliptic partial differential equations in "bad" domains, generalized analytic functions, and so on. To keep the length of this paper reasonable, we will not discuss these topics here.

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