Relationship between the effective thermal properties of linear and nonlinear doubly periodic composites

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The present paper is devoted to the study of the effective properties of 2D unbounded composite materials with temperature dependent conductivities. We consider a special case of nonlinear composites, when the conductivity coefficients of the matrix and the composite constituencies are proportional. This allows us to transform the problem for the nonlinear composite to a problem for an equivalent linear composite and then to find a solution of the nonlinear type. Analyzing the effective properties of the composites we derive relationships between their average properties. We show that, when computing the effective properties of the representative cell of the nonlinear composite, the result may depend not only on the temperature but also on its gradient.

1 Introduction

Composite models with constant conductivities (linear models) have been exhaustively investigated by many authors, using various methods dependent on the composite structures. In the case of randomly distributed components, effective properties of such composites were successfully studied, for example, in [8, 10, 11, 13, 14, 29–31], while analytical and numerical results for composites with periodic structure can be found in [4, 12, 15, 16, 32, 33]. An extensive and complete overview of the methods employed can be found in the fundamental work [17].

The results for both types of the composite materials can be compared to each other in the case of small concentrations of the components, when the interactions between them do not play an essential role (cf. [15, 19]). The upper and lower bounds are extremely useful tools for practical evaluation of the effective constants (cf. [20–22]). One of the important features of the linear problems (composites with constant physical properties) is the fact that, although the temperature can be defined here with an accuracy up to a constant, the effective properties are independent of the applied external field (cf. [4]).

Nonlinear models of the components can be divided into two major classes. The first class, when the material properties depend on the gradient of the temperature, has been discussed in many works. The respective mathematical methods are well developed and can be found in [2, 23–27].

The second class of nonlinear composites corresponds to the case when the material properties depend only on the solution and is less developed in the literature. The probable first fundamental theoretical result concerning the homogenization problem for nonlinear composites of that type was published in [3]. In [7], the elementary and Hashin-Shtrikman type bounds were extended for nonlinear models and further analyzed in [28]. The prevailing conclusion is that homogenisation of nonlinear composites with temperature dependent components can be treated in the same manner as the linear ones. Unfortunately, there are few examples when the solution can be found effectively. In the one dimensional case such results can be found in [1], while in the two dimensional case such results have been recently obtained for composites with the component conductivities proportional to that in the matrix in [2] for random composites, and in [6] for periodic composites. However, even in this rather special case, the Eshelby approach that is so effective for linear composites is not applicable (cf. [2]).

This paper is devoted to further analysis of nonlinear composites of periodic structure, when material characteristics depend on the temperature and are proportional to each other. Specifically, using the fact that the solution can be obtained analytically, we compute the effective conductivity of a separate cell of the nonlinear composites and establish a relationship between its average properties and the corresponding model of a linear composite.

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We show that, even for this very specific case of nonlinear composites, the theory is much more complex than for the linear equivalent. For instance, formally computed effective properties may depend not only on temperature but also on its gradient. An exact solution to the problem allows us to evaluate a formula relating effective properties of the representative cells of linear and nonlinear composites and to demonstrate that the conclusions reached on the average properties discussed in [2, 7, 28] are valid only when the heat flux is negligibly small. We also deliver an estimate of those properties when the difference between those results and the exact value is small.

The paper is organized as follows. In Sect. 2, we describe a geometry for the considered periodic composite materials, and formulate the nonlinear boundary value problem and its linear equivalent. In Sect. 3, we establish a relationship between the effective conductivity tensors of the representative cells of the linear and nonlinear composites. We then give and discuss numerical examples in Sect. 4. The paper then closes with the Conclusion.

2 Statement of the problem and preliminary results

We consider a lattice defined by the two fundamental translation vectors 1 and $\imath$ (where $\imath^2 = -1$) in the complex plane $\mathbb{C} \cong \mathbb{R}^2$ of the complex variable $z = x + \imath y$. The representative cell is the unit square

$$Q_{(0,0)} := \left\{ z = t_1 + \imath t_2 \in \mathbb{C} : -\frac{1}{2} < t_p < \frac{1}{2},\ p = 1, 2 \right\}.$$

Let $\mathcal{E} := \bigcup_{m_1,m_2} \{m_1 + \imath m_2\}$ be the set of the lattice points, where $m_1, m_2 \in \mathbb{Z}$. The cells corresponding to the points of the lattice $\mathcal{E}$ are denoted by

$$Q_{(m_1,m_2)} = Q_{(0,0)} + m_1 + \imath m_2 := \left\{ z \in \mathbb{C} : z - m_1 - \imath m_2 \in Q_{(0,0)} \right\}.$$

Mutually non-overlapping disks (inclusions) of different radii $D_k := \{z \in \mathbb{C} : |z - a_k| < r_k\}$ with boundaries $\partial D_k := \{z \in \mathbb{C} : |z - a_k| = r_k\}$ $(k = 1, 2, \ldots, N)$ are located inside the cell $Q_{(0,0)}$ and periodically repeated in all cells $Q_{(m_1,m_2)}$. Let

$$D_0 := Q_{(0,0)} \setminus \left(\bigcup_{k=1}^{N} D_k \cup \partial D_k \right)$$

be the connected domain obtained by removal of the inclusions from the cell $Q_{(0,0)}$. We consider the situation when the matrix and inclusions occupy domains

$$D_{matrix} = \bigcup_{m_1,m_2} ((D_0 \cup \partial Q_{(0,0)}) + m_1 + \imath m_2)$$

and

$$D_{inc} = \bigcup_{m_1,m_2} \bigcup_{k=1}^{N} (D_k + m_1 + \imath m_2)$$

with thermal-sensitive conductivities $\lambda_m = \lambda_m(T)$ and $\lambda_k = \lambda_k(T)$, respectively. Here, temperature T is defined in the whole of $\mathbb{R}^2$. In general, the conductivities λ_m, λ_k $(k = 1, \ldots, N)$ are continuous bounded positive functions on $\mathbb{R}$.

The purpose of this paper is to investigate properties of the effective conductivity tensor for steady-state distribution of the temperature and heat flux within such a nonlinear composite when the temperature function $T = T(x, y)$ satisfies the nonlinear differential equations

$$\nabla(\lambda_m(T)\nabla T) = 0,\ \ (x, y) \in D_{matrix}, \tag{1}$$

$$\nabla(\lambda_k(T)\nabla T) = 0, \quad (x, y) \in D_{inc}. \tag{2}$$

We assume that the perfect (ideal) contact conditions on the boundaries between the matrix and inclusions hold:

$$T(s) = T_k(s), \quad s \in \bigcup_{m_1,m_2} (\partial D_k + m_1 + \imath m_2), \tag{3}$$

$$\lambda_m(T(s))\frac{\partial T(s)}{\partial n} = \lambda_k(T_k(s))\frac{\partial T_k(s)}{\partial n}, \quad s \in \bigcup_{m_1,m_2} (\partial D_k + m_1 + \imath m_2). \tag{4}$$

Here, the vector n is the outward unit normal vector to ∂D_k. According to the formulation, the flux and the temperature are continuous functions in the entire structure. The average flux vector of intensity A is directed at an angle θ to axis Ox which does not coincide, in general, with the orientation of the periodic cell:

$$\int_{\partial Q^{(top)}_{(m_1,m_2)}} \lambda_m(T) T_y ds = -A\sin\theta, \tag{5}$$

$$\int_{\partial Q^{(right)}_{(m_1,m_2)}} \lambda_m(T) T_x ds = -A\cos\theta. \tag{6}$$

2.1 Linear composite

Note that, if the conductivities λ_m and λ_k are constants, we have the linear partial differential equation

$$\Delta T = 0, \ (x, y) \in D_{matrix} \cup D_{inc}, \tag{7}$$

with linear boundary conditions

$$T(s) = T_k(s), \quad s \in \bigcup_{m_1,m_2} (\partial D_k + m_1 + \imath m_2), \tag{8}$$

$$\lambda_m \frac{\partial T(s)}{\partial n} = \lambda_k \frac{\partial T_k(s)}{\partial n}, \quad s \in \bigcup_{m_1,m_2} (\partial D_k + m_1 + \imath m_2), \tag{9}$$

$$\lambda_m \int\limits_{\partial Q^{(top)}_{(m_1,m_2)}} T_y ds = -A\sin\theta, \quad \lambda_m \int\limits_{\partial Q^{(right)}_{(m_1,m_2)}} T_x ds = -A\cos\theta. \tag{10}$$

As we discussed in the introduction, such a problem can be completely solved by various methods. Here, we present only some results important for the nonlinear composite considered below.

Theorem 2.1. *For any given data θ and A the linear boundary value problem* (7)–(10) *with ideal (perfect) contact conditions has a unique (modulo real constants) real analytic solution.*

For the proof cf. [4, 5].

Note that an additional condition on the temperature T such as

$$T(x_0, y_0) = t_0, \quad (x_0, y_0) \in D_{matrix} \cup D_{inc}, \tag{11}$$

where t_0 is a given value of the temperature at any point (x_0, y_0), will give us a unique solvability result. Specifically, the following corollary holds true.

Corollary 2.2. *The boundary value problem* (7)–(11) *has a unique real analytic solution.*

Remark 2.3. Note that the calculation of the effective conductivity tensor Λ defined by the formula

$$\langle \lambda \nabla T \rangle = \Lambda \langle \nabla T \rangle, \tag{12}$$

where

$$\lambda = \lambda(z) = \begin{cases} \lambda_m, & z \in D_{matrix}, \\ \lambda_k, & z \in D_{inc}, \end{cases}$$

does not depend on the chosen cell, cf. [4, 8]. Moreover, since the constants disappear in the expression ∇T, the effective properties of the linear composites are independent of the additional condition (11) and, crucially, of the applied flux A.

2.2 Nonlinear composite with proportional components

In this section, investigation of the effective conductivity tensor Λ_n corresponding to the nonlinear boundary value problem (1)–(6) is carried out under the assumption that the ratio of the component conductivities is a given constant C:

$$\frac{\lambda_m(T)}{\lambda_k(T)} = C \quad \text{for all} \quad k = 1, \dots, N. \tag{13}$$

In this case, the problem (1)–(6) is linearized and solved in [6].

Theorem 2.4 (cf. [6]). *The boundary value problem* (1)–(6), (11) *has a unique real analytic solution.*

Let us recall that we have established in [6] a bijection between solutions of the linear and nonlinear boundary value problems via the Kirchhoff transformation (cf. [9] and (26), (27) below). This fact, together with Theorem 2.1, allows us to describe a solution of the boundary value problem (1)–(6) complemented by the condition (11), and to prove the effectiveness of the numerical algorithm for evaluation of the effective properties of the composite.

Let us denote by T_n a solution to the nonlinear boundary value problem (1)–(6), and

$$\lambda_n(T_n(z)) = \begin{cases} \lambda_m(T_n(z)), & z \in D_{matrix}, \\ \lambda_k(T_n(z)), & z \in D_{inc}. \end{cases} \tag{14}$$

Following [6], we define the effective conductivity tensor Λ_n of the representative cell of a nonlinear composite by the same formula (12):

$$\langle \lambda_n(T_n)\nabla T_n \rangle = \Lambda_n \langle \nabla T_n \rangle. \tag{15}$$

It was shown in [6] that, in contrast to the linear composites, the effective conductivity tensor Λ_n varies from cell to cell and thus represents a function of the problem solution. The question arises: what would be a minimal set of variables that defines this function uniquely? For a chosen example of the nonlinear composite and the averaged flux flowing through it, it was shown in [6] that the effective properties of the composite can be attributed to the average temperature:

$$\Lambda_n = \Lambda_n(\langle T_n \rangle). \tag{16}$$

Note that in case of holes ($\lambda_k(T) = 0$), this paper's assumption (13) is satisfied automatically, with $1/C = 0$. This special case was considered in [18] where, when considering average properties, an alternative reference parameter to the average temperature $\langle T \rangle$ was utilized. Specifically, the author considered the jump of the temperature over the unit cell as the parameter.

Below we will show that the tensor of the effective properties, Λ_n, defined according to (14), may depend not only on the average temperature, $\langle T_n \rangle$, but also the flux intensity, A,

$$\Lambda_n = \Lambda_n(\langle T_n \rangle, |A|). \tag{17}$$

The natural question may then be asked of whether such a periodic structure can be represented as a composite material possessing average properties, or whether it is only the nonlinear periodic structure and the respective physical problems that should be considered in the original formulation, without reference to its effective properties.

3 Relationship between the effective conductivity tensors of non linear and respective linear models

We denote by $T_l = T_l(\lambda_k, \lambda_m)$ and $\Lambda_l = \Lambda_l(\lambda_k, \lambda_m)$ a temperature and the effective conductivity tensor for the respective linear problem (7)–(10), respectively. Here, we establish the relationship between Λ_n and the effective conductivity tensor Λ_l for the linear problem with the same ratio of constant conductivities λ_m and λ_k:

$$\frac{\lambda_m}{\lambda_k} = C \quad \text{for all} \quad k = 1, \dots, N. \tag{18}$$

In this section, we assume that data r_k, a_k, C, θ, A defined in Sect. 2 are given and fixed.

Let $\lambda_l = \lambda_l(z)$ and $\lambda_0 = \lambda_0(z)$ be piecewise constant functions defined as follows

$$\lambda_l(z) = \begin{cases} \lambda_m, & z \in D_{matrix}, \\ \lambda_k, & z \in D_{inc}, \end{cases} \qquad \lambda_0(z) = \begin{cases} 1, & z \in D_{matrix}, \\ \frac{1}{C}, & z \in D_{inc}. \end{cases}$$

Lemma 3.1. *Let λ_k, $k = 1, \dots, N$, and λ_m be arbitrary positive constants satisfying the condition* (18). *Then, we have*

$$\Lambda_l(\lambda_k, \lambda_m) = \lambda_m \Lambda_l\left(C^{-1}, 1\right). \tag{19}$$

Lemma 3.1 is an immediate consequence of the definition of the effective conductivity and Theorem 2.1.

For further references, we recall the representation of the solution in terms of complex potentials. The linear boundary value problem (7)–(10) can be equivalently reduced to the corresponding R-linear conjugation problem on each contour $|t - a_k| = r_k$,

$$\varphi(t) = \varphi_k(t) - \rho_k \overline{\varphi_k(t)} - Bt, \tag{20}$$

for unknown complex analytic functions $\varphi, \varphi_1, \ldots, \varphi_N$ connected with the function T_l by the following equality (cf. [4]):

$$T_l(\lambda_k, \lambda_m; z) = \begin{cases} \mathrm{Re}\,(\varphi(z) + Bz), & z \in D_{matrix}, \\ \frac{2\lambda_m}{\lambda_m+\lambda_k}\,\mathrm{Re}\,\varphi_k(z), & z \in D_{inc}. \end{cases} \tag{21}$$

Here, $\rho_k = \frac{\lambda_k-\lambda_m}{\lambda_k+\lambda_m} = \frac{1-C}{1+C}$, $B = \frac{1}{\lambda_m}\widetilde{B}$, where

$$\widetilde{B} = -\frac{A\cos\theta}{I+1} - \frac{A\cos\theta}{I^{\perp}-1}i, \tag{22}$$

is a constant complex number with some real constants I and $I^{\perp}$ uniquely determined by r_k, a_k, C (cf. [4]). Note that the constant $\widetilde{B}$ does not depend on λ_m or λ_k but does depend on C. Then, from (20) we get

$$\widetilde{\varphi}(t) = \widetilde{\varphi}_k(t) - \rho_k\overline{\widetilde{\varphi}_k(t)} - \widetilde{B}t,$$

where $\widetilde{\varphi}(t) = \lambda_m\varphi(t)$, $\widetilde{\varphi}_k(t) = \lambda_m\varphi_k(t)$, and

$$T_0(z) = \begin{cases} \mathrm{Re}\,(\widetilde{\varphi}(z) + \widetilde{B}z), & z \in D_{matrix}, \\ \frac{2C}{C+1}\,\mathrm{Re}\,\widetilde{\varphi}_k(z), & z \in D_{inc}. \end{cases} \tag{23}$$

Thus, we have

$$\lambda_m T_l(\lambda_k, \lambda_m; z) = T_0(z). \tag{24}$$

Lemma 3.2. *Let the functions λ_m and λ_k satisfy the condition* (13). *Then*

$$\langle\lambda_n(T_n)\nabla T_n\rangle = \langle\lambda_0\nabla T_0\rangle. \tag{25}$$

P r o o f . Using the same ideas as in [4], the nonlinear boundary value problem under the condition (13) can be reduced to the boundary R-linear conjugation problem on each contour $|t - a_k| = r_k$:

$$\frac{2}{C+1}\varphi(t) = \varphi_k(t) + \frac{C-1}{C+1}\overline{\varphi_k(t)} - \frac{2\widetilde{B}}{C+1}t,$$

where $\widetilde{B}$ is defined in (22). Denoting $\frac{C+1}{2}\varphi_k(t) = \widetilde{\varphi}_k(t)$ and $\varphi(t) = \widetilde{\varphi}(t)$, we get

$$\widetilde{\varphi}(t) = \widetilde{\varphi}_k(t) - \rho_k\overline{\widetilde{\varphi}_k(t)} - \widetilde{B}t,$$

where $\rho_k = \frac{\lambda_k(T)-\lambda_m(T)}{\lambda_k(T)+\lambda_m(T)} = \frac{1-C}{1+C}$ (cf. (20)).

Following [4] and [6], a solution of the nonlinear problem (1)–(6) can be found in the form

$$u(z) = \begin{cases} \mathrm{Re}\,(\varphi(z) + \widetilde{B}z), & z \in D_{matrix}, \\ \mathrm{Re}\,\varphi_k(z), & z \in D_{inc}, \end{cases}$$

where

$$f_m(T) = \int_0^T \lambda_m(\xi)\,d\xi, \quad f_k(T) = \int_0^T \lambda_k(\xi)\,d\xi, \tag{26}$$

and

$$u(x,y) = f_m(T(x,y)), \quad u_k(x,y) = f_k(T_k(x,y)). \tag{27}$$

Or, in terms of $\widetilde{\varphi}$ and $\widetilde{\varphi}_k$, we can rewrite

$$u(z) = \begin{cases} \mathrm{Re}\,(\widetilde{\varphi}(z) + \widetilde{B}z), & z \in D_{matrix}, \\ \frac{2}{C+1}\mathrm{Re}\,\widetilde{\varphi}_k(z), & z \in D_{inc}. \end{cases} \tag{28}$$

Comparing equalities (23) and (28), we get

$$u(z) = T_0(z), \quad \text{for} \quad z \in D_{matrix}, \quad u_k(z) = \frac{1}{C}T_0(z), \quad \text{for} \quad z \in D_{inc}.$$

For a solution T_n of the nonlinear boundary value problem (1)–(6), we have

$$T_n(z) = f_m^{-1}(u(z)) = f_m^{-1}(T_0(z)), \quad \text{for} \quad z \in D_{matrix}, \tag{29}$$

$$T_n(z) = f_k^{-1}(u_k(z)) = f_k^{-1}(C^{-1}T_0(z)), \quad \text{for} \quad z \in D_{inc}, \tag{30}$$

or

$$f_m(T_n(z)) = T_0(z), \quad \text{for} \quad z \in D_{matrix}, \quad Cf_k(T_n(z)) = T_0(z), \quad \text{for} \quad z \in D_{inc}. \tag{31}$$

Since $f_m'(T) = \lambda_m(T)$ and $f_k'(T) = \lambda_k(T)$, after differentiating the equalities (31) by x and y, and taking into account condition (13), we arrive at

$$\lambda_m(T_n)\nabla T_n = \nabla T_0, \quad \text{for} \quad z \in D_{matrix},$$

$$\lambda_k(T_n)\nabla T_n = \frac{1}{C}\nabla T_0, \quad \text{for} \quad z \in D_{inc}, \tag{32}$$

which imply (25). □

We can now establish a relation between the effective conductivity tensors of the linear and nonlinear models.

Theorem 3.3. *If the conditions* (13) *and* (18) *are fulfilled then the effective conductivity tensor* Λ_n *of a nonlinear model is related to the effective conductivity tensor* Λ_l *of a linear model by the following equality:*

$$\Lambda_n\langle\nabla T_n\rangle = \Lambda_l(C^{-1}, 1)\langle\nabla T_0\rangle. \tag{33}$$

P r o o f. Using equality (25) from Lemma 3.2, definition (15) can be rewritten as

$$\Lambda_n\langle\nabla T_n\rangle = \langle\lambda_0\nabla T_0\rangle = \Lambda_l(C^{-1}, 1)\langle\nabla T_0\rangle.$$

□

Taking (13) into account, the equalities (32) give

$$\lambda_m(T_n)\nabla T_n = \nabla T_0. \tag{34}$$

Thus, equality (33) can be rewritten as

$$\Lambda_n\left\langle\frac{\nabla T_0}{\lambda_m(T_n)}\right\rangle = \Lambda_l(C^{-1}, 1)\langle\nabla T_0\rangle. \tag{35}$$

Here, it is worth mentioning that in some practical problems we may have sufficiently small $|\lambda_m(T_n(z)) - \lambda_m(\langle T_n(z)\rangle)|$ for all $z \in Q_{(0,0)}$. In this case we can substitute $\lambda_m(\langle T_n(z)\rangle)$ for $\lambda_m(T_n(z))$ in (34), which gives

$$\langle\nabla T_n(z)\rangle \approx \frac{1}{\lambda_m(\langle T_n(z)\rangle)}\langle\nabla T_0(z)\rangle,$$

where the latter becomes

$$\Lambda_n \approx \lambda_m(\langle T_n(z)\rangle)\Lambda_l(C^{-1}, 1). \tag{36}$$

A similar relation can be obtained for an arbitrary cell $Q_{(m_1,m_2)}$. This relation was first obtained in [2] for random composites without discussion on its applicability.

Theorem 3.4. *Let the functions* f_m *defined in* (26) *and* λ_m *be Lipschitz continuous with Lipschitz constants* C_f *and* C_λ*, respectively. Then for any arbitrarily small value* $\varepsilon > 0$*, there exists* $A_* > 0$*,* ($A_* = A_*(\varepsilon)$) *such that for any intensity satisfying an equality* $|A| < A_*$ *the following estimate holds,*

$$\left|(\Lambda_n)_{ij} - \lambda_m(\langle T_n(z)\rangle)\Lambda_l(C^{-1}, 1)_{ij}\right| < \varepsilon, \tag{37}$$

for all $i, j = 1, 2$.

P r o o f. Let us denote by $T_{0,1}(z)$ a solution $T_0(z)$ in (23) when $A = -1$, then we have

$$T_0(z) = |A|\, T_{0,1}(z),$$

for any $A < 0$. Further, let us note that

$$|T_{0,1}(z) - \langle T_{0,1}(z)\rangle| \le M, \tag{38}$$

for any $z \in Q_{(m_1,m_2)}$, where $< \cdot >$ is taken with respect to the cell $Q_{(m_1,m_2)}$ and for some positive constant $M > 0$ which is independent of the chosen cell $Q_{(m_1,m_2)}$. Note that the constant M depends on the contrast parameter C as well as the specific distribution of the inclusions within the unit cell for the respective linear composite.

Since the function f_m^{-1} is Lipschitz continuous with a Lipschitz constant C_f, we get

$$|f_m^{-1}(|A|\,T_{0,1}(z)) - f_m^{-1}(\langle |A|\,T_{0,1}(z)\rangle)| \le C_f \left||A|\,T_{0,1}(z) - \langle |A|\,T_{0,1}(z)\rangle\right| \le C_f|A|M.$$

Thus,

$$|f_m^{-1}(|A|\,T_{0,1}(z))| \le |f_m^{-1}(\langle |A|\,T_{0,1}(z)\rangle)| + C_f|A|M,$$

and, obviously,

$$\begin{aligned}|\langle f_m^{-1}(|A|\,T_{0,1}(z))\rangle| &\le \langle |f_m^{-1}(|A|\,T_{0,1}(z))|\rangle \\ &\le \langle |f_m^{-1}(\langle |A|\,T_{0,1}(z)\rangle)|\rangle + \langle C_f|A|\,M\rangle \\ &= |f_m^{-1}(\langle |A|\,T_{0,1}(z)\rangle)| + C_f|A|\,M.\end{aligned}$$

Further, using the Lipschitz continuity of the function λ_m, together with (29) and the estimates obtained above, we get

$$\begin{aligned}|\lambda_m(T_n(z)) - \lambda_m(\langle T_n(z)\rangle)| &\le C_\lambda |T_n(z) - \langle T_n(z)\rangle)| \\ &= C_\lambda |f_m^{-1}(T_0(z)) - \langle f_m^{-1}(T_0(z))\rangle| \\ &= C_\lambda |f_m^{-1}(|A|\,T_{0,1}(z)) - f_m^{-1}(\langle |A|\,T_{0,1}(z)\rangle) \\ &\quad + f_m^{-1}(\langle |A|\,T_{0,1}(z)\rangle) - \langle f_m^{-1}(|A|\,T_{0,1}(z))\rangle| \\ &\le 2C_\lambda C_f|A|\,M.\end{aligned}$$

Let us set

$$\begin{aligned}\lambda_{\min} &:= \inf_{\xi\in\mathbb{R}}\{\lambda_m(\xi)\}, \\ \lambda_{\max} &:= \sup_{\xi\in\mathbb{R}}\{\lambda_m(\xi)\}, \\ \lambda_{k,\max} &:= \sup_{\xi\in\mathbb{R}}\{\lambda_k(\xi)\}, \\ \Lambda_{\max} &:= \max\{\lambda_{max}, \lambda_{k,max}\}.\end{aligned}$$

Since

$$\begin{aligned}\left|\frac{1}{\lambda_m(T_n(z))} - \frac{1}{\lambda_m(\langle T_n(z)\rangle)}\right| &= \frac{|\lambda_m(\langle T_n(z)\rangle) - \lambda_m(T_n(z))|}{\lambda_m(T_n(z))\lambda_m(\langle T_n(z)\rangle)} \\ &\le \frac{|\lambda_m(\langle T_n(z)\rangle) - \lambda_m(T_n(z))|}{\lambda_{min}^2} \\ &\le \frac{2C_\lambda C_f|A|\,M}{\lambda_{min}^2},\end{aligned}$$

we obtain

$$\left|\left\langle \frac{(\nabla T_0)_j}{\lambda_m(T_n(z))} - \frac{(\nabla T_0)_j}{\lambda_m(\langle T_n(z)\rangle)}\right\rangle\right| \le \frac{2C_\lambda C_f|A|\,M}{\lambda_{min}^2}|\langle(\nabla T_0)_j\rangle|$$

and, consequently,

$$\left|\left\langle \frac{(\nabla T_0)_j}{\lambda_m(T_n(z))}\right\rangle - \frac{\langle(\nabla T_0)_j\rangle}{\lambda_m(\langle T_n(z)\rangle)}\right| \le \frac{2C_\lambda C_f|A|\,M}{\lambda_{min}^2}|\langle(\nabla T_0)_j\rangle|.$$

Further, we have

$$\left|(\Lambda_l(C^{-1},1))_{ij}\langle(\nabla T_0)_j\rangle - (\Lambda_n)_{ij}\frac{\langle(\nabla T_0)_j\rangle}{\lambda_m(\langle T_n(z)\rangle)}\right|$$

$$= \left|(\Lambda_n)_{ij}\left\langle\frac{(\nabla T_0)_j}{\lambda_m(T_n)}\right\rangle - (\Lambda_n)_{ij}\frac{\langle(\nabla T_0)_j\rangle}{\lambda_m(\langle T_n(z)\rangle)}\right|$$

$$\le \Lambda_{\max}\left|\left\langle\frac{(\nabla T_0)_j}{\lambda_m(T_n)}\right\rangle - \frac{\langle(\nabla T_0)_j\rangle}{\lambda_m(\langle T_n(z)\rangle)}\right|$$

$$\le \frac{2C_\lambda C_f|A|\,M\Lambda_{max}}{\lambda_{min}^2}|\langle(\nabla T_0)_j\rangle|,$$

and we finally obtain

$$\left|(\Lambda_l(C^{-1},1))_{ij}\lambda_m(\langle T_n(z)\rangle) - (\Lambda_n)_{i,j}\right| \le \frac{|A|}{a_*} \tag{39}$$

for all $i, j = 1, 2$, where

$$a_* := \frac{\lambda_{min}^2}{2\lambda_{max}\Lambda_{max}C_\lambda C_f M}.$$

Thus, we have (37) provided $|A| \le A_* = \varepsilon a_*$. □

One can also conclude that for small values of $\varepsilon \ll 1$ (or equivalently for small values of the heat flux $|A| \ll 1$, the following estimate holds true

$$\delta_\Lambda(A) = |A|^{-1}\max_{z\in Q_{(0,0)}}\left|(\Lambda_l(C^{-1},1))_{ij}\lambda_m(\langle T_n(z)\rangle) - (\Lambda_n)_{i,j}\right| = O(1), \quad A\to 0. \tag{40}$$

$$\lim_{|A|\to 0}\delta_\Lambda(A) = \delta_\Lambda^* \le 1/a_*.$$

Analogously to (39), we define

$$\delta_M(A) = |A|^{-1}\max_{z\in Q_{(0,0)}}\left|\langle T_n(z)\rangle - T_n(z)\right| \le 2C_f\,M, \qquad \lim_{|A|\to 0}\delta_M(A) = \delta_M^*. \tag{41}$$

4 Numerical examples

In this section, we analyze the nonlinear composite considered in the paper [6] for various values of the flux intensity A. Specifically, we will show that in the case when the temperature changes dramatically within cells, the average properties formally defined by (35) will be different depending not only on the average temperature but on the flux intensity, A, which contradicts the statements in [2, 7] and [28].

We choose for calculations the representative cell $Q_{(0,0)}$, with one central inclusion of the radius $r_k = R = 0.29$, i.e. the volume fraction of the inclusion is 0.2642. We consider the same material properties, as discussed in [6], where the material conductivities are characterized by identical pike shapes satisfying the condition (13), and defined in the following form:

$$\lambda(T) = \begin{cases} \alpha_1, & T < \beta_1, \\ \alpha_2 + \dfrac{\alpha_1-\alpha_2}{\beta_1}T, & \beta_1 \le T \le 0, \\ \alpha_2 + \dfrac{\alpha_1-\alpha_2}{\beta_2}T, & 0 \le T \le \beta_2, \\ \alpha_1, & T > \beta_2, \end{cases} \tag{42}$$

where $\beta_1 = -2$, $\beta_2 = 2$, $\alpha_1 = 4.5$, $\alpha_2 = 13.5$ for the matrix conductivity, λ_m, $\alpha_1 = 50$, $\alpha_2 = 150$ for the inclusion conductivities, λ_k, and $C = 0.09$. We will examine the cases of flux intensities $|A| = 0.01; 0.1; 30; 100$.

The effective conductivity tensor $\Lambda_l(C^{-1}, 1)$ of the corresponding linear problem is

$$\Lambda_l(C^{-1},1) = 1.56676\,\mathbf{I}, \tag{43}$$

where **I** is the unit tensor.

In Figs. 1 and 2, we present the temperature distributions within one cell for three values of the flux intensity. It is clearly seen that the gradient of the temperature within the cell is closely correlated to the average heat flux intensity and essentially depends on the latter.

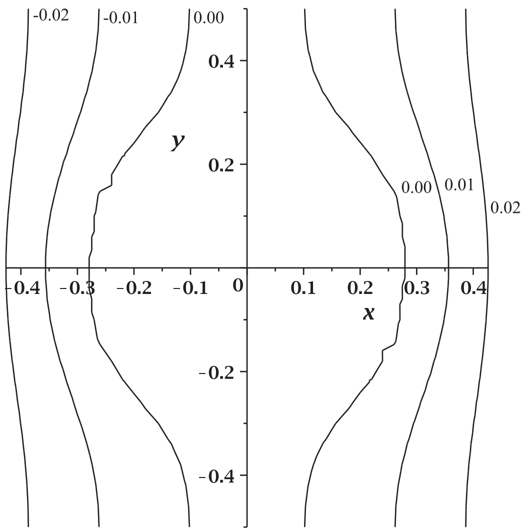

Fig. 1 Temperature distribution in the cell $Q_{(0,0)}$, with flux intensity $|A| = 1$, for the nonlinear composite.

In Fig. 3 we show the properties of the nonlinear composite as evaluated by the formula (37) for 3 different values of the average heat flux intensity: $|A| = 1, 30, 100$. Since, according to the composite geometry, we always obtain the unit tensor, only the diagonal component of the tensor Λ_n is presented, $(\Lambda_n(\langle T_n \rangle))_{11}$. For comparison, we also depict the value $\lambda_m(\langle T_n \rangle) \cdot \Lambda_l(C^{-1}, 1)$ corresponding to the model from [2]. Note that, for the smaller values of the average heat flux intensity, the difference between those solutions and the result obtained by the model from [2] is invisible in the scale of the graph.

Finally, we can evaluate the constants from the estimates (39) and (41) in order to determine the accuracy of the inequalities. For this reason, let us note that for the composite under consideration the functions λ_m and f_m^{-1} are characterized by their Lipschitz constants $C_\lambda = 4.5$, $C_f = 2/9$, while the constant M seen in inequality (38) can be estimated as $M \approx 0.077$. As a result, the constant $A_* = \varepsilon a_*$ in Theorem 3.4 can be computed to give $a_* \approx 0.06$. Additionally, we calculate constants $\delta_\Lambda^* \approx 16.67$ and $\delta_M^* \approx 0.034$.

We can expect that those two important estimates, (39) and (41), are rather conservative. Our computations confirm this at least for the chosen configuration of the composite. Indeed, the values of the constants from the estimate are given in Fig. 4.

Moreover, as a result of the computations we can suggest more accurate and even double-sided inequalities for this particular composite

$$\underline{\delta}_\Lambda \leq \delta_\Lambda(A) \leq \overline{\delta}_\Lambda, \quad \underline{\delta}_M \leq \delta_M(A) \leq \overline{\delta}_M, \tag{44}$$

where $\underline{\delta}_\Lambda \approx 0.09$, $\overline{\delta}_\Lambda \approx 4.46$ and $\underline{\delta}_M \approx 0.0195$, $\overline{\delta}_M \approx 0.0242$.

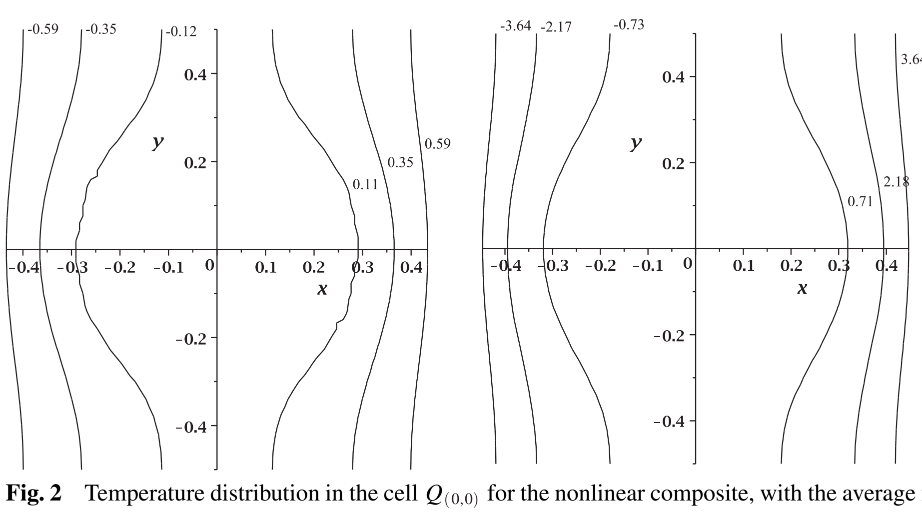

Fig. 2 Temperature distribution in the cell $Q_{(0,0)}$ for the nonlinear composite, with the average flux intensities $|A| = 30$ and $|A| = 100$.

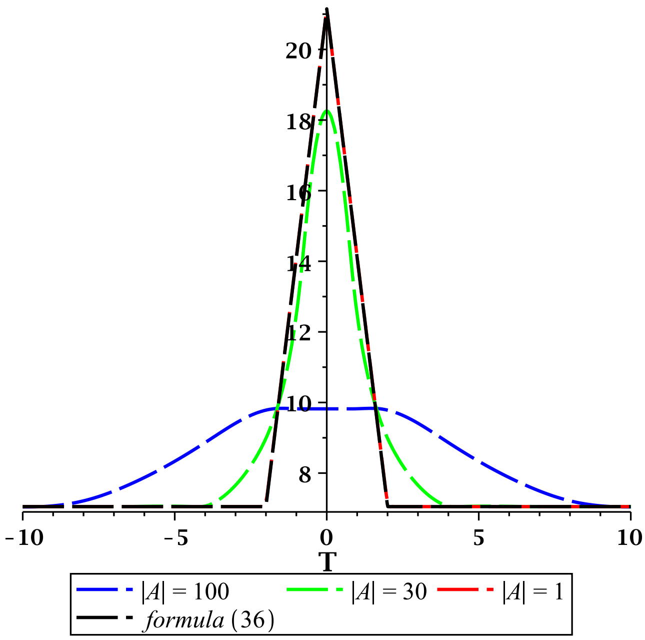

Fig. 3 Comparison of the diagonal components of the tensor $\lambda_m(\langle T_n\rangle)\cdot\Lambda_l(C^{-1},1)$ corresponding to the respective linear composite to the diagonal components of the effective conductivity tensors Λ_n of the nonlinear model, for different intensities of the flux.

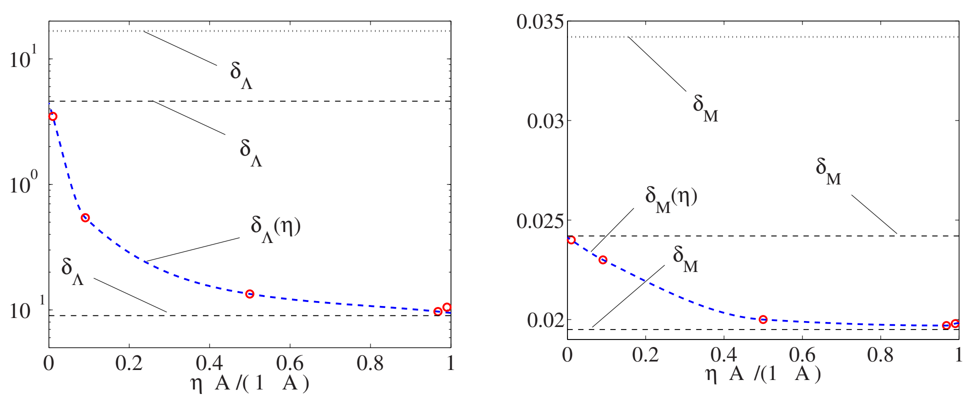

Fig. 4 Constants calculated from the estimates (39) and (41), for different values of the heat flux in the chosen geometry.

5 Conclusions

In the special case of proportional conductivities, we have shown that the effective properties of the representative cell of a nonlinear temperature dependent composite computed by the standard definition (15) may essentially depend on the intensity of the flux penetrating the composite. We have proved that, for a sufficiently small flux intensity, the result for such a nonlinear composite is indistinguishable from another obtained in [2]. However, if the flux intensity becomes sufficiently large, the result developed for the periodic model differs from the corresponding result obtained in [2], where the classic homogenization methods were utilised. In other words, if we use the simplest formula from [2] to solve a boundary value problem for the respective composite, it is of crucial importance to analyse at the postprocessing stage the level of heat flux locally developed in different parts of the nonlinear composite. In those parts of the composite where the flux is high enough, the average properties cannot be assumed without taking into account a specific structure of the composite and the flux level.

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