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Boundary-contact problems for domains with edge singularities

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Abstract

We study boundary-contact problems for elliptic equations (and systems) with interfaces that have edge singularities. Such problems represent continuous operators between weighted edge spaces and subspaces with asymptotics. Ellipticity is formulated in terms of a principal symbolic hierarchy, containing interior, transmission, and edge symbols. We construct parametrices, show regularity with asymptotics of solutions in weighted edge spaces and illustrate the results by boundary-contact problems for the Laplacian with jumping coefficients.

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1. Introduction and formulation of the problems

1.1. Edge boundary-contact problems

This paper is aimed at studying boundary-contact problems with singularities at the interfaces. Problems of this kind have been investigated by several authors, in different context, partly under specific assumptions on the geometry or the involved dimensions, cf. Lemrabet [14], Escauri-

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aza, Fabes, and Verchota [6], Torres and Welland [21], Chkadua [3,4], Li and Vogelius [13], Li and Nirenberg [12], Nicaise and Sändig [16] (numerical method), Heinrich, Nicaise and Weber [8] (the Fourier-finite-element method and singular functions of non-tensorial type), Kapanadze and Schulze [10] (the latter paper studies the case with conical singularities at the interfaces).

In the present paper we study the case with edge singularities: let G be a bounded domain in the Euclidean space (first of any dimension, in the example below of dimension 3) of the form $G = G_+ \cup G_- \cup S$ for open subdomains $G_\pm$ of G such that $\overline{G}_+ \cap \overline{G}_- = S$ is an interface of codimension 1. More precisely, we assume that $\partial G_+ = S$, $S \cap \partial G = \emptyset$, which has the consequence that $\partial G_- = S \cup \partial G$. Starting from a pair of elliptic systems of differential operators $A_\pm$ of order μ in $G_\pm$ (with smooth coefficients up to the respective boundaries) our problems have the form

$$A_\pm u_\pm = f_\pm \quad \text{in } G_\pm, \tag{1}$$

$$Tu_- = h \quad \text{on } \partial G, \tag{2}$$

$$T_+ u_+ + T_- u_- = g \quad \text{on } S. \tag{3}$$

Here T is (Shapiro–Lopatinskij) elliptic with respect to the operator A_-, and $T_\pm$ are trace operators of the form $T_\pm = {}^{\mathrm{t}}(T_{\pm,j})_{j=1,\ldots,N}$,

$$T_{\pm,j} u_\pm := (B_{\pm,j} u_\pm)|_S \tag{4}$$

for differential operators $B_{\pm,j}$ of order m_j with smooth coefficients, defined in a tubular neighborhood V of S in G. The restriction to S refers to the corresponding plus or minus side. The trace operator $T = {}^{\mathrm{t}}(T_1, \ldots, T_{N'})$ is given in an analogous form, i.e., $T_j u_- = B_j u_-|_{\partial G}$ for smooth differential operators of order m'_j in a collar neighborhood of ∂G. The numbers N and N' are known from the context. For instance, if $A_\pm$ are $L \times L$ systems of operators of order $2m$ in dimension $\geqslant 3$, then we have $N = 2mL$ and $N' = mL$ (under some standard conditions on the principal symbols of the operators near S and ∂G, respectively, see Agmon, Douglis, and Nirenberg [1]). For our approach is not essential whether we consider scalar operators or systems.

As it was mentioned above the main focus of the paper is the case when $G_\pm$ are manifolds with edges $Y \subset S$ and boundary; in this case S itself is a closed manifold with edge singularity Y. So, we assume that

(i) $\partial G_\pm \setminus Y$ and Y are C^∞ manifolds, and $\dim Y = q$;
(ii) every $y \in Y$ has a neighborhood modeled on a wedge $\varXi_\pm^\Delta \times \varOmega$, where $\varXi_\pm^\Delta := (\overline{\mathbb{R}}_+ \times \varXi_\pm)/(\{0\} \times \varXi_\pm)$ for a certain closed compact C^∞ manifold $\varXi_\pm = \varXi_\pm(y)$, $\dim \varXi_\pm = n$ and an open set $\varOmega \subset \mathbb{R}^q$.

The behavior of solutions far from S is known from the standard theory of elliptic boundary value problems when we assume ∂G to be smooth. To illustrate the situation we mainly look at the case of scalar operators. Note also that when S is smooth the problem (1)–(3) represents continuous operators

$$\mathcal{A}=\begin{pmatrix} A_+ & 0 \\ 0 & A_- \\ T_+ & T_- \\ 0 & T \end{pmatrix}: \begin{matrix} H^s(G_+) \\ \oplus \\ H^s(G_-) \end{matrix} \to \begin{matrix} H^{s-\mu}(G_+) \\ \oplus \\ H^{s-\mu}(G_-) \\ \oplus \\ \bigoplus_{l=1}^{N} H^{s-m_l-\frac{1}{2}}(S) \\ \oplus \\ \bigoplus_{j=1}^{N'} H^{s-m'_j-\frac{1}{2}}(\partial G) \end{matrix} \tag{5}$$

for arbitrary $s > \max\{m_l + \frac{1}{2}, m'_j + \frac{1}{2}\}$ (in the system case we would have everywhere the $\mathbb{C}^L$-valued analogues of the Sobolev spaces). Such transmission problems are well investigated, cf. [15], or [17, Section 4.3.3], and [10, Section 3.2].

If S has an edge singularity Y it is adequate to replace the standard Sobolev spaces by weighted edge spaces and subspaces with asymptotics. Let us introduce more convenient notation. We set $W_\pm = \overline{G}_\pm$, then $S = \partial W_+ = W_- \backslash \partial G$. By virtue of the nature of the singular charts in the above condition (ii) we can interpret the set $W_\pm \backslash Y$ as a subspace of a space $\mathbb{W}_\pm$ that is locally near Y modeled on open stretched wedges of the form $[0,1) \times \Xi_\pm \times Y$, where $r \in [0,1)$ is the axial variable of the respective cone with $\Xi_\pm$ as the base manifold. The example is Section 3.2 will concern the case $\dim \Xi_\pm = 1$, and we then assume that $\Xi_+ = [0,\alpha]$ and $\Xi_- = [\alpha, 2\pi]$ for $0 < \alpha < \pi$.

For the interface S we use the following local representation $[0,1) \times \Sigma \times Y$ (more details will be explained below). The global stretched 'surface' $\mathbb{S}$ obtained from S by blowing up the singularity near Y then has the property

$$\partial \mathbb{W}_{+,\mathrm{reg}} = \mathbb{S}_{\mathrm{reg}}, \qquad \partial \mathbb{W}_{-,\mathrm{reg}} = \mathbb{S}_{\mathrm{reg}} \cup \partial G,$$

where subscript 'reg' denotes the stretched space minus the bottom $r = 0$. There are now weighted edge spaces $\mathcal{W}^{s,\gamma}(\mathbb{W}_\pm)$ and $\mathcal{W}^{s,\gamma}(\mathbb{S})$ of smoothness s and weight γ (and subspaces with asymptotics for $r \to 0$, also to be introduced below). Then our boundary-contact problem locally represents continuous operators

$$\mathcal{A}: \begin{matrix} \mathcal{W}^{s,\gamma}(\mathbb{W}_+) \\ \oplus \\ \mathcal{W}^{s,\gamma}(\mathbb{W}_-) \end{matrix} \to \begin{matrix} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}_+) \\ \oplus \\ \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}_-) \\ \oplus \\ \bigoplus_{l=1}^{N} \mathcal{W}^{s-m_l-\frac{1}{2},\gamma-m_l-\frac{1}{2}}(\mathbb{S}) \\ \oplus \\ \bigoplus_{j=1}^{N'} H^{s-m'_j-\frac{1}{2}}(\partial G) \end{matrix} \tag{6}$$

for arbitrary $s > \max\{m_l + \frac{1}{2}, m'_j + \frac{1}{2}\}$ and $\gamma \in \mathbb{R}$.

We assume that $A_\pm$ near $r = 0$ are operators of edge-degenerate type. This includes the case of operators with smooth coefficients up to the interface from the respective side which easily follows by introducing polar coordinates transversal to Y. Moreover, the trace operators $T_{\pm,j}$ (in (4)) are assumed to be of the form of a composition of an edge-degenerate differential operator $B_{\pm,j}$ with the restriction to $\mathrm{int}\,\mathbb{S}$, cf. the formulas (11), (12) below.

The program of the paper is to solve problems of type (1)–(3) in terms of a parametrix construction under a natural condition of ellipticity (referring to the weights). We show regularity and obtain asymptotics of solutions in weighted edge spaces. The necessary material will be given in Section 2. The result will be applied to example, cf. Section 3.2 which could be easily generalized to, say, second order elliptic equations.

1.2. The principal symbolic structure

Recall that when S is smooth (and also the coefficients of the involved operators up to the interface) the ellipticity of $\mathcal{A}$ refers to a principal symbolic hierarchy

$$\sigma(\mathcal{A}) := \big(\sigma_\psi(A_+), \sigma_\psi(A_-), \sigma_{\mathrm{tr}}(\mathcal{A}), \sigma_\partial(\mathcal{A})\big),$$

where $\mathcal{A}$ is regarded as an operator (5). The first two components $\sigma_\psi(A_\pm)$ are the homogeneous principal symbols of the operators $A_\pm$ over $\mathrm{int}\,\Xi_\pm$ (smooth up to respective boundaries). The boundary symbol $\sigma_\partial(\mathcal{A}) := {}^{\mathrm{t}}(\sigma_\partial(A_-)\sigma_\partial(T))$ comes from the standard calculus of boundary value problems. Recall that when (x', t) is a local splitting of variables in a collar neighborhood $\partial G \times [0, 1)$ of the boundary, with the covariables (ξ', τ), then

$$\sigma_\partial(A_-)(x', \xi') := \sigma_\psi(A_-)(x', 0, \xi', D_t)$$

interpreted as an operator family $\sigma_\partial(A_-)(x', \xi') : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+)$. If $T = {}^{\mathrm{t}}(T_1, \dots, T_{N'})$ is given in terms of expressions $T_k u_- = B_k u_-|_{\partial G}$ we set

$$\sigma_\partial(T_k)(x, \xi')f := \big(\sigma_\psi(B_k)(x', 0, \xi', D_t)f\big)\big|_{t=0}$$

and $\sigma_\partial(T) := {}^{\mathrm{t}}(\sigma_\partial(T_k))_{k=1,\dots,N'}$.

The principal transmission symbol $\sigma_{\mathrm{tr}}(\mathcal{A})$ is defined as follows: let us choose a tubular neighborhood $V \subset G$ of S, set $V_\pm := V \cap \Xi_\pm$, and let $\varepsilon : V_- \to V_+$ be defined by $\varepsilon(x', t) = (x', -t)$. Then we can pass to the operator

$$\mathcal{A}_{V_+} := \begin{pmatrix} A_+|_{\mathrm{int}\,V_+} & 0 \\ 0 & \varepsilon_*(A_-|_{\mathrm{int}\,V_-}) \\ T_+ & \varepsilon_* T_- \end{pmatrix}.$$

Here

$$\varepsilon_*(A_-|_{\mathrm{int}\,V_-}) := (\varepsilon^*)^{-1} A_-|_{\mathrm{int}\,V_-}\varepsilon^*, \tag{7}$$

with ε^* being the function pull back under ε and

$$(\varepsilon_* T_-)u := (\varepsilon_* B_{-,j}|_{\mathrm{int}\,V_-} u)|_S \tag{8}$$

for a function u on V_+. Then the operator $\mathcal{A}_{V_+}$ represents a boundary value problem on V_+ with the boundary symbol

$$\sigma_\partial(\mathcal{A}_{V_+})(x', \xi') : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ H^s(\mathbb{R}_+) \end{matrix} \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^\mu \end{matrix}, \quad (x', \xi') \in T^*S \setminus 0. \tag{9}$$

Now we obtain $\sigma_{\mathrm{tr}}(\mathcal{A})$ (the so-called principal transmission symbol of $\mathcal{A}$) from $\sigma_\partial(\mathcal{A}_{V_+})$ by applying the push forward $(\varepsilon^{-1})_*$ to the operators of the second column of (9) from $\mathbb{R}_+$ to $\mathbb{R}_-$, similarly as the relation between operators (7), (8). This gives rise to an operator family

$$\sigma_{\mathrm{tr}}(\mathcal{A})(x',\xi'):\begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ H^s(\mathbb{R}_-)\end{matrix} \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ H^{s-\mu}(\mathbb{R}_-) \\ \oplus \\ \mathbb{C}^\mu \end{matrix}, \quad (x',\xi')\in T^*S\backslash 0. \tag{10}$$

The transmission problem (5) is called elliptic if the symbols $\sigma_\psi(\mathcal{A}_\pm)(x,\xi)$ are non-vanishing for all $(x,\xi)\in T^*\overline{G}_\pm\backslash 0$ and if (9) and (10) are bijective operators for all sufficiently large s. For more details, cf. [10].

Let us now return to our singular configuration (i.e., S has an edge singularity Y). Denote by $\mathcal{A}_{\mathrm{reg}}$ the restriction of the operator (6) to distributions in the complement of Y. Then $\sigma(\mathcal{A}_{\mathrm{reg}})$ is as before. Close to the edge singularity we have to add a so-called principal edge symbol $\sigma_\wedge(\mathcal{A})$, which comes from the theory of pseudo-differential boundary value problems on a manifolds with edges. As noted before by inserting polar coordinates (r,ϕ,y) we pass to the stretched domains $[0,1)\times\Xi_\pm\times Y$. Then we obtain the operators $A_\pm$ in the form

$$A_\pm = r^{-\mu}\sum_{k+|\beta|\leqslant\mu} a^\pm_{j\beta}(r,y)(-r\partial_r)^k(rD_y)^\beta \tag{11}$$

with coefficients $a^\pm_{k\beta}(r,y)\in C^\infty(\overline{\mathbb{R}}_+\times Y,\mathrm{Diff}^{\mu-(k+|\beta|)}(\Xi_\pm))$. Here $\mathrm{Diff}^\nu(\cdot)$ denotes the space of all differential operators of order ν on the manifold in the brackets. Similarly, for the trace operators $T_\pm$ we assume

$$T_\pm = {}^{\mathrm{t}}\Bigg(\mathrm{r}_{\mathrm{int}\,\mathbb{S}}\, r^{-m_l}\sum_{k+|\beta|\leqslant m_j} b^\pm_{j,k\beta}(r,y)(-r\partial_r)^k(rD_y)^\beta\Bigg)_{j=1,\dots,N}, \tag{12}$$

with coefficients $b^\pm_{j,k\beta}(r,y)\in C^\infty(\overline{\mathbb{R}}_+\times Y,\mathrm{Diff}^{m_j-(k+|\beta|)}(\Xi_\pm))$, and $\mathrm{r}_{\mathrm{int}\,\mathbb{S}}$ denotes the operator of restriction to $\mathrm{int}\,\mathbb{S}$.

The representation of the operators in edge degenerate form is just the reason for the continuity of (6) in weighted edge spaces. The typical Fuchs type differentiation $-r\partial_r$ in (11) can be regarded as a Mellin operator with symbol z, i.e., $-r\partial_r = \mathcal{M}^{-1}z\mathcal{M}$, where $\mathcal{M}u(z) = \int_0^\infty r^{z-1}u(r)\,dr$ is the Mellin transform. Then the variable z will often be considered on the 'weight line'

$$\Gamma_\beta = \{z\in\mathbb{C}\colon \operatorname{Re} z = \beta\}$$

for some $\beta\in\mathbb{R}$. The Mellin transform will also be applied to vector-valued functions on $\mathbb{R}_+$, first with compact support and then extended various function and distribution spaces. A Mellin pseudo-differential operator with respect to some weight $\gamma\in\mathbb{R}$ is defined as follows:

$$\mathrm{op}_M^\gamma(h)u(r) := (2\pi)^{-1}\iint_0^\infty \left(\frac{r'}{r}\right)^{\frac{1}{2}-\gamma+i\varrho} h\left(r,r',\frac{1}{2}-\gamma+i\varrho\right)u(r')\frac{dr'}{r'}\,d\varrho,$$

where $h(r, r', z)$ is a parameter-dependent (operator-valued) amplitude function with covariables $z \in \Gamma_{1/2-\gamma}$. Writing

$$\tilde{h}_\pm(r, y, z, \tilde{\eta}) := \sum_{k+|\beta|\leqslant\mu} a^\pm_{k\beta}(r, y) z^k \tilde{\eta}^\beta \Big|_{\tilde{\eta}=r\eta}$$

then the Mellin amplitude functions in our case have the form

$$h_\pm(r, y, z, \eta) = \tilde{h}_\pm(r, y, z, \tilde{\eta})|_{\tilde{\eta}=r\eta}.$$

Similarly we have

$$h'_\pm(r, y, z, \eta) = \tilde{h}'_\pm(r, y, z, \tilde{\eta})|_{\tilde{\eta}=r\eta} := {}^{\mathrm{t}}\!\left(r_\Sigma \sum_{k+|\beta|\leqslant m_j} b^\pm_{k\beta}(r, y) z^k \tilde{\eta}^\beta\right)_{j=1,\ldots,N}\Big|_{\tilde{\eta}=r\eta}.$$

Then $A_\pm = \mathrm{Op}_y(a_\pm)$, where

$$a_\pm(y, \eta) = r^{-\mu}\,\mathrm{op}_M^{\gamma-\frac{1}{2}}(h_\pm) = r^{-\mu} \sum_{k+|\beta|\leqslant\mu} a^\pm_{k\beta}(r, y)(-r\partial_r)^j (r\eta)^\beta \tag{13}$$

and $\mathrm{Op}_y(a)u(y) := \iint e^{i(y-y')\eta} a(y, \eta) u(y')\, dy\, đ\eta$, $đ\eta := (2\pi)^{1-q}\, d\eta$.

Writing $T_\pm = \mathrm{diag}(r^{-m_j})\,\mathrm{Op}_y(\mathrm{op}_M^{\gamma-\frac{n}{2}}(h'_\pm))$ we obtain that the operator $\mathcal{A}$ close to $r = 0$ has the form

$$\mathcal{A} = m(r)\,\mathrm{Op}_y\big(\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\big)$$

for a matrix $m(r) := \mathrm{diag}(r^{-\mu}, r^{-\mu}, \mathrm{diag}(m_j))$ and the matrix of Mellin amplitude functions $h(r, y, z, \eta) = \tilde{h}(r, y, z, \tilde{\eta})|_{\tilde{\eta}=r\eta}$ where

$$\tilde{h}(r, y, z, \tilde{\eta}) := \begin{pmatrix} \tilde{h}_+(r, y, z, \tilde{\eta}) & 0 \\ 0 & \tilde{h}_-(r, y, z, \tilde{\eta}) \\ \tilde{h}'_+(r, y, z, \tilde{\eta}) & \tilde{h}'_-(r, y, z, \tilde{\eta}) \end{pmatrix}. \tag{14}$$

The function $\tilde{h}(r, y, z, \eta)$ is smooth up to $r = 0$ and takes values in the space of transmission problems on S^n with respect to the subdivision $S^n = \Xi_+ \cup \Xi_-$ with the interface $\Sigma = \Xi_+ \cap \Xi_-$. The adequate choice of γ depends on the so-called principal edge symbol $\sigma_\wedge(\mathcal{A})(y, \eta)$ which is in our case is defined by the expression

$$\sigma_\wedge(\mathcal{A})(y, \eta) := m(r)\,\mathrm{op}_M^{\gamma-\frac{n}{2}}\big(\tilde{h}(0, y, z, r\eta)\big).$$

This is an operator function parametrized by $(y,\eta) \in Y \times (\mathbb{R}^q \setminus \{0\})$, acting as

$$\sigma_\wedge(\mathcal{A})(y,\eta): \begin{array}{c} \mathcal{K}^{s,\gamma}(\Xi_+^\wedge) \\ \oplus \\ \mathcal{K}^{s,\gamma}(\Xi_-^\wedge) \end{array} \to \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}(\Xi_+^\wedge) \\ \oplus \\ \mathcal{K}^{s-\mu,\gamma-\mu}(\Xi_-^\wedge) \\ \oplus \\ \bigoplus_{l=1}^N \mathcal{K}^{s-m_l-\frac{1}{2},\gamma-m_l-\frac{1}{2}}(\Sigma^\wedge) \end{array} \tag{15}$$

where $\mathcal{K}^{s,\gamma}(X^\wedge)$ denote weighted Sobolev spaces on the cone $X^\wedge = \mathbb{R}_+ \times X$ (where X stands for $\Xi_\pm$ or Σ), of smoothness $s \in \mathbb{R}$ and weight $\gamma \in \mathbb{R}$ (concerning the definition, cf. Section 2.1 below). The operators $\sigma_\wedge(\mathcal{A})(y,\eta)$ take values in the transmission cone algebra with a corresponding symbolic structure. The ellipticity of $\mathcal{A}$ also requires the bijectivity of (15) for all $(y,\eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$. However, this cannot be expected to hold true without additional information. The necessary and sufficient condition for the Fredholm property of (15) is that the subordinate principal conormal symbol

$$\sigma_M\sigma_\wedge(\mathcal{A})(y,z) = \tilde{h}(0,y,z,0): \begin{array}{c} H^s(\Xi_+) \\ \oplus \\ H^s(\Xi_-) \end{array} \to \begin{array}{c} H^{s-\mu}(\Xi_+) \\ \oplus \\ H^{s-\mu}(\Xi_-) \\ \oplus \\ \bigoplus_{l=1}^N H^{s-m_l-\frac{1}{2}}(\Sigma) \end{array} \tag{16}$$

is invertible for all z with $\operatorname{Re} z = \frac{n+1}{2} - \gamma$.

Summing up a boundary-contact problem (6) with an interface S with edge singularity has a principal symbolic hierarchy

$$\sigma(\mathcal{A}) = \big(\sigma_\psi(A_+), \sigma_\psi(A_-), \sigma_{\mathrm{tr}}(\mathcal{A}_{\mathrm{reg}}), \sigma_\wedge(\mathcal{A}), \sigma_\partial(\mathcal{A})\big), \tag{17}$$

where $\sigma_{\mathrm{tr}}(\mathcal{A}_{\mathrm{reg}})$ was defined before.

1.3. Outline of the results

In this paper we study boundary-contact problems $\mathcal{A}$ which are elliptic with respect to $\sigma(\mathcal{A})$, that is, $\sigma_\psi(A_\pm)$ are non-vanishing as usual, and the other components are bijective families. The precise conditions are given in Definition 3.1. The new results are as follows. We construct parametrices within an 'algebra' of pseudo-differential boundary-contact problems with a similar principal symbolic structure as (17), cf. Theorem 3.3. The operators act in weighted edge spaces and subspaces with asymptotics. We single out a specific subalgebra of so-called Green operators (the notation comes from Boutet de Monvel's calculus [2]), combined with asymptotic data close to the edge. Outside any neighborhood of the edge these operators are smoothing (and of some type), while close to edge they are pseudo-differential with so-called Green symbols, acting in weighted spaces on the infinite cones, cf. the notation in Section 3.3.

Green operators are the left-over terms in parametrices, and their continuity in weighted edge spaces with asymptotics in the image gives rise to the regularity of solutions with asymptotics, cf. Theorem 3.4. A reference for the technique from Boutet de Monvel's calculus are the monographs Rempel, Schulze [17], Grubb [7], or Schulze [20]; the latter also contains many details on (pseudo-differential) edge operators, including the role of the operator-valued edge symbolic structure. Another reference is the author's joint paper [10]. It contains an accessible exposition on pseudo-differential transmission problems with parameters, which is systematically employed here.

2. Boundary-contact operators

2.1. Cone Sobolev spaces with asymptotics

Let X be a C^∞ manifold of dimension n and denote by $L^\mu_{\mathrm{cl}}(X;\mathbb{R}^l)$ the space of all parameter-dependent (with parameter $\lambda \in \mathbb{R}^l$) classical pseudo-differential operators on X in its natural Fréchet topology. We first consider the case that X is closed, compact.

Let $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s,\gamma \in \mathbb{R}$ denote the completion of $C_0^\infty(X^\wedge)$ with respect to the norm

$$\left\{\frac{1}{2\pi i}\int_{\Gamma_{(n+1)/2-\gamma}} \left\|R^s(\operatorname{Im} z)\mathcal{M}u(z)\right\|^2_{L^2(X)}\,dz\right\}^{1/2}.$$

Here, $R^s(\varrho)$ is a parameter-dependent elliptic operator belonging to the space $L^s_{\mathrm{cl}}(X;\mathbb{R})$ that induces isomorphisms $R^s(\varrho): H^t(X) \to H^{t-s}(X)$ for all $t,s \in \mathbb{R}$. $H^s(X)$ is the standard Sobolev space on X of smoothness $s \in \mathbb{R}$, and $H^0(X)$ is identified with $L^2(X)$.

In the present paper a cut-off function on $\mathbb{R}_+$ is any real-valued function $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\omega = 1$ near 0. We then define

$$\mathcal{K}^{s,\gamma}\left(X^\wedge\right) := \left\{\omega u + (1-\omega)v\colon\ u \in \mathcal{H}^{s,\gamma}\left(X^\wedge\right),\ v \in H^s_{\mathrm{cone}}\left(X^\wedge\right)\right\}$$

for any cut-off function ω. Here we used a version of weighted Sobolev spaces $H^s_{\mathrm{cone}}(X^\wedge)$ that are standard ones near infinity, for details, cf. [20]. In particular, for $X = S^n$ (the unit sphere in $\mathbb{R}^{n+1}$) we have $(1-\omega)H^s_{\mathrm{cone}}(X^\wedge) = (1-\omega)H^s(\mathbb{R}^{n+1})$. We endow the spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ with a Hilbert space structure in a natural way. Observe that when we set $\kappa_\lambda: u(r,x) \to \lambda^{\frac{n+1}{2}}u(\lambda r,x)$ for $\lambda \in \mathbb{R}_+$, we obtain a strongly continuous group of isomorphisms operating on $\mathcal{K}^{s,\gamma}(X^\wedge)$.

Remark 2.1. We will also need these spaces in the variant when X is a compact C^∞ manifold with boundary. In this case we first consider the double $2X$ (obtained by gluing together two copies $X_\pm$ of X along the common boundary ∂X together to a closed compact C^∞ manifold; we then identify the original X with X_+). Then we define

$$\mathcal{K}^{s,\gamma}\left(X^\wedge\right) := \left\{u|_{(\operatorname{int} X)^\wedge}\colon\ u \in \mathcal{K}^{s,\gamma}\left((2X)^\wedge\right)\right\}.$$

In particular, we have the spaces $\mathcal{K}^{s,\gamma}(\Xi^\wedge_\pm)$, and (13) induce families of continuous operators

$$a_\pm(y,\eta): \mathcal{K}^{s,\gamma}\left(\Xi^\wedge_\pm\right) \to \mathcal{K}^{s-\mu,\gamma-\mu}\left(\Xi^\wedge_\pm\right)$$

are continuous for all $s,\gamma \in \mathbb{R}$.

Here we always assume the coefficients $a^\pm_{k\beta}$ in (13) to be independent of r for large r, which is adequate in our context. The families $a^\pm_{k\beta}$ are C^∞ in (y,η), and they are operator-valued symbols in the following sense: If E is a Hilbert space and $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ a strongly continuous group of isomorphisms on E, $\kappa_\lambda\kappa_\delta = \kappa_{\lambda\delta}$ for all $\lambda, \delta \in \mathbb{R}_+$, we say that E is endowed with a group action. Given Hilbert spaces E and $\widetilde{E}$ with group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively,

$$S^\mu\big(\Omega \times \mathbb{R}^q; E, \widetilde{E}\big) \tag{18}$$

will denote the subspace of all $a(y, \eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$ such that

$$\sup_{y\in K, \eta\in\mathbb{R}^q} \langle\eta\rangle^{-\mu+|\beta|} \big\| \tilde{\kappa}^{-1}_{\langle\eta\rangle} D^\alpha_y D^\beta_\eta a(y, \eta)\kappa_{\langle\eta\rangle} \big\|_{\mathcal{L}(E,\widetilde{E})}$$

is finite for every $K \subset \Omega$ and all multi-indices $\alpha, \beta \in \mathbb{N}^q$. Here $\langle\eta\rangle = (1 + |\eta|^2)^{1/2}$. Note that we obtain an equivalent definition of (18) when we replace η by, for instance, a C^∞ function $\eta \to [\eta]$ that is strictly positive and satisfies $[\eta] = |\eta|$ for $|\eta| > C$ for a $C > 0$.

A function $a_{(\mu)}(y, \eta) \in C^\infty(\Omega \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(E, \widetilde{E}))$ is called ('twisted') homogeneous in $\eta \neq 0$ of order μ if

$$a_{(\mu)}(y, \lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda a_{(\mu)}(y, \eta)\kappa^{-1}_\lambda \tag{19}$$

for all $(y, \eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$, $\lambda \in \mathbb{R}_+$. Note that when $\chi(\eta)$ is an arbitrary excision function in $\mathbb{R}^q$ (i.e., in $C^\infty(\mathbb{R}^q)$, zero in a neighborhood of the origin, 1 for $|\eta| > R$ for some $R > 0$) we have $\chi(\eta)a_{(\mu)}(y, \eta) \in S^\mu(\Omega \times \mathbb{R}^q; E, \widetilde{E})$ when $a_{(\mu)}$ is homogeneous in the former sense. This gives rise to

$$S^\mu_{\mathrm{cl}}\big(\Omega \times \mathbb{R}^q; E, \widetilde{E}\big), \tag{20}$$

the subspace of (18) of all elements $a(y, \eta)$ which admit an asymptotic expansion into terms of the kind $\chi(\eta)a_{(\mu-j)}(y, \eta)$, with homogeneous functions $a_{(\mu-j)}(y, \eta)$ of order $\mu - j$, $j \in \mathbb{N}$. In this case we set

$$\sigma_\wedge(a)(y, \eta) := a_{(\mu)}(y, \eta).$$

The concept of operator-valued symbols in that sense is very close to the scalar case where $E = \widetilde{E} = \mathbb{C}$ and the group actions are trivial (i.e., identity operators for all $\lambda \in \mathbb{R}_+$).

The definition has a generalization to pairs of Fréchet spaces E and $\widetilde{E}$. For instance, let $\widetilde{E}$ be defined as a projective limit $\varprojlim_{j\in\mathbb{N}} \widetilde{E}^j$ of Hilbert spaces $\widetilde{E}^j$ with continuous embeddings $\widetilde{E}^{j+1} \hookrightarrow \widetilde{E}^j \hookrightarrow \cdots \hookrightarrow \widetilde{E}^0$ such that $\widetilde{E}^0$ is endowed with a group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ such that $\{\kappa_\lambda|_{\widetilde{E}^j}\}_{\lambda\in\mathbb{R}_+}$ defines a group action $\widetilde{E}^j$ for every j. Then $\widetilde{E}$ is said to be equipped with the group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$. We have the spaces $S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; E, \widetilde{E}^j)$ for all j, and $S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; E, \widetilde{E})$ is the projective limit of these spaces over j (subscript '(cl)' means that we are talking about the classical or the general case).

We are interested in subspaces $\mathcal{K}^{s,\gamma}_P(X^\wedge)$ of $\mathcal{K}^{s,\gamma}(X^\wedge)$ with asymptotics for $r \to 0$ of type P, i.e.,

$$u(r, x) \sim \sum_j \sum_{k=0}^{m_j} c_{jk}(x) r^{-p_j} \log^k r$$

with $p_j \in \mathbb{C}$, $m_j \in \mathbb{N}$, and coefficients c_{jk} that we control as elements in certain finite-dimensional subspaces $L_j \subset C^\infty(X)$. In this connection we set

$$P = \{(p_j, m_j, L_j)\}_{j=0,\dots,N} \tag{21}$$

and assume that $\pi_{\mathbb{C}} P := \{p_j\}_{j=0,\dots,N} \subset \{z\colon \operatorname{Re} z < \frac{n+1}{2} - \gamma\}$. We may talk about finite or infinite asymptotics. In the finite case we fix a weight interval $\Theta = [0, \vartheta)$ for some $\vartheta > 0$ and set

$$\mathcal{K}_\Theta^{s,\gamma}(X^\wedge) := \varprojlim_{j\in\mathbb{N}} \mathcal{K}^{s,\gamma+\vartheta-\frac{1}{1+j}}(X^\wedge).$$

This space is regarded as the subspace of functions which are flat of order Θ with respect to the reference weight γ. Assuming N to be finite for finite Θ and $\pi_{\mathbb{C}} P \subset \{z\colon \frac{n+1}{2} - \gamma - \vartheta < \operatorname{Re} z < \frac{n+1}{2} - \gamma\}$ we form the space

$$\mathcal{E}_P(X^\wedge) := \left\{\omega(r) \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk}(x) r^{-p_j} \log^k r\colon\ c_{jk} \in L_j \text{ for } 0 \leqslant k \leqslant m_j,\ 0 \leqslant j \leqslant N\right\}$$

which is of finite dimension and contained in $\mathcal{K}^{\infty,\gamma}(X^\wedge)$. We then define

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) := \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) + \mathcal{E}_P(X^\wedge) \tag{22}$$

(which is a direct sum). In the case of infinite Θ we admit N to be ∞ and assume in this case $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$. Then, setting $P_k = \{(p, m, L) \in P\colon \operatorname{Re} p > \frac{n+1}{2} - \gamma - (1+k)\}$ and $\Theta_k := [0, 1+k)$, $k \in \mathbb{N}$, we have the spaces $\mathcal{K}_{P_k}^{s,\gamma}(X^\wedge)$ and define

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) = \varprojlim_{k\in\mathbb{N}} \mathcal{K}_{P_k}^{s,\gamma}(X^\wedge)$$

in the Fréchet topology of the projective limit.

Let P be an asymptotic type of the kind (21), associated with (γ, Θ) (i.e., a weight plus a fixed (finite or infinite) weight interval $\Theta = [0, \vartheta)$), and set

$$\mathcal{S}_P^\gamma(X^\wedge) := \{\omega u + (1-\omega)v\colon\ u \in \mathcal{K}_P^{\infty,\gamma}(X^\wedge),\ v \in \mathcal{S}(\overline{\mathbb{R}}_+, C^\infty(X))\}. \tag{23}$$

This is a (nuclear) Fréchet space. Recall that the spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ with the group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, defined by $\kappa_\lambda : u(r,x) \to \lambda^{\frac{n+1}{2}} u(\lambda r, x)$, $\lambda \in \mathbb{R}_+$.

The space of $\mathcal{S}_P^\gamma(X^\wedge)$ can be represented as a projective limit of Hilbert spaces contained in $\mathcal{K}^{\infty,\gamma}(X^\wedge)$, in which $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, $\kappa_\lambda : u(r,x) \to \lambda^{\frac{n+1}{2}} u(\lambda r, x)$, $\lambda \in \mathbb{R}_+$, are group actions. According to our general notation $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ is a group action in (23). For references below we also form the spaces

$$\mathcal{S}^\gamma(X^\wedge) := \{\omega u + (1-\omega)v\colon\ u \in \mathcal{K}^{\infty,\gamma}(X^\wedge),\ v \in \mathcal{S}(\overline{\mathbb{R}}_+, C^\infty(X))\}.$$

The definitions of the spaces $\mathcal{K}_P^{s,\gamma}(X^\wedge)$, $\mathcal{S}_P^\gamma(X^\wedge)$ for the case of smooth manifold X with smooth boundary ∂X are analogous. For details see [20] or [9].

2.2. Edge spaces with asymptotics

Let E be a Hilbert space equipped with a group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$.

Definition 2.2. The space $\mathcal{W}^s(\mathbb{R}^q, E)$ for $s \in \mathbb{R}$ is the completion of $\mathcal{S}(\mathbb{R}^q, E)$ (the Schwartz space of E-valued functions) with respect to the norm

$$\left\{\int \langle\eta\rangle^s \left\|\kappa^{-1}(\eta)\hat{u}(\eta)\right\|_E^2\right\}^{1/2}.$$

Here $\kappa(\eta) = \kappa_{\langle\eta\rangle}$ and $\hat{u}(\eta)$ is the Fourier transform in $\mathbb{R}^q$.

If $E = \varprojlim_{j\in\mathbb{N}} E^j$ is a Fréchet space written as a projective limit of Hilbert spaces E^j with continuous embeddings $E^{j+1} \hookrightarrow E^j$, and let $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ be a group action on E^0 which restricts to group actions on E^j for every j. In that case we have continuous embeddings $\mathcal{W}^s(\mathbb{R}^q, E^{j+1}) \hookrightarrow \mathcal{W}^s(\mathbb{R}^q, E^j)$, and we write

$$\mathcal{W}^s(\mathbb{R}^q, E) = \varprojlim_{j\in\mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E^j).$$

Similarly as standard Sobolev spaces we also have 'comp' and 'loc' versions $\mathcal{W}^s_{\mathrm{comp}}(\Omega, E)$ and $\mathcal{W}^s_{\mathrm{loc}}(\Omega, E)$ for any open set $\Omega \subset \mathbb{R}^q$. More details on the nature of abstract edge spaces may be found in [19] or [20].

Let X be a compact C^∞ manifold with boundary ∂X and apply Definition 2.2 to the spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$, $\mathcal{K}^{s,\gamma}((\partial X)^\wedge)$ with group actions $\kappa_\lambda^{(n)}u(r,x) = \lambda^{(n+1)/2}u(\lambda r, x)$ for $u \in \mathcal{K}^{s,\gamma}(X^\wedge)$ and $\kappa_\lambda^{(n-1)}u(r,x') = \lambda^{n/2}v(\lambda r, x')$ for $v \in \mathcal{K}^{s,\gamma}((\partial X)^\wedge)$, respectively. Then the spaces $\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$ and $\mathcal{W}^{s,\gamma}((\partial X)^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}((\partial X)^\wedge))$ are called edge spaces of smoothness s and weight γ.

We now introduce subspaces of $\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) \ni u(r,x,y)$ with asymptotics for $r \to 0$, which are discrete and constant with respect to the edge variable y.

Note that we can write $\mathcal{K}_P^{s,\gamma}(X^\wedge)$ as a projective limit of $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$-invariant Hilbert spaces E^k, $k \in \mathbb{N}$, which gives us the edge spaces $\mathcal{W}^s(\mathbb{R}^q, E^k)$ with continuous embeddings $\mathcal{W}^s(\mathbb{R}^q, E^{k+1}) \hookrightarrow \mathcal{W}^s(\mathbb{R}^q, E^k)$ for all k, and then we define

$$\mathcal{W}_P^{s,\gamma}(X^\wedge \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_P^{s,\gamma}(X^\wedge)) \tag{24}$$

as the projective limit $\varprojlim_{k\in\mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E^k)$ with the corresponding Fréchet structure. It can easily be proved that (24) is independent of the specific choice of the sequence $\{E^k\}_{k\in\mathbb{N}}$.

To characterize the singular functions of the edge asymptotics we first observe that when E is a Hilbert (or Fréchet space) with group action, we have canonical isomorphisms

$$T(\eta) := \mathcal{F}^{-1}\kappa_{\langle\eta\rangle}^{-1}\mathcal{F} : \mathcal{W}^s(\mathbb{R}^q, E) \to H^s(\mathbb{R}^q, E)$$

for all $s \in \mathbb{R}$, cf. [19]. Let $E = E_0 \oplus E_1$ be a direct decomposition of E into closed subspaces, not necessarily invariant under the group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ on E. We then obtain $H^s(\mathbb{R}^q, E) =$

$H^s(\mathbb{R}^q, E_0) \oplus H^s(\mathbb{R}^q, E_1)$ which generates a direct decomposition

$$\mathcal{W}^s(\mathbb{R}^q, E) = T^{-1}H^s(\mathbb{R}^q, E_0) \oplus T^{-1}H^s(\mathbb{R}^q, E_1) \tag{25}$$

into closed subspaces.

Let us apply this construction to the space $E = \mathcal{K}_P^{s,\gamma}(X^\wedge)$, decomposed as (22) with $E_0 = \mathcal{K}_\Theta^{s,\gamma}(X^\wedge)$, $E_1 = \mathcal{E}_P(X^\wedge)$, for an element P as in (21), where the weight interval Θ is finite. The space $\mathcal{K}_\Theta^{s,\gamma}(X^\wedge)$ is closed with respect to $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$; which gives us

$$T^{-1}H^s(\mathbb{R}^q, \mathcal{K}_\Theta^{s,\gamma}(X^\wedge)) = \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_\Theta^{s,\gamma}(X^\wedge)),$$

also denoted by $\mathcal{W}_\Theta^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$. However, $\mathcal{E}_P(X^\wedge)$ is not preserved under the group action, but we can form

$$\mathcal{V}_P^s(X^\wedge \times \mathbb{R}^q) := T^{-1}\mathcal{E}_P(X^\wedge)$$

which is as a closed subspace of $\mathcal{W}_P^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$. In other words, we have a direct decomposition

$$\mathcal{W}_P^{s,\gamma}(X^\wedge \times \mathbb{R}^q) = \mathcal{W}_\Theta^{s,\gamma}(X^\wedge \times \mathbb{R}^q) + \mathcal{V}_P^s(X^\wedge \times \mathbb{R}^q)$$

into a component of distributions of edge-flatness Θ and a space of singular functions with discrete (and constant in y) edge asymptotics of type P.

Remark 2.3. Every $f(r,x,y) \in \mathcal{W}_P^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ for a (discrete) asymptotic type P of the kind (21), $\Theta = [0, \vartheta)$ finite (i.e., $N < \infty$), can be written in the form

$$f(r,x,y) = f_{\text{sing}}(r,x,y) + f_\Theta(r,x,y)$$

for singular functions

$$f_{\text{sing}}(r,x,y) = \sum_{j=0}^{N}\sum_{k=0}^{m_j} \mathcal{F}_{\eta\to y}^{-1}[\eta]^{\frac{n+1}{2}}\omega(r[\eta])c_{jk}(x)(r[\eta])^{-p_j}\log^k(r[\eta])\hat{v}_{jk}(\eta)$$

with suitable $v_{jk} \in H^s(\mathbb{R}^q)$, coefficients $c_{jk} \in L_j$, $0 \leqslant k \leqslant m_j$, for all j, and a flat remainder $f_\Theta(r,x,y) \in \mathcal{W}_\Theta^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$. Note that in the case $s = \infty$ we may write

$$f_{\text{sing}}(r,x,y) = \sum_{j=0}^{N}\sum_{k=0}^{m_j} \omega(r)c_{jk}(x)w_{jk}(y)r^{-p_j}\log^k r$$

mod $\mathcal{W}_\Theta^{\infty,\gamma}(X^\wedge \times \mathbb{R}^q) = H^\infty(\mathbb{R}^q, \mathcal{K}_\Theta^{\infty,\gamma}(X^\wedge))$ with elements $w_{jk} \in H^\infty(\mathbb{R}^q)$.

One may ask to what extent our notation of singular functions of the edge asymptotics depends on the choice of the function $\eta \to [\eta]$. One can prove, cf. [9], that when $p(\eta)$ is any other element of $C^\infty(\mathbb{R}^q)$ such that $c_1[\eta] \leqslant p(\eta) \leqslant c_2[\eta]$ for all $\eta \in \mathbb{R}^q$, with constants $c_1 < c_2$, then $f_{\text{sing}}(r,x,y)$ can be reformulated into an equivalent expression with $p(\eta)$ in place of $[\eta]$ and other coefficients c_{jk}, v_{jk}, mod $\mathcal{W}^{s,\gamma}_{\Theta}(X^\wedge \times \mathbb{R}^q)$. Also the choice of ω is unessential modulo such flat remainders.

Let us return to our configuration. Asymptotics of solutions (also to be expressed explicitly for specific examples) will be formulated in terms of the spaces $\mathcal{W}^{s,\gamma}_{P_+}(\Xi_+^\wedge \times \mathbb{R}^q) \oplus \mathcal{W}^{s,\gamma}_{P_-}(\Xi_-^\wedge \times \mathbb{R}^q)$, i.e., for given $\gamma \in \mathbb{R}$ such that our operator is Fredholm, the components of a solution $u(r,\phi,y) = {}^{\mathrm{t}}(u_+(r,\phi,y), u_-(r,\phi,y))$ can be written as

$$\begin{aligned} u_\pm(r,\phi,y) = {} & \sum_{j=0}^{N}\sum_{k=0}^{m_{\pm,j}} \mathcal{F}^{-1}_{\eta\to y}[\eta]^{\frac{n+1}{2}}\omega\big(r[\eta]\big)c_{\pm,jk}(\phi)\big(r[\eta]\big)^{-p_{\pm,j}}\log^k\big(r[\eta]\big)\hat{v}_{\pm,jk}(\eta) \\ & + u_{\pm,\Theta}(r,\phi,y) \end{aligned}$$

for asymptotic types $P_\pm = \{(p_{\pm,j}, m_{\pm,j}, L_{\pm,j})\}_{j\in\mathbb{N}}$, coefficients $c_{\pm,jk} \in L_{\pm,j}$, $v_{\pm,jk} \in H^s(\mathbb{R}^q)$ for all j and $0 \leqslant k \leqslant m_{\pm,j}$ and flat remainder $u_{\pm,\Theta}$.

2.3. *Mellin quantization of transmission symbols*

Our next objective is to establish some pseudo-differential formalities which express the structure of parametrices of our boundary-contact problems for differential operators.

Similarly as in the calculus of boundary value problems in a domain with edges the main information comes from a neighborhood of the edge. In localized form we have (stretched) wedges $\Xi_\pm \times \mathbb{R}^q$ and $\Sigma^\wedge \times \mathbb{R}^q$, respectively, with $\mathbb{R}^q$ being the local model of the edge of dimension q, and $\Xi_\pm$ and Σ are the base manifolds of the respective model cones.

By assumption there is a closed compact C^∞ manifold M such that $\Xi_\pm \subset M$ are compact C^∞ manifolds with common boundary $\Sigma = \Xi_+ \cap \Xi_-$, $M := \Xi_+ \cup \Xi_-$. It will be convenient to formulate operators for the case that Σ has only one connected component (although in the example below we have $M = S^1$ with $\Xi_+ = [0,\alpha]$, $\Xi_- = [\alpha, 2\pi]$ for $0 < \alpha < \pi$ with Σ consisting of two point $\phi = 0$ and $\phi = \alpha$; the corresponding modification will be straightforward). The main ingredient of the symbolic structure of parametrices of elliptic boundary-contact problems are parameter-dependent transmission problems of the class $B^{\mu,d}(\Xi_+, \Xi_-; \mathbb{R}^l)$, $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, where $\lambda \in \mathbb{R}^l$ is parameter (in our case needed for the case $l = 1, 2$). The spaces $B^{\mu,d}(\Xi_+, \Xi_-; \mathbb{R}^l)$ consist of families

$$p(\lambda): H^s(\Xi_+) \oplus H^s(\Xi_-) \oplus H^{s-\frac{1}{2}}(\Sigma) \to H^{s-\mu}(\Xi_+) \oplus H^{s-\mu}(\Xi_-) \oplus H^{s-\mu-\frac{1}{2}}(\Sigma) \tag{26}$$

$s > d - \frac{1}{2}$, cf. also (16). The precise definition is given in [10]. The technique in connection with transmission problems in the case of smooth interfaces is close to the calculus of pseudo-differential boundary value problems with the transmission property, see [2,5,7,17]. For convenience we consider block matrices $p(\lambda) = (p_{ij}(\lambda))_{i,j=1,2,3}$ such that the entries are scalar and of order as in the formula (26); more precisely, $p_{11}(\lambda)$ is of order μ, $p_{31}(\lambda)$ of order $\mu + \frac{1}{2}$, etc. In general, we may have larger matrices with entries of arbitrary orders. However this case only needs trivial modifications and will be tacitly used later on. Let us also note that in the pseudo-differential characterization of boundary and transmission problems we have to expect trace and potential entries at the same time (in contrast to (16)) where we only have trace terms; the potential terms are generated in parametrices.

The spaces $B^{\mu,d}(\Xi_+, \Xi_-; \mathbb{R}^l)$ are Fréchet, and there are subspaces $B^{\mu,d}_G(\Xi_+, \Xi_-; \mathbb{R}^l)$ of so-called Green elements where $p_{11}(\lambda) \in L^{-\infty}(\operatorname{int}\Xi_+; \mathbb{R}^l)$, $p_{22}(\lambda) \in L^{-\infty}(\operatorname{int}\Xi_-; \mathbb{R}^l)$.

Remark 2.4. There is a straightforward analogue of the spaces $B^{\mu,d}(\Xi_+, \Xi_-; \mathbb{R}^l)$ for the case of non-compact $\Xi_\pm$ with common boundary $\Xi_+ \cap \Xi_-$, decomposing a manifold M. Then the Sobolev spaces in (26) have to be replaced by corresponding 'comp' and 'loc' variants. Spaces of transmission operators in the non-compact case will occur in the versions $B^{\mu,d}(\Xi_+^\wedge, \Xi_-^\wedge; \mathbb{R}^l)$ and

$$B^{\mu,d}\big(\Xi_+^\wedge \times \mathbb{R}^q, \Xi_-^\wedge \times \mathbb{R}^l\big). \tag{27}$$

Motivated by the form of (11), (12) we consider, in particular, families of operators

$$p(r, y, \varrho, \eta) := \tilde{p}(r, y, r\varrho, r\eta) \tag{28}$$

for $\tilde{p}(r, y, \tilde{\varrho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, B^{\mu,d}(\Xi_+, \Xi_-; \mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}))$ (the weight factors $r^{-\mu}$, etc., in front of the operators (11), (12) are ignored for the moment). In contrast to the case of differential transmission problems as in Section 1.2 we have not at once associated families (14) that are holomorphic in z, but we need a so-called Mellin quantization to pass from p to families h of that kind. In order to formulate a corresponding result we need to say what we understand by a holomorphic family of transmission problems.

By $B^{\mu,d}(\Xi_+, \Xi_-; \mathbb{C} \times \mathbb{R}^q)$ for any $q \in \mathbb{N}$ we denote the space of all $h(z, \eta) \in \mathcal{A}(\mathbb{C}, B^{\mu,d}(\Xi_+, \Xi_-; \mathbb{R}^q))$ such that

$$h(\beta + i\varrho, \eta) \in B^{\mu,d}\big(\Xi_+, \Xi_-; \mathbb{R}^{1+q}_{\varrho,\eta}\big)$$

for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals. The space $B^{\mu,d}(\Xi_+, \Xi_-; \mathbb{C} \times \mathbb{R}^q)$ is Fréchet as well. So we can talk about C^∞ functions in $(r, \eta) \in \overline{\mathbb{R}}_+ \times \mathbb{R}^q$ with values there.

Theorem 2.5. [10, Theorem 3.10] *Given any* $\tilde{p}(r, y, \tilde{\varrho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, B^{\mu,d}(\Xi_+, \Xi_-; \mathbb{R}^{1+q}_{\tilde{\varrho},\tilde{\eta}}))$ *there exists an* $\tilde{h}(r, y, z, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, B^{\mu,d}(\Xi_+, \Xi_-; \mathbb{C} \times \mathbb{R}^q))$ *such that* $p(r, y, \varrho, \eta)$ *defined by* (28) *and* $h(r, y, z, \eta) := \tilde{h}(r, y, z, r\eta)$ *satisfy the relation*

$$\mathrm{op}_r(p)(y, \eta) = \mathrm{op}_M^\delta(h)(y, \eta) \mod C^\infty\big(\mathbb{R}^q, B^{-\infty,d}\big(\Xi_+^\wedge, \Xi_-^\wedge; \mathbb{R}^q\big)\big)$$

for every $\delta \in \mathbb{R}$.

Observe that $\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y, \eta) \in C^\infty(\mathbb{R}^q, B^{\mu,d}(\Xi_+^\wedge, \Xi_-^\wedge; \mathbb{R}^q))$ and

$$\begin{aligned}\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y, \eta) : {} & \mathcal{H}^s\big(\Xi_+^\wedge\big) \oplus \mathcal{H}^s\big(\Xi_-^\wedge\big) \oplus \mathcal{H}^{s-\frac{1}{2}}\big(\Sigma^\wedge\big) \\ & \to \mathcal{H}^{s-\mu}\big(\Xi_+^\wedge\big) \oplus \mathcal{H}^{s-\mu}\big(\Xi_-^\wedge\big) \oplus \mathcal{H}^{s-\mu-\frac{1}{2}}\big(\Sigma^\wedge\big)\end{aligned}$$

are continuous operators for all $s > d - \frac{1}{2}$.

Remark 2.6. If p and h are as in Theorem 2.5 and

$$p_0(r, y, \varrho, \eta) := \tilde{p}(0, y, r\varrho, r\eta), \qquad h_0(r, y, z, \eta) := \tilde{h}(0, y, z, r\eta)$$

we also have

$$\mathrm{op}_r(p_0)(y,\eta)=\mathrm{op}_M^{\delta}(h_0)(y,\eta) \quad \mathrm{mod}\; C^{\infty}\big(\mathbb{R}^q, B^{-\infty,d}\big(\Xi_+^{\wedge}, \Xi_-^{\wedge}; \mathbb{R}^q\big)\big).$$

Transmission amplitude functions in $(y,\eta)\in\mathbb{R}^q\times\mathbb{R}^q$ (as variables and covariables on the edge) are defined as follows. Let $\omega_j(r)$, $j=1,2,3$, be cut-off functions such that $\omega_2\equiv 1$ on $\mathrm{supp}\,\omega_1$ and $\omega_1\equiv 1$ on $\mathrm{supp}\,\omega_3$. Moreover, let $\sigma(r)$, $\tilde\sigma(r)$ be other cut-off functions. Then we form

$$\begin{aligned} a(y,\eta) := \sigma(r)r^{-\mu}\big\{&\omega_1\big(r[\eta]\big)\,\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta)\omega_2\big(r[\eta]\big) \\ &+\big(1-\omega_1\big(r[\eta]\big)\big)\,\mathrm{op}_r(p)(y,\eta)\big(1-\omega_3\big(r[\eta]\big)\big)\big\}\tilde\sigma(r). \end{aligned} \tag{29}$$

Similarly as before in the case of differential transmission problems we have

$$a(y,\eta)\in S^{\mu}\big(\mathbb{R}^q\times\mathbb{R}^q; E, \widetilde{E}\big)$$

with

$$\begin{aligned} E &= \mathcal{K}^{s,\gamma}\big(\Xi_+^{\wedge}\big)\oplus\mathcal{K}^{s,\gamma}\big(\Xi_-^{\wedge}\big)\oplus\mathcal{K}^{s-\frac12,\gamma-\frac12}\big(\Sigma^{\wedge}\big), \\ \widetilde{E} &= \mathcal{K}^{s-\mu,\gamma-\mu}\big(\Xi_+^{\wedge}\big)\oplus\mathcal{K}^{s-\mu,\gamma-\mu}\big(\Xi_-^{\wedge}\big)\oplus\mathcal{K}^{s-\mu-\frac12,\gamma-\mu-\frac12}\big(\Sigma^{\wedge}\big). \end{aligned}$$

The space of symbols refers to the group action

$$\boldsymbol{\kappa}_{\lambda}: u_+(r,\cdot)\oplus u_-(r,\cdot)\oplus v(r,\cdot)\to\lambda^{\frac{n+1}{2}}u_+(\lambda r,\cdot)\oplus u_-(\lambda r,\cdot)\oplus v(\lambda r,\cdot). \tag{30}$$

This allows us to form $\mathrm{Op}_y(a)$. Observe that when we consider the family of (pseudo-differential) transmission problems $r^{-\mu}\tilde p(r,y,rD_r,rD_y)$ (obtained by applying the operator convention based on the Fourier transform in $(r,y)\in\mathbb{R}_+\times\mathbb{R}^q$) we have

$$\sigma(r)r^{-\mu}\tilde p(r,y,rD_r,rD_y)\tilde\sigma(r)=\mathrm{Op}_y(a) \quad \mathrm{mod}\; B^{-\infty,d}\big(\Xi_+^{\wedge}\times\mathbb{R}^q, \Xi_-^{\wedge}\times\mathbb{R}^q\big).$$

Thus $p\to\mathrm{Op}_y(a)$ can be regarded as a quantization of the {transmission problem on (Ξ_+,Ξ_-)}-valued amplitude function $r^{-\mu}p(r,y,\varrho,\eta)$, now based on the Mellin transform in r-direction near $r=0$. At the same time we took holomorphic representatives in the quantization in r near 0. In order to reflect asymptotic phenomena in the pseudo-differential context we therefore add so-called smoothing Mellin plus Green symbols. The definitions will be given in Section 3.3 below. The similar construction for the step $r^{-\mu}p\to a(y,\eta)$ may be found in [18].

Let us now describe the principal symbolic structure of the operator functions $a(y,\eta)$. From the definition we see that $a(y,\eta)\in C^{\infty}(\mathbb{R}^q, B^{\mu,d}(\Xi_+^{\wedge},\Xi_-^{\wedge};\ \mathbb{R}^q))$. Writing $a(y,\eta)=(a_{ij}(y,\eta))_{i,j=1,2,3}$ we have that $a_{11}(y,\eta)\in C^{\infty}(\mathbb{R}^q,\ L_{\mathrm{cl}}^{\mu}(\mathrm{int}\,\Xi_+^{\wedge};\mathbb{R}^q))$, and $a_{22}(y,\eta)\in C^{\infty}(\mathbb{R}^q,$ $L_{\mathrm{cl}}^{\mu}(\mathrm{int}\,\Xi_-^{\wedge};\mathbb{R}^q))$, where the operators have the transmission property at the interface $\Sigma^{\wedge}$.

Let $\sigma_{\psi,\pm}(a)$ denote the parameter-dependent (with parameter $\eta\in\mathbb{R}^q$) homogeneous principal symbol of $a_{11}(y,\eta)$ and $a_{22}(y,\eta)$, respectively, with the $+$ sign for the first, the $-$ sign for the second operator. Let us consider, for instance, the plus case. We have $\sigma_{\psi,+}(a)\in C^{\infty}(T^*\Xi_+^{\wedge}\times$

$\mathbb{R}^q \setminus 0$) (with additional smoothness in $y \in \mathbb{R}^q$ and 0 denoting the covector $(\varrho, \xi, \eta) = 0$). In addition in the splitting of variables $(r, x, y) \in \mathbb{R}_+ \times \Xi_+ \times \mathbb{R}^q$ we have the representation

$$\sigma_{\psi,+}(a)(r, x, y, \varrho, \xi, \eta) = r^{-\mu} \tilde{\sigma}_{\psi,+}(a)(r, x, y, r\varrho, \xi, r\eta)$$

for a homogeneous function $\tilde{\sigma}_{\psi,+}(a)(r, x, y, r\varrho, \xi, r\eta)$ in $(\tilde{\varrho}, \xi, \tilde{\eta}) \neq 0$, smooth up to $r = 0$. In a similar manner we have $\sigma_{\psi,-}(a)$ together with $\tilde{\sigma}_{\psi,-}(a)$.

In order to define the principal transmission symbol of $a(y, \eta)$ we consider a tubular neighborhood $V \subset M$ of Σ, $V \cong \Sigma \times (-1, 1)$, set $V_\pm = \Xi_\pm \cap V$ and define a reflection diffeomorphism $\varepsilon : V_-^\wedge \to V_+^\wedge$ by $\varepsilon(r, x', t) := (r, x', -t)$, $x' \in \Sigma$. This allows us to pass to the operators

$$a_{V_+^\wedge}(y, \eta) := \operatorname{diag}\big(\mathrm{id}, (\varepsilon^*)^{-1}, \mathrm{id}\big) a(y, \eta)|_{V^\wedge} \operatorname{diag}(\mathrm{id}, \varepsilon^*, \mathrm{id}).$$

We thus obtain a family of (pseudo-differential) boundary-value problems $a_{V_+^\wedge}(y, \eta)$ on $V_+^\wedge$ with the boundary $\Sigma^\wedge$. As such it is a principal boundary symbol

$$\sigma_\partial(a_{V_+^\wedge})(r, x', y, \varrho, \xi', \eta) : H^s(\mathbb{R}_+) \oplus H^s(\mathbb{R}_+) \oplus \mathbb{C} \to H^{s-\mu}(\mathbb{R}_+) \oplus H^{s-\mu}(\mathbb{R}_+) \oplus \mathbb{C},$$

$s > d - \frac{1}{2}$; here (x', ξ') denotes the points in $T^*\Sigma$, and the definition refers to $(\varrho, \xi', \eta) \neq 0$. Passing to

$$\sigma_{\mathrm{tr}}(a)(r, x', y, \varrho, \xi', \eta) := \operatorname{diag}(\mathrm{id}, \varepsilon^*, \mathrm{id}) \sigma_\partial(a_{V_+^\wedge})(r, x', y, \varrho, \xi', \eta) \operatorname{diag}\big(\mathrm{id}, (\varepsilon^*)^{-1}, \mathrm{id}\big)$$

we obtain the homogeneous principal transmission symbol of a of order μ, namely,

$$\sigma_{\mathrm{tr}}(a)(r, x', y, \varrho, \xi', \eta) : H^s(\mathbb{R}_+) \oplus H^s(\mathbb{R}_-) \oplus \mathbb{C} \to H^{s-\mu}(\mathbb{R}_+) \oplus H^{s-\mu}(\mathbb{R}_-) \oplus \mathbb{C}.$$

From the definition it follows that there is another operator function $\tilde{\sigma}_{\mathrm{tr}}(a)(r, x', y, \tilde{\varrho}, \xi', \tilde{\eta})$, homogeneous in $(\tilde{\varrho}, \xi', \tilde{\eta}) \neq 0$ and smooth up to $r = 0$, such that

$$\sigma_{\mathrm{tr}}(a)(r, x', y, \varrho, \xi', \eta) = r^{-\mu} \tilde{\sigma}_{\mathrm{tr}}(a)(r, x', y, r\varrho, \xi', r\eta).$$

2.4. *The algebra of boundary-contact operators*

The category of operators $\mathcal{A}$ that we observe in this section are a pseudo-differential analogue of the boundary-contact problems of Section 1.1. Because of the expected shape of parametrices of elliptic elements and in order to carry out compositions within our class of operators we start from 4×4 block matrices $\mathcal{A} = (A_{ij})_{i,j=1,\dots,4}$ which contains trace and potential operators with respect to $\mathbb{S}$, and ∂G at the same time. Our operators will be continuous as maps

$$\begin{aligned} \mathcal{A} : {} & \mathcal{W}^{s,\gamma}_{(P_+)}(\mathbb{W}_+) \oplus \mathcal{W}^{s,\gamma}_{(P_-)}(\mathbb{W}_-) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}_{(S)}\big(\mathbb{S}, \mathbb{C}^L\big) \oplus H^{s-\frac{1}{2}}\big(\partial G, \mathbb{C}^{I'}\big) \\ & \to \mathcal{W}^{s-\mu,\gamma-\mu}_{(Q_+)}(\mathbb{W}_+) \oplus \mathcal{W}^{s-\mu,\gamma-\mu}_{(Q_-)}(\mathbb{W}_-) \oplus \mathcal{W}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}_{(T)}\big(\mathbb{S}, \mathbb{C}^N\big) \oplus H^{s-\mu-\frac{1}{2}}\big(\partial G, \mathbb{C}^{J'}\big) \end{aligned} \tag{31}$$

for all $s > d - \frac{1}{2}$. Subscripts '$(P_\pm)$', etc., mean that we have continuity between spaces with (or without) the corresponding asymptotic types.

We will concentrate on the regularity of solutions with asymptotics, starting from solutions in weighted edge spaces without asymptotics. This allows us to ignore any extra edge entries of trace and potential type with respect to Y (those may occur in the general edge pseudo-differential calculus). The regularity including the smoothness s would require observing also these conditions; however, this is voluminous; so we ignore this aspect here. In order to understand the typical contributions to the asymptotics it is enough to consider operators close to the edge, i.e., the localized on a (stretched) 'wedge' of the form $M^\wedge \times \mathbb{R}^q$ for $M = \Xi_+ \cup \Xi_-$ with $\Sigma = \Xi_+ \cap \Xi_-$. Since ∂G does not intersect the edge we omit the components referring to ∂G (although the smoothing operators in the global edge calculus may also contribute asymptotic information). In other words the essential information is coming from 3×3 block matrices $(A_{ij})_{i,j=1,2,3}$.

Remark 2.7. As noted before in the pseudo-differential descriptions we mainly content ourselves with scalar entries, especially, $L = N = 1$, because the entries of larger block matrices are completely characterized by this case.

In the localized situation we have $\mathbb{W}_+ = (\overline{\mathbb{R}}_+ \times \Xi_\pm) \times \mathbb{R}^q$ (with Y being identified with $\mathbb{R}^q$) and $\mathbb{S} = (\overline{\mathbb{R}}_+ \times \Sigma) \times \mathbb{R}^q$. To unify some notation we set $\mathbb{W} = \overline{\mathbb{R}}_+ \times M \times \mathbb{R}^q$ which is a stretched wedge, subdivided into $\mathbb{W}_\pm$, i.e., $\mathbb{W} = \mathbb{W}_+ \cup \mathbb{W}_-$ with $\mathbb{S} = \mathbb{W}_+ \cap \mathbb{W}_-$. Then a (pseudo-differential) boundary-contact operator on $\mathbb{W}$ has the form

$$\mathcal{A} = \mathrm{Op}_y(a + m + g) + \mathcal{A}_{\mathrm{int}} + \mathcal{C} \tag{32}$$

where $a(y,\eta)$ is an amplitude function of the form (29) with respect to the spaces E and $\widetilde{E}$, furthermore, $m(y,\eta)$ is a smoothing Mellin symbol of the form (47) below, and $g(y,\eta)$ is a Green symbol, cf. Section 3.3. For convenience, we always assume that the involved amplitude functions are independent of y for large $|y|$. Moreover, let

$$\mathcal{A}_{\mathrm{int}} \in (1-\sigma) B^{\mu,d}\big(\Xi_+^\wedge \times \mathbb{R}^q, \Xi_-^\wedge \times \mathbb{R}^q\big)(1-\tilde{\tilde{\sigma}}),$$

where $\sigma(r)$ is as in (29) and $\tilde{\tilde{\sigma}}$ is another cut-off function such that $\sigma \equiv 1$ on $\operatorname{supp}\tilde{\tilde{\sigma}}$. There is no reason to admit a particularly general behavior of $\mathcal{A}_{\mathrm{int}}$ for large r or $|y|$; therefore, we simply assume that the operators are continuous in the above mentioned edge spaces.

The operator $\mathcal{C}$ is smoothing. For type $d = 0$ such operators are characterized by the following properties: $\mathcal{C}$ induces continuous operators

$$\begin{aligned} \mathcal{C}: {}&\mathcal{W}^{s,\gamma}(\mathbb{W}_+) \oplus \mathcal{W}^{s,\gamma}(\mathbb{W}_-) \oplus \mathcal{W}^{s',\gamma-\frac{1}{2}}(\mathbb{S}) \\ &\to \mathcal{W}^{\infty,\gamma-\mu}_{Q_+}(\mathbb{W}_+) \oplus \mathcal{W}^{\infty,\gamma-\mu}_{Q_-}(\mathbb{W}_-) \oplus \mathcal{W}^{\infty,\gamma-\mu-\frac{1}{2}}_{T}(\mathbb{S}) \end{aligned}$$

for all $s > \frac{1}{2}$, $s' \in \mathbb{R}$, with certain asymptotic types $Q_\pm$, T depending on $\mathcal{C}$; a similar behavior is required for the formal adjoint $\mathcal{C}^*$. For arbitrary $d \in \mathbb{N}$ the structure is

$$\mathcal{C} = \mathcal{C}_0 + \sum_{j=1}^{d} \mathcal{C}_j \operatorname{diag}\big(D^j, 0, 0\big) \tag{33}$$

for smoothing $\mathcal{C}_j$ of type 0, $0 \leqslant j \leqslant d$, and a first order differential operator D which differentiates in normal direction to Σ $(= \partial \Xi_\pm)$, cf. the above local description of $\mathbb{W}_\pm$.

Observe that when $\omega(y, y') \in C^\infty(\mathbb{R}^q \times \mathbb{R}^q)$ is a function which is equal to 1 in a neighborhood of $\operatorname{diag}(\mathbb{R}^q \times \mathbb{R}^q)$ and 0 outside another neighborhood of the diagonal, then the operator

$$\operatorname{Op}_y\big((1-\omega)(a+m+g)\big) \tag{34}$$

is smoothing in the above mentioned sense.

Let us also note that an evident global analogue of smoothing operators in the edge calculus on G with the given boundary-contact configuration, encodes global asymptotic properties of solutions, contributed by remainders after the local characterization of asymptotics.

Operators of the form (32) will be called (local) boundary-contact operators. Let $\mu = \operatorname{ord} \mathcal{A}$ (the order of $\mathcal{A}$). Compositions between such operators are possible if one factor is properly supported in a suitable sense (which is an obvious modification of a corresponding notion in the scalar pseudo-differential calculus). For instance, $\operatorname{Op}(\omega(a+m+g))$ with $\omega(y, y')$ as above, is properly supported with respect to (y, y')-variables. Moreover, $\mathcal{A}_{\text{int}}$ can be replaced by a properly supported representative in the class $(1-\sigma)B^{\mu,d}(\Xi_+^\wedge \times \mathbb{R}^q, \Xi_-^\wedge \times \mathbb{R}^q)(1-\tilde{\tilde{\sigma}})$ modulo some smoothing operator $\mathcal{G}$ of the kind (33).

Given an operator $\mathcal{A}$ of the form (32) we set

$$\sigma(\mathcal{A}) = \big(\sigma_{\psi,+}(\mathcal{A}), \sigma_{\psi,-}(\mathcal{A}), \sigma_{\text{tr}}(\mathcal{A}), \sigma_\wedge(\mathcal{A})\big) \tag{35}$$

where $\sigma_{\psi,\pm}(\mathcal{A}) := \sigma_{\psi,\pm}(a) + \sigma_{\psi,\pm}(\mathcal{A}_{\text{int}})$, $\sigma_{\text{tr}}(\mathcal{A}) := \sigma_{\text{tr}}(a) + \sigma_{\text{tr}}(\mathcal{A}_{\text{int}})$, and $\sigma_\wedge(\mathcal{A}) := \sigma_\wedge(a + m + g)$, cf. the notation in Section 3.3. Apart from the principal symbol (35) our operators also have a subordinate (complete) conormal symbols $\sigma_M(\mathcal{A})$; it will be defined in Section 3.3 below.

Remark 2.8. In the latter definition we used the fact that the space of transmission operators $B^{\mu,d}(\Xi_+^\wedge \times \mathbb{R}^q, \Xi_-^\wedge \times \mathbb{R}^q)$ has principal interior symbols $\sigma_{\psi,\pm}(\cdot)$ as functions on $T^*(\Xi_\pm^\wedge \times \mathbb{R}^q)\setminus 0$ as usual (smooth up to $\Sigma^\wedge \times \mathbb{R}^q$ from the respective sides) and a principal transmission symbol $\sigma_{\text{tr}}(\cdot)$ parametrized by $T^*(\Sigma \times \mathbb{R}^q)\setminus 0$, which is a natural analogue of the transmission symbol (10) in the case of differential transmission problems. Observe that also the elements (32) belong to $B^{\mu,d}(\Xi_+^\wedge \times \mathbb{R}^q, \Xi_-^\wedge \times \mathbb{R}^q)$, and they have a specific 'edge-degenerate' structure near the edge $\mathbb{R}^q$. The definition of the class of all operators (32) is independent of the choice of the cut-off functions $\sigma, \tilde{\sigma}, \tilde{\tilde{\sigma}}$.

Clearly in the global calculus on G the tuple of the symbol $\sigma(\mathcal{A})$ also contains a corresponding principal boundary symbol $\sigma_\partial(\mathcal{A})$ associated with the boundary ∂G.

Theorem 2.9. *The composition of two boundary-contact operators $\mathcal{A}$ and $\tilde{\mathcal{A}}$ (one of them properly supported) is again an boundary-contact operator, where* $\operatorname{ord}(\mathcal{A}\tilde{\mathcal{A}}) = \operatorname{ord}\mathcal{A} + \operatorname{ord}\tilde{\mathcal{A}}$ *and* $\sigma(\mathcal{A}\tilde{\mathcal{A}}) = \sigma(\mathcal{A})\sigma(\mathcal{A})$ (*with componentwise composition*).

Definition 2.10. A boundary-contact operator (32) is called Green (of order μ and type d) if both $a(y,\eta)$ and $m(y,\eta)$ in (32) vanish, and $\mathcal{A}_{\text{int}} \in (1-\sigma)B^{-\infty,d}(\Xi_+^\wedge \times \mathbb{R}^q, \Xi_-^\wedge \times \mathbb{R}^q)(1-\tilde{\tilde{\sigma}})$.

Remark 2.11. A boundary-contact operator $\mathcal{G}$ is Green if and only if

(i) $\mathcal{G} \in B^{-\infty,d}(\Xi_+^\wedge \times \mathbb{R}^q, \Xi_-^\wedge \times \mathbb{R}^q)$;
(ii) $\sigma_M(\mathcal{A}) \equiv 0$ (cf. the corresponding notation in Section 3.3).

Proposition 2.12.

(i) *Let $\mathcal{G}$ be as in Definition* 2.10 (*associated with the weights γ, $\gamma-\mu$ and the weight data Θ*). *Then $\mathcal{G}$ induces continuous operator*

$$\begin{aligned}\mathcal{G}:\ &\mathcal{W}^{s,\gamma}(\mathbb{W}_+) \oplus \mathcal{W}^{s,\gamma}(\mathbb{W}_-) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{S}) \\ &\to \mathcal{W}_{Q_+}^{s-\mu,\gamma-\mu}(\mathbb{W}_+) \oplus \mathcal{W}_{Q_-}^{s-\mu,\gamma-\mu}(\mathbb{W}_-) \oplus \mathcal{W}_T^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}(\mathbb{S})\end{aligned} \tag{36}$$

for every $s > d - \frac{1}{2}$, with asymptotic types $Q_\pm$ and T, associated with the weight data $(\gamma-\mu, \Theta)$ and $(\gamma-\mu-\frac{1}{2}, \Theta)$, respectively ($Q_\pm$ and T depending on $\mathcal{G}$).
(ii) *If $\mathcal{A}$ or $\tilde{\mathcal{A}}$ in Theorem* 2.9 *is a Green operator, then so is the composition.*

Proof. A Green operator can be equivalently characterized by $\mathcal{G} = \mathrm{Op}_y(g) + \mathcal{C}$ for a Green symbol $g(y,\eta)$, cf. the notation of Section 3.3, and a smoothing operator $\mathcal{C}$, cf. formula (33). Since $\mathcal{C}$ has the desired mapping property, the assertion follows from the fact that $g(y,\eta)$ is an operator-valued symbol (48) between the spaces E and $\widetilde{E}$ (given in connection with (48)) with group action (30), and

$$\mathrm{Op}_y(g): \mathcal{W}^s(\mathbb{R}^q, E)_{\boldsymbol{\kappa}} \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{E})_{\boldsymbol{\kappa}}, \tag{37}$$

$\boldsymbol{\kappa} = \{\kappa_\lambda\}$, is continuous. Subscripts '$\boldsymbol{\kappa}$' indicate edge spaces modeled on E with the group action $\boldsymbol{\kappa}$, cf. Definition 2.2. Now the space on the left of (37) just coincides with the space on the left of (36) while the space on the right of (37) which is equal to

$$\mathcal{W}^{s-\mu}\big(\mathbb{R}^q, \mathcal{S}_{Q_+}^{\gamma-\mu}(\Xi_+^\wedge)\big) \oplus \mathcal{W}^{s-\mu}\big(\mathbb{R}^q, \mathcal{S}_{Q_-}^{\gamma-\mu}(\Xi_-^\wedge)\big) \oplus \mathcal{W}^{s-\mu-\frac{1}{2}}\big(\mathbb{R}^q, \mathcal{S}_T^{\gamma-\mu-\frac{1}{2}}(\Sigma^\wedge)\big)$$

(where the $\mathcal{W}^s$-spaces refer to the 'standard' group actions on the respective spaces) are continuously embedded into ones on the right of (36). □

3. Asymptotics of solutions

3.1. Ellipticity and regularity of solutions

We now turn to the ellipticity of boundary-contact problems. The structures can be motivated by the fact that the pseudo-differential representatives in the algebra of boundary-contact (or transmission) operators formulate in advance the structure of parametrices of elliptic problems for differential operators, while the weighted edge spaces a priori formulate the nature of elliptic regularity of solutions (with or without asymptotics). To see the results in principle it suffices to assume that the operators are 3×3 block matrices with scalar entries, cf. Remark 2.7. A simple modification then admits the study of arbitrary block matrices (also several kinds of row and column matrices where some components may simply disappear). As in the preceding sections the dimension of $\Xi_\pm$ may be arbitrary; here for convenience > 1.

Definition 3.1. A boundary-contact operator $\mathcal{A}$ of the form (32) of order $\mu \in \mathbb{Z}$ and type $d \in \mathbb{N}$ on a stretched wedge $\mathbb{W} = \mathbb{W}_+ \cup \mathbb{W}_-$ (with scalar entries) is called elliptic (with respect to a fixed weight $\gamma \in \mathbb{R}$) if the components of the principal symbolic hierarchy (35) have the following properties:

(i) The interior symbols $\sigma_{\psi,\pm}(\mathcal{A})$ do not vanish on $T^*(\mathbb{W}_{\pm,\mathrm{reg}})\setminus 0$; similarly, we have $\tilde{\sigma}_{\psi,\pm}(\mathcal{A})(r,x,y,\varrho,\xi,\eta) \neq 0$ for all $(\varrho,\xi,\eta) \neq 0$, up to $r = 0$;
(ii) the transmission symbol $\sigma_{\mathrm{tr}}(\mathcal{A})$ defines a family of bijections

$$\sigma_{\mathrm{tr}}(\mathcal{A}): H^s(\mathbb{R}_+) \oplus H^s(\mathbb{R}_-) \oplus \mathbb{C} \to H^{s-\mu}(\mathbb{R}_+) \oplus H^{s-\mu}(\mathbb{R}_-) \oplus \mathbb{C}$$

parametrized by the points of $T^*(\mathbb{S}_{\mathrm{reg}})\setminus 0$; similarly, $\tilde{\sigma}_{\mathrm{tr}}(\mathcal{A})(r,x',y,\varrho,\xi',\eta)$ are bijections for all $(\varrho,\xi',\eta) \neq 0$, up to $r = 0$;
(iii) the edge symbol $\sigma_\wedge(\mathcal{A})$, parametrized by $(y,\eta) \in T^*\mathbb{R}^q\setminus 0$, defines a family of Fredholm operators

$$\begin{aligned}\sigma_\wedge(\mathcal{A}): &\mathcal{K}^{s,\gamma}\big(\Xi_+^\wedge\big) \oplus \mathcal{K}^{s,\gamma}\big(\Xi_-^\wedge\big) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}\big(\Sigma^\wedge\big)\\ &\to \mathcal{K}^{s-\mu,\gamma-\mu}\big(\Xi_+^\wedge\big) \oplus \mathcal{K}^{s-\mu,\gamma-\mu}\big(\Xi_-^\wedge\big) \oplus \mathcal{K}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}\big(\Sigma^\wedge\big).\end{aligned}$$

The conditions (ii), (iii) are required for all $s > \max(\mu,d) - \frac{1}{2}$; they are then independent of s.

Remark 3.2. Condition (iii) in Definition 3.1 together with (i), (ii) is equivalent to the bijectivity of the subordinate conormal symbol

$$\begin{aligned}\sigma_M\sigma_\wedge(\mathcal{A})(y,z): &H^s(\Xi_+) \oplus H^s(\Xi_-) \oplus H^{s-\frac{1}{2}}(\Sigma)\\ &\to H^{s-\mu}(\Xi_+) \oplus H^{s-\mu}(\Xi_-) \oplus H^{s-\mu-\frac{1}{2}}(\Sigma)\end{aligned}$$

for any (and, then, equivalently, all) $s > \max(\mu,d) - \frac{1}{2}$, for all $y \in \mathbb{R}^q$ and all $z \in \Gamma_{(n+1)/2-\gamma}$, $n = \dim \Xi_\pm$.

Observe that we have $\sigma_M\sigma_\wedge(\mathcal{A})(y,z) \in C^\infty(\mathbb{R}^q, M_R^{\mu,d}(\Xi_+,\Xi_-))$ for some Mellin asymptotic type R as described in Section 3.3 below. Recall that the role of the conormal symbols for the asymptotics of solution in the simpler case of conical singularities has been emphasized in the work of Kondratyev [11].

The y-wise inverse $(\sigma_M\sigma_\wedge(\mathcal{A})(y,z))$ defines a family in $M_S^{-\mu,\max(d-\mu,0)}(\Xi_+,\Xi_-)$, where the Mellin asymptotic type S may depend on y. We do not study this effect here; this would require continuous asymptotic types or refined version of variable and pointwise discrete asymptotic types, cf. [19] or [9]. Therefore, in the following Theorems 3.3 and 3.4 we assume that S is independent of y.

Theorem 3.3. *Let $\mathcal{A}$ be a boundary-contact operator which is elliptic of order μ and type d in the sense of Definition* 3.1. *Then there exists an elliptic* (*properly supported*) *boundary-contact operator $\mathcal{P}$ of order $-\mu$ and type* $\max(d-\mu,0)$ (*with respect to the weight $\gamma-\mu$*) *which is a parametrix of $\mathcal{A}$ in the sense that*

$$\mathcal{I}-\mathcal{P}\mathcal{A}=\mathcal{G}_l \quad \textit{and} \quad \mathcal{I}-\mathcal{A}\mathcal{P}=\mathcal{G}_r$$

are Green operators of order 0 *and types* $d_l=\max(\mu,d)$ *and* $d_r=\max(d-\mu,0)$, *respectively.*

Proof. First observe that the operator $\mathcal{A}$ is elliptic in $B^{\mu,d}(\Xi_+^\wedge\times\mathbb{R}^q,\Xi_-^\wedge\times\mathbb{R}^q)$ with respect to $\sigma_{\psi,\pm}$ and σ_{tr}. We use the fact that there is then a (properly supported) parametrix $\mathcal{P}_1\in B^{-\mu,\max(d-\mu,0)}(\Xi_+^\wedge\times\mathbb{R}^q,\Xi_-^\wedge\times\mathbb{R}^q)$ of $\mathcal{A}$ such that $\sigma_{\psi,\pm}(\mathcal{P}_1)=\sigma_{\psi,\pm}(\mathcal{A})^{-1}$, $\sigma_{\mathrm{tr}}(\mathcal{P}_1)=\sigma_{\mathrm{tr}}(\mathcal{A})^{-1}$. We now improve $\mathcal{P}_1$ near the edge $\mathbb{R}^q$, i.e., near $r=0$, by setting

$$\mathcal{P}:=\mathrm{Op}_y(b+l+g)+(1-\sigma)\mathcal{P}_1(1-\tilde{\tilde{\sigma}}). \tag{38}$$

Here $b(y,\eta)$ is given by

$$\begin{aligned} b(y,\eta):=\sigma(r)r^{\mu}\big\{&\omega_1\big(r[\eta]\big)\mathrm{op}_M^{\gamma-\mu-\frac{n}{2}}(f)(y,\eta)\omega_2\big(r[\eta]\big)\\ &+\big(1-\omega_1\big(r[\eta]\big)\big)\,\mathrm{op}_r(t)(y,\eta)\big(1-\omega_3\big(r[\eta]\big)\big)\big\}\tilde{\sigma}(r)\end{aligned}$$

with cut-off functions $\sigma,\tilde{\sigma},\omega_1,\omega_2,\omega_3$ as in (29), and

$$t(r,y,\varrho,\eta)=\tilde{t}(r,y,r\varrho,r\eta) \tag{39}$$

for a suitable $\tilde{t}(r,y,r\varrho,r\eta)\in C^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}^q,B^{-\mu,e}(\Xi_+,\Xi_-;\mathbb{R}^{1+q}))$, $e:=\max(d-\mu,0)$ and an associated $f(r,y,z,\eta)=\tilde{f}(r,y,z,r\eta)$, $\tilde{f}(r,y,z,r\eta)\in C^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}^q,B^{-\mu,e}(\Xi_+,\Xi_-;\mathbb{C}\times\mathbb{R}^q))$ in the sense of Theorem 2.5. Moreover, $l(y,\eta)$ is a suitable Mellin edge symbol of analogous form as (47) below, here with $-\mu$ and $\gamma-\mu$ instead of μ and γ, respectively. The choice of l will be explained later on in this proof.

In order to construct the operator family (39) (which we only need in a neighborhood of $r=0$, since the operators (32) are independent of the cut-off functions σ, $\tilde{\sigma}$, etc., modulo smoothing elements) we first recall that the essential contribution to $\mathcal{A}$ near the edge is given by the operator function

$$r^{-\mu}\tilde{p}(r,y,r\varrho,r\eta) \tag{40}$$

for $\tilde{p}(r,y,\tilde{\varrho},\tilde{\eta})\in C^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}^q,B^{\mu,d}(\Xi_+,\Xi_-;\mathbb{R}^{1+q}))$. Let us ignore for the moment the way to obtain (32) in terms of (40) and Theorem 2.5. From the assumption of ellipticity with respect to $\sigma_{\psi,\pm}$ and σ_{tr} (including the $\tilde{\sigma}_{\psi,\pm}$ and $\tilde{\sigma}_{\mathrm{tr}}$ objects up to $r=0$) we know that $\tilde{p}(r,y,\tilde{\varrho},\tilde{\eta})$ is parameter-dependent elliptic in $B^{\mu,d}(\Xi_+,\Xi_-;\mathbb{R}^{1+q}))$ with the parameters $\tilde{\varrho}$, $\tilde{\eta}$, for every $(r,y)\in\overline{\mathbb{R}}_+\times\mathbb{R}^q$ (up to $r=0$). This allows us to construct a parameter-dependent elliptic family $\tilde{p}^{(-1)}(r,y,\tilde{\varrho},\tilde{\eta})\in C^\infty(\overline{\mathbb{R}}_+\times\mathbb{R}^q,B^{-\mu,e}(\Xi_+,\Xi_-;\mathbb{R}^{1+q}))$ such that for the pointwise composition (in $(r,y)\in\overline{\mathbb{R}}_+\times\mathbb{R}^q$) we have

$$\tilde{p}^{(-1)}(r,y,\tilde{\varrho},\tilde{\eta})\tilde{p}(r,y,\tilde{\varrho},\tilde{\eta})=1+\tilde{c}(r,y,\tilde{\varrho},\tilde{\eta}) \tag{41}$$

for an element $\tilde{c}(r, y, \tilde{\varrho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, B^{-1,\max(\mu,d)}(\Xi_+, \Xi_-; \mathbb{R}^{1+q}))$. In the following considerations we carry out operators in terms of Leibniz products of operator functions depending on $(r, y, r\varrho, r\eta)$ which imitate compositions of associated pseudo-differential operators $\mathrm{Op}_{r,y}(\cdot)$ with such amplitude functions. Denoting the Leibniz multiplication between such operator functions $\boldsymbol{b}$ and $\boldsymbol{c}$ by #, i.e.,

$$\boldsymbol{b}(r, y, r\varrho, r\eta) \# \boldsymbol{c}(r, y, r\varrho, r\eta) \sim \sum_{\alpha \in \mathbb{N}^{1+q}} \frac{1}{\alpha!} \big(\partial^\alpha_{\varrho,\eta} \boldsymbol{b}(r, y, r\varrho, r\eta)\big) D^\alpha_{r,y} \boldsymbol{c}(r, y, r\varrho, r\eta)$$

(with ∂ indicating differentiation without the factor i^{-1}) we see some very convenient properties. In the asymptotic summation the 'edge-degenerate' shape of the terms remains preserved, and also the representative modulo an operator function of order $-\infty$ can be chosen in such a way that it is smooth in the first r-variable up to zero. Another feature of this kind of operations is that $\{r^\mu \boldsymbol{b}(r, y, r\varrho, r\eta)\} \# \{r^{-\mu} \boldsymbol{c}(r, y, r\varrho, r\eta)\}$ is of analogous behavior, i.e., the extra r-powers are canceling out, such that there only remain smooth terms in the first r-variable up to zero. From (41) it follows that

$$\big\{r^\mu \tilde{p}^{(-1)}(r, y, r\varrho, r\eta)\big\}\big\{r^{-\mu} \tilde{p}(r, y, r\varrho, r\eta)\big\} = 1 + \tilde{c}(r, y, r\varrho, r\eta).$$

This implies that

$$\big\{r^\mu \tilde{p}^{(-1)}(r, y, r\varrho, r\eta)\big\} \# \big\{r^{-\mu} \tilde{p}(r, y, r\varrho, r\eta)\big\} = 1 + \tilde{c}_1(r, y, r\varrho, r\eta)$$

for a $\tilde{c}_1(r, y, r\varrho, r\eta)$ of analogous property as $\tilde{c}(r, y, r\varrho, r\eta)$. There is now a $\tilde{d}_1(r, y, r\varrho, r\eta)$, again of the same structure, such that

$$\big\{1 + \tilde{d}_1(r, y, r\varrho, r\eta)\big\} \# \big\{1 + \tilde{c}_1(r, y, r\varrho, r\eta)\big\} = 1 + \tilde{k}(r, y, r\varrho, r\eta)$$

for a $\tilde{k}(r, y, r\varrho, r\eta) \in B^{-\infty,\max(\mu,d)}(\Xi_+, \Xi_+; \mathbb{R}^{1+q})$. This gives us

$$\begin{aligned}&\big[\big\{1 + \tilde{d}_1(r, y, r\varrho, r\eta)\big\} \# \big\{r^\mu \tilde{p}^{(-1)}(r, y, r\varrho, r\eta)\big\}\big] \# \big\{r^{-\mu} \tilde{p}(r, y, r\varrho, r\eta)\big\}\\ &\quad = 1 + \tilde{k}(r, y, r\varrho, r\eta).\end{aligned}$$

The expression in $[\ldots]$ is nothing other than $r^\mu \tilde{t}(r, y, r\varrho, r\eta)$ with an operator function $\tilde{t}(r, y, \tilde{\varrho}, \tilde{\eta})$ as required. As announced before we want to express our parametrix near $r = 0$ in a form analogous to (32). Since in the final result we admit Green remainders of order 0 the only point is to find the Mellin amplitude function $l(y, \eta)$ which is expected to be of the form

$$l(y, \eta) = r^\mu \omega\big(r[\eta]\big) \sum_{j=0}^{k} r^j \sum_{|\alpha| \leqslant j} \mathrm{op}_M^{\gamma_j - \mu - \frac{n}{2}}(l_{j\alpha})(y) \eta^\alpha \tilde{\omega}\big(r[\eta]\big) \tag{42}$$

for suitable γ_j such that $\gamma - j \leqslant \gamma_j \leqslant \gamma$ (especially, $\gamma_0 = \gamma$) and smoothing Mellin symbols $l_{j\alpha} \in C^\infty(\mathbb{R}^q, M^{-\infty,e}_{R_{j\alpha}}(\Xi_+, \Xi_-))$ with certain Mellin asymptotic types $R_{j\alpha}$. The main issue is to find $l_0 := l_{00}$; the other $l_{j\alpha}$ then follow afterwards. Similarly as in the general edge operator calculus we have

$$\sigma_M\sigma_\wedge(b)(y, z+\mu)\sigma_M\sigma_\wedge(a)(y, z) = 1 + h_0(y, z+\mu)$$

for some $h_0(y, z) \in C^\infty(\mathbb{R}^q, M_R^{-\infty,\max(\mu,d)}(\Xi_+, \Xi_-))$ and certain R. The principal conormal symbol of $b(y, \eta)$ has the form $\sigma_M\sigma_\wedge(b)(y, z) = \tilde{f}(0, y, z, 0)$. This gives us

$$\big(1 + h_0(y, z+\mu)\big)^{-1}\tilde{f}(0, y, z+\mu, 0) = \big(\sigma_M\sigma_\wedge(a)(y, z)\big)^{-1}.$$

We have $(1 + h_0(y, z+\mu))^{-1} - 1 = k_0(y, z+\mu) \in C^\infty(\mathbb{R}^q, M_R^{-\infty,\max(\mu,d)}(\Xi_+, \Xi_-))$; here and in the sequel by R we denote different Mellin asymptotic types. We have $l_0(y, z) := k_0(y, z)\tilde{f}(0, y, z, 0) \in C^\infty(\mathbb{R}^q, M_R^{-\infty,e}(\Xi_+, \Xi_-))$. The inevitability of $\sigma_M\sigma_\wedge(a)(y, z)$ for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ shows us that $l_0(y, z+\mu)$ has no poles on the weight line $\Gamma_{\frac{n+1}{2}-(\gamma-\mu)}$. Thus we can form (42) with the constructed $l_0(y, z)$ and unknown Mellin symbols $l_{j\alpha}(y, z)$ for $j > 0$, $|\alpha| \leqslant j$. In any case it follows that

$$\sigma_M\sigma_\wedge\big[(b+l)\,\#_y\, a\big](y, z) = 1$$

for all $y \in \mathbb{R}^q$, $z \in \mathbb{C}$; here $\#_y$ denotes the Leibniz multiplication of (operator-valued) amplitude functions in y. The complete conormal symbol of $b(y, \eta) + l(y, \eta)$ has the form

$$\boldsymbol{l}(y, z, \eta) := \left(\frac{1}{j!}\left(\frac{\partial^j}{\partial r^j}\tilde{f}(r, y, z, r\eta)\right)\bigg|_{r=0} + \sum_{|\alpha|\leqslant j} l_{j\alpha}(y, z)\eta^\alpha\right)_{0\leqslant j\leqslant k};$$

we see that the components are polynomials in η of order j. Similarly, $a(y, \eta)$ has the complete conormal symbol

$$\boldsymbol{f}(y, z, \eta) := \left(\frac{1}{j!}\left(\frac{\partial^j}{\partial r^j}\tilde{h}(r, y, z, r\eta)\right)\bigg|_{r=0} + \sum_{|\alpha|\leqslant j} f_{j\alpha}(y, z)\eta^\alpha\right)_{0\leqslant j\leqslant k}.$$

The sequence $\boldsymbol{e}(y, z, \eta)$ of conormal symbols of $(b+l)\,\#_y\, a$ then follows by a combination of $\#_y$ with the Mellin translation product in z, cf., analogously, [20, Theorem 2.4.15] or [22]. Setting $\boldsymbol{e}(y, z, \eta) = (1, 0, \dots, 0)$ we obtain (since $l_0(y, z)$ is already calculated and $\boldsymbol{f}(y, z, \eta)$ is given by the original operator) a recursive formula to uniquely determine the components $l_{j\alpha}$ for $j > 0$ and $|\alpha| \leqslant j$. This allows us to form (42) for any choice of cut-off functions ω, $\tilde{\omega}$ (and of the function $\eta \to [\eta]$). Any other choice generates remainders in form of Green symbols $g(y, \eta)$ of order $-\mu$, cf. the corresponding observation in Section 3.3 below. Thus, without loss of generality we may assume that $\sigma(r)\omega(r[\eta]) = \omega(r[\eta])$ and $\tilde{\sigma}(r)\tilde{\omega}(r[\eta]) = \tilde{\omega}(r[\eta])$ for all $r \in \mathbb{R}_+$, $\eta \in \mathbb{R}^q$ with $\sigma, \tilde{\sigma}$ being the cut-off functions in the expression for $b(y, \eta)$. In order to obtain a properly supported parametrix also near $r = 0$ we choose a cut-off factor $\omega(y, y')$ as mentioned in connection with (34) and observe that for every Green symbol $g_1(y, \eta)$ (here of order $-\mu$) there is another Green symbol $g(y, \eta)$ such that

$$\mathcal{P}_0 := \operatorname{Op}_y\big(\omega(b+l+g_1)\big) = \operatorname{Op}(b+l+g) + \mathcal{C} \tag{43}$$

for a smoothing operator $\mathcal{C}$ (cf. the notation in Section 2.4). Since in the final parametrix we accept Green remainders of order zero the expression (38) with the above mentioned $\mathcal{P}_1$ and (43) gives us a left parametrix of $\mathcal{A}$. In a similar manner we find a right parametrix modulo a Green remainder. Thus our parametrix is two-sided; this completes the proof. □

We now formulate the regularity of solutions u to elliptic boundary-contact equations $\mathcal{A}u = f$ with asymptotics.

Theorem 3.4. *Let $\mathcal{A}$ be an elliptic boundary-contact operator of order $\mu \in \mathbb{Z}$ and type $d \in \mathbb{N}$, and let $u \in \mathcal{W}^{s,\gamma}(\mathbb{W}_+) \oplus \mathcal{W}^{s,\gamma}(\mathbb{W}_-) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{S})$ be a solution of*

$$\mathcal{A}u = f \in \mathcal{W}_{Q_+}^{s-\mu,\gamma-\mu}(\mathbb{W}_+) \oplus \mathcal{W}_{Q_-}^{s-\mu,\gamma-\mu}(\mathbb{W}_-) \oplus \mathcal{W}_T^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}(\mathbb{S})$$

for some $s > \max(\mu, d) - \frac{1}{2}$ and asymptotic types $Q_\pm$ and T, associated with the weight data $(\gamma - \mu, \Theta)$ and $(\gamma - \mu - \frac{1}{2}, \Theta)$, respectively, $\Theta = [0, k+1)$ for any $k \in \mathbb{N}$. Then we have

$$u \in \mathcal{W}_{P_+}^{s,\gamma}(\mathbb{W}_+) \oplus \mathcal{W}_{P_-}^{s,\gamma}(\mathbb{W}_-) \oplus \mathcal{W}_S^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{S})$$

for resulting asymptotic types $P_\pm$ and S, associated with the weight data (γ, Θ) and $(\gamma - \frac{1}{2}, \Theta)$, respectively.

Proof. Applying Theorem 3.3 the operator $\mathcal{A}$ has a (properly supported) parametrix $\mathcal{P}$. From $\mathcal{A}u = f$ we obtain $\mathcal{P}\mathcal{A}u = (\mathcal{I} - \mathcal{G}_l)u = \mathcal{P}f$. According to (31) the function $\mathcal{P}f$ is of the same smoothness as u and has asymptotics. By virtue of Proposition 2.12 also $\mathcal{G}_l u$ is of the required smoothness and has asymptotics. □

Remark 3.5. Definition 3.1 easily extends to the case of boundary-contact operators (including systems in the upper left corner) with an arbitrary number of trace and potential transmission conditions; those may also have different orders, cf. the examples in Section 1.2. We then have corresponding analogues of Theorems 3.3, 3.4, and of the auxiliary structures in Section 3.3.

3.2. An example

Let us consider a simple example, namely,

$$A_+ = \Delta|_{W_+}, \qquad A_- = c\Delta|_{W_-} \tag{44}$$

for a constant $c \neq 0$, with Δ being the Laplace operator in $\mathbb{R}^3$, and

$$T_\pm = {}^{\mathrm{t}}(T_{\pm,1}, T_{\pm,2}) \quad \text{for } T_{\pm,1}u := \pm u|_{\mathrm{int}\,\mathbb{S}} \text{ and } T_{\pm,2}u := \frac{\partial}{\partial \nu_\pm} u|_{\mathrm{int}\,\mathbb{S}}, \tag{45}$$

where $\nu_\pm$ are the outward normal directions to the boundaries of $W_\pm \setminus \{0\}$.

Theorem 3.6. *The boundary-contact operator*

$$\mathcal{A} = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \\ T_+ & T_- \end{pmatrix}$$

is elliptic for all $\gamma \in \mathbb{R} \setminus \{1 - \frac{\pi}{\pi-\alpha} j\colon\ j \in \mathbb{Z}\}$.

Proof. The ellipticity conditions (i), (ii) of Definition 3.1 are obviously satisfied for our problem and also the condition on the Mellin asymptotic type S. It remains to find the non-bijectivity points for the corresponding conormal symbol, cf. Remark 3.2.

The Laplace operator in polar coordinates $r^{-2}\{\partial_\phi^2 + (-r\partial_r)^2 - (rD_y)^2\}$ gives rise to the principal edge and conormal symbols

$$\sigma_\wedge(\Delta)(\eta) = r^{-2}\big\{\partial_\phi^2 + (-r\partial_r)^2 - (r\eta)^2\big\}$$

and

$$\sigma_M \sigma_\wedge(\Delta)(z) = \partial_\phi^2 + z^2,$$

respectively. Then (16) has the form

$$\sigma_M \sigma_\wedge(\mathcal{A})(z) = \begin{pmatrix} z^2 + \partial_\phi^2 & 0 \\ 0 & c(z^2 + \partial_\phi^2) \\ r_0' & -r_0' \\ r_0'\partial_\phi & r_0'\partial_\phi \\ r_\alpha' & -r_\alpha' \\ r_\alpha'\partial_\phi & r_\alpha'\partial_\phi \end{pmatrix} : \begin{matrix} H^s(\Xi_+) \\ \oplus \\ H^s(\Xi_-) \end{matrix} \to \begin{matrix} H^{s-2}(\Xi_+) \\ \oplus \\ H^{s-2}(\Xi_-) \\ \oplus \\ \mathbb{C}^2 \\ \oplus \\ \mathbb{C}^2 \end{matrix} \tag{46}$$

where $\Xi_+ = [0, \alpha]$, $\Xi_- = [\alpha, 2\pi]$. The admissible weight γ for our boundary-contact problem follows from the set D of those points $z \in \mathbb{C}$ where $h(z)$ in not bijective. Calculations in [10] shows that $D = \{\frac{\pi}{\pi-\alpha} j\}_{j\in\mathbb{Z}}$. Since $\dim \Xi_\pm = 1$ we obtain $\gamma \in \mathbb{R} \setminus \{1 - \frac{\pi}{\pi-\alpha} j\colon\ j \in \mathbb{Z}\}$. $\square$

Theorem 3.4 can be specialized to the present situation. In particular, let $u \in \mathcal{W}^{s,\gamma}(\mathbb{W}_+) \oplus \mathcal{W}^{s,\gamma}(\mathbb{W}_-) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{S})$ be a solution of $\mathcal{A}u = 0$. Then near the edge, in the splitting of variables (r, ϕ, y) we obtain asymptotics of $u(r, \phi, y) = {}^{\mathrm{t}}(u_+(r, \phi, y), u_-(r, \phi, y))$ of the form

$$\begin{aligned} &u_\pm(r, \phi, y) \\ &\quad \sim \sum_{j\in\mathbb{Z}\setminus 0,\ \frac{\pi}{\pi-\alpha} j < 1-\gamma} \mathcal{F}^{-1}_{\eta\to y}[\eta]\omega\big(r[\eta]\big)c_{\pm,j}(\phi)\big(r[\eta]\big)^{-\frac{\pi}{\pi-\alpha} j}\hat{v}_{\pm,j}(\eta) \\ &\qquad + \mathcal{F}^{-1}_{\eta\to y}[\eta]\omega\big(r[\eta]\big)c_{\pm,00}(\phi)\hat{v}_{\pm,00}(\eta) + \mathcal{F}^{-1}_{\eta\to y}[\eta]\omega\big(r[\eta]\big)c_{\pm,01}(\phi)\log\big(r[\eta]\big)\hat{v}_{\pm,01}(\eta) \end{aligned}$$

with coefficients $c_{\pm,j}, c_{\pm,00}, c_{\pm,01} \in C^\infty(\Xi_\pm)$, $v_{\pm,j}, v_{\pm,00}, v_{\pm,01} \in H^s(\mathbb{R})$. The second two terms only occur in the case $\gamma < 1$.

3.3. The asymptotic contribution in transmission operators

As we saw in Section 3.1 the asymptotics of solutions to an elliptic boundary-contact problem is 'generated' by a specific ingredient of the parametrices, namely, by Mellin operators with meromorphic amplitude functions. In the present case they take values in the space $B^{-\infty,d}(\Xi_+,\Xi_-)$ of smoothing transmission operators of type d on the base $\Xi_+\cup\Xi_-$ with respect to the interface Σ. Recall that the parameter-dependent analogue $B^{-\infty,d}(\Xi_+,\Xi_-;\mathbb{R}^l)$ is defined as $\mathcal{S}(\mathbb{R}^l,B^{-\infty,d}(\Xi_+,\Xi_-))$. If $U\subset\mathbb{C}$ is an open set and E a Fréchet space, by $\mathcal{A}(U,E)$ we denote the space of all holomorphic functions in U with values in E.

A sequence $R=\{(r_j,n_j,N_j)\}_{j\in\mathbb{Z}}$ of triples $r_j\in\mathbb{C}$, $n_j\in\mathbb{N}$, $N_j\subset B^{-\infty,d}(\Xi_+,\Xi_-)$ is called a Mellin asymptotic type if $\pi_{\mathbb{C}}R:=\{r_j\}_{j\in\mathbb{Z}}$ intersects every strip $\{z\colon c<\operatorname{Re}z<c'\}$, $c<c'$, in a finite set, and if N_j is a finite-dimensional subspace of operators of finite rank. Then $M_R^{-\infty,d}(\Xi_+,\Xi_-)$ denotes the subspace of all $f(z)\in\mathcal{A}(\mathbb{C}\setminus\pi_{\mathbb{C}}R,B^{-\infty,d}(\Xi_+,\Xi_-))$ such that $(\chi_R f)(\beta+i\varrho)\in B^{-\infty,d}(\Xi_+,\Xi_-;\mathbb{R}_\varrho)$ for every $\beta\in\mathbb{R}$, uniformly in compact β-intervals (where χ_R is an arbitrary $\pi_{\mathbb{C}}R$-excision function, i.e., $\chi_R\in C^\infty(\mathbb{C})$, $\chi_R(z)=0$ for $\operatorname{dist}(\pi_{\mathbb{C}}R,z)<c_0$, $\chi_R(z)=1$ for $\operatorname{dist}(\pi_{\mathbb{C}}R,z)>c_1$ for certain $0<c_0<c_1$), and, moreover, $f(z)$ is meromorphic with poles at r_j of multiplicity n_j+1 and Laurent coefficients at $(z-r_j)^{-(k+1)}$ belonging to N_j for $0\leqslant k\leqslant n_j$. Let $M_R^{\mu,d}(\Xi_+,\Xi_-):=M_{\mathcal{O}}^{\mu,d}(\Xi_+,\Xi_-)+M_R^{-\infty,d}(\Xi_+,\Xi_-)$ where $M_{\mathcal{O}}^{\mu,d}(\Xi_+,\Xi_-):=B^{-\infty,d}(\Xi_-,\Xi_+;\mathbb{C})$, cf. Section 2.3.

We now fix a weight interval $\Theta=[0,k+1)$, $k\in\mathbb{N}$, and form operator functions

$$m(y,\eta):=r^{-\mu}\omega\big(r[\eta]\big)\sum_{j=0}^{k}r^j\sum_{|\alpha|\leqslant j}\operatorname{op}_M^{\gamma_j-\frac{n}{2}}(f_{j\alpha})(y)\eta^\alpha\tilde{\omega}\big(r[\eta]\big) \tag{47}$$

for arbitrary cut-off functions ω, $\tilde{\omega}$, and $f_{j\alpha}\in C^\infty(\mathbb{R}^q,M_{R_{j\alpha}}^{-\infty,d}(\Xi_+,\Xi_-))$ for certain Mellin asymptotic types $R_{j\alpha}$ and weight γ_j such that $\gamma-j\leqslant\gamma_j\leqslant\gamma$ for all and $\pi_{\mathbb{C}}R_{j\alpha}\cap\Gamma_{\frac{n+1}{2}-\gamma_j}=\emptyset$. Every such $m(y,\eta)$ defines a C^∞ family in $(y,\eta)\in\mathbb{R}^q\times\mathbb{R}^q$ of continuous operator $m(y,\eta)$: $E\to\widetilde{E}$ for

$$E=\mathcal{K}_{(P_+)}^{s,\gamma}\big(\Xi_+^\wedge\big)\oplus\mathcal{K}_{(P_-)}^{s,\gamma}\big(\Xi_-^\wedge\big)\oplus\mathcal{K}_{(S)}^{s-\frac{1}{2},\gamma-\frac{1}{2}}\big(\Sigma^\wedge\big)$$

and

$$\widetilde{E}=\mathcal{K}_{(Q_+)}^{\infty,\gamma-\mu}\big(\Xi_+^\wedge\big)\oplus\mathcal{K}_{(Q_-)}^{\infty,\gamma-\mu}\big(\Xi_-^\wedge\big)\oplus\mathcal{K}_{(T)}^{\infty,\gamma-\mu-\frac{1}{2}}\big(\Sigma^\wedge\big)$$

for arbitrary $s>d-\frac{1}{2}$ and asymptotic types $P_\pm$, S, with some resulting asymptotic types $Q_\pm$, T (associated with the weight data (γ,Θ) and $(\gamma-\mu,\Theta)$, respectively), determined by the choice of Mellin asymptotic types $R_{j\alpha}$ (similarly as before, subscripts '$(P_\pm)$', etc., indicate subspaces with asymptotics of type $P_\pm$ or without asymptotics). Observe that we have

$$m(y,\eta)\in S_{\mathrm{cl}}^\mu\big(\mathbb{R}^q\times\mathbb{R}^q;E,\widetilde{E}\big),$$

based on the group action (30). The homogeneous principal part of $m(y,\eta)$ is given by

$$\sigma_\wedge(m)(y,\eta)=r^{-\mu}\omega\big(r|\eta|\big)\sum_{j=0}^{k}r^j\sum_{|\alpha|=j}\mathrm{op}_M^{\gamma_j-\frac{n}{2}}(f_{j\alpha})(y)\eta^\alpha\tilde{\omega}\big(r|\eta|\big),$$

$(y,\eta)\in T^*\mathbb{R}^q\setminus 0$. The analogue of symbols of the kind (47) in standard boundary value problems are well investigated. We do not consider here all the useful properties. Let us only observe that when we change the cut-off functions ω, $\tilde{\omega}$ or the weights $\gamma_{j\alpha}$ we only change $m(y,\eta)$ by a so-called Green symbol. The definition is as follows.

A C^∞ family in $(y,\eta)\in\mathbb{R}^q\times\mathbb{R}^q$ of operators $g(y,\eta)$ is called a Green symbol of order μ and type $d=0$ if it represents a symbol

$$g(y,\eta)\in S_{\mathrm{cl}}^\mu\big(\mathbb{R}^q\times\mathbb{R}^q;E,\widetilde{E}\big) \tag{48}$$

for $E:=\mathcal{K}^{s,\gamma}(\Xi_+^\wedge)\oplus\mathcal{K}^{s,\gamma}(\Xi_-^\wedge)\oplus\mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\Sigma^\wedge)$ and $\widetilde{E}:=\mathcal{S}_{Q_+}^{\gamma-\mu}(\Xi_+^\wedge)\oplus\mathcal{S}_{Q_-}^{\gamma-\mu}(\Xi_-^\wedge)\oplus\mathcal{S}_T^{\gamma-\mu-\frac{1}{2}}(\Sigma^\wedge)$ for asymptotic types $Q_\pm$, T depending on g (and associated with the weight data $(\gamma-\mu,\Theta)$ and $(\gamma-\mu-\frac{1}{2},\Theta)$, respectively), for all real $s>-\frac{1}{2}$, and if the pointwise adjoint $g^*(y,\eta)$ satisfies an analogous condition with respect to spaces of opposite weights and corresponding asymptotic types $P_\pm$, S in the image. Moreover, a Green symbol $g(y,\eta)$ of order μ and type $d\in\mathbb{N}$ is an operator family of the form

$$g(y,\eta)=g_0(y,\eta)+\sum_{j=1}^{d}g_j(y,\eta)\,\mathrm{diag}\big(D^j,0,0\big)$$

for arbitrary Green symbols $g_j(y,\eta)$ of order μ and type 0 and a first order differential operator D of similar meaning as in (33) (i.e., differentiating transversally to Σ). By $\sigma_\wedge(g)(y,\eta)$, $(y,\eta)\in T^*\mathbb{R}^q\setminus 0$, we denote the homogeneous principal component of $g(y,\eta)$ of order μ.

Remark 3.7. Let $g(y,\eta)$ be a Green symbol which is independent of y for $|y|>C$ for a constant $C>0$. Then $\mathcal{G}=\mathrm{Op}_y(g)$ is a Green operator in the sense of Definition 2.10.

We now define the complete conormal symbol $\sigma_{\boldsymbol{M}}(\mathcal{A})$ belonging to a boundary contact operator $\mathcal{A}$ of the form (32). The symbols $a(y,\eta)$ and $m(y,\eta)$ are given by (29) and (47), respectively. In (47) we have fixed a weight strip $\Theta=[0,k+1)$. We set

$$\sigma_{\boldsymbol{M}}(\mathcal{A}):=\left(\frac{1}{j!}\left(\frac{\partial^j}{\partial r^j}\tilde{h}(r,y,z,r\eta)\right)\bigg|_{r=0}+\sum_{|\alpha|\leqslant j}f_{j\alpha}(y,z)\eta^\alpha\right)_{0\leqslant j\leqslant k}.$$

The definition is motivated in a similar manner as in the general calculus of operators on a configuration with edges.

Remark 3.8. If $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are as in Theorem 2.9 the complete conormal symbol $\sigma_{\boldsymbol{M}}(\mathcal{A}\tilde{\mathcal{A}})$ can be computed in terms of $\sigma_{\boldsymbol{M}}(\mathcal{A})$ and $\sigma_{\boldsymbol{M}}(\tilde{\mathcal{A}})$ as the Leibniz–Mellin translation product.

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