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Wave diffraction by a 45 degree wedge sector with Dirichlet and Neumann boundary conditions☆

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Abstract

The problems of wave diffraction by a plane angular screen occupying an infinite 45 degree wedge sector with Dirichlet and/or Neumann boundary conditions are studied in Bessel potential spaces. Existence and uniqueness results are proved in such a framework. The solutions are provided for the spaces in consideration, and higher regularity of solutions are also obtained in a scale of Bessel potential spaces.

Keywords: Helmholtz equation; Wedge diffraction problem; Boundary value problem; Potential method; Pseudo-differential equations

1. Introduction

The first solved plane wave diffraction problems for wedges date from the beginning of the twentieth century. Indeed, in 1901 Sommerfeld [1] solved the problem of diffraction of an electromagnetic wave by a wedge whose angle was a rational multiple of 180°. In 1902, Macdonald [2] provided a complex contour integral (obtained by following Poincaré [3] and summing the Fourier series representation of the Green function) which was the exact solution of the problem of diffraction by a soft or hard wedge of any angle, in the two-dimensional case of cylindrical acoustic wave incidence. Later on, in 1920, the Sommerfeld suggestion of generalizing his method to irrational angles (by considering the corresponding irrational number as the limit of a sequence of rational numbers) was successfully implemented by Carslaw [4].

A very important contribution was also provided in the 1950s by Malyuzhinets when the acoustic wedge diffraction problem was solved for impedance boundary conditions, and for arbitrary angles. In the mean time, the so-called factorization technique was also more developed in the second half of the twentieth century, providing therefore several other possibilities of finding corresponding solutions.

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Regardless of these developments (and several other more recent ones like in [5–7]), for several cases a complete space setting description for those problems with a consequent analysis of solvability, and the eventual obtainment of more regular solutions is missing. It was mainly because of these reasons that such kind of problems gained an increased importance in the last years, and several authors turned to consider those problems in appropriate Sobolev or Bessel potential spaces. Presently, several results in this direction are known for the right-angle wedge [8–12] (with different kinds of boundary conditions), but not so many for other angles. For the Neumann–Neumann problem within the framework of H^1 Sobolev spaces, the work [13] generalizes the results of [8] to an arbitrary angle of magnitude less than 180°.

The present paper is devoted to the analysis of the boundary value problem originated by the problem of diffraction by a wedge with a 45° angle, and for the three different cases of Dirichlet–Dirichlet, Neumann–Dirichlet and Neumann–Neumann data in Sobolev spaces. Our method is based upon a sort of doubling process for appropriate potential operators, and corresponding pseudo-differential equations. Besides providing the unique solution in the natural order Sobolev spaces it is also proved that the same solution can be interpreted in higher regularity Sobolev spaces (/Bessel potential spaces).

2. Basic notations and formulation of the problems

In the present section the three problems under consideration are going to be formulated, and the necessary notations will be introduced.

As usual, $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of all rapidly vanishing functions and $\mathcal{S}'(\mathbb{R}^n)$ the dual space of tempered distributions on $\mathbb{R}^n$. The Bessel potential space $H^s(\mathbb{R}^n)$, with $s \in \mathbb{R}$, is formed by the elements $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|\varphi\|_{H^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2} \cdot \mathcal{F}\varphi\|_{L_2(\mathbb{R}^n)}$ is finite. As the notation indicates, $\|\cdot\|_{H^s(\mathbb{R}^n)}$ is a norm for the space $H^s(\mathbb{R}^n)$ which makes it a Banach space. Here, $\mathcal{F} = \mathcal{F}_{x\mapsto\xi}$ denotes the Fourier transformation in $\mathbb{R}^n$.

For a given domain, $\mathcal{D}$, on $\mathbb{R}^n$ we denote by $\widetilde{H}^s(\mathcal{D})$ the closed subspace of $H^s(\mathbb{R}^n)$ whose elements have supports in $\overline{\mathcal{D}}$, and $H^s(\mathcal{D})$ denotes the space of generalized functions on $\mathcal{D}$ which have extensions into $\mathbb{R}^n$ that belong to $H^s(\mathbb{R}^n)$. The space $\widetilde{H}^s(\mathcal{D})$ is endowed with the subspace topology, and on $H^s(\mathcal{D})$ we put the norm of the quotient space $H^s(\mathbb{R}^n)/\widetilde{H}^s(\mathbb{R}^n \setminus \overline{\mathcal{D}})$. Obviously, these definitions are valid for L_2 spaces. Note that the spaces $H^0(\mathbb{R}^n_+)$ and $\widetilde{H}^0(\mathbb{R}^n_+)$ can be identified, and we will denote them by $L_2(\mathbb{R}^n_+)$.

Let $\Omega := \{x := (x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 < x_1\}$, $\Gamma_1 := \{(x_1, 0) : x_1 \in \mathbb{R}\}$, $\Gamma_2 := \{(x_1, x_1) : x_1 \in \mathbb{R}\}$. Further let $\Gamma_{1,+} := \{(x_1, 0) : x_1 \in \mathbb{R}_+\}$, $\Gamma_{2,+} := \{(x_1, x_1) : x_1 \in \mathbb{R}_+\}$ and $\partial\Omega := \Gamma_{1,+} \cup \Gamma_{2,+} \cup \{(0, 0)\}$. Denote by $n_1 = \overrightarrow{(0, -1)}$, $n_2 = \overrightarrow{(-1/\sqrt{2}, 1/\sqrt{2})}$ the unit normal vectors to Γ_1 and Γ_2, respectively. For our purposes below, let us also define $\Gamma_3 := \{(0, x_2) : x_2 \in \mathbb{R}\}$ and associate with it the unit normal vector $n_3 := \overrightarrow{(-1, 0)}$.

We are interested in studying the problem of existence and uniqueness of an element $u \in H^{1+\varepsilon}(\Omega)$, $0 \le \varepsilon < 1$, such that

$$\left(\Delta + k^2\right) u = 0 \quad \text{in } \Omega, \tag{2.1}$$

(where the wave number $k \in \mathbb{C} \setminus \mathbb{R}$ is given), and satisfy one of the following boundary conditions:

$$\text{(D–D)} \qquad [u]^+_{\Gamma_{1,+}} = h_{1,D} \quad \text{on } \Gamma_{1,+} \quad \text{and} \quad [u]^+_{\Gamma_{2,+}} = h_{2,D} \quad \text{on } \Gamma_{2,+}; \tag{2.2}$$

$$\text{(N–N)} \quad [\partial_{n_1} u]^+_{\Gamma_{1,+}} = h_{1,N} \quad \text{on } \Gamma_{1,+} \quad \text{and} \quad [\partial_{n_2} u]^+_{\Gamma_{2,+}} = h_{2,N} \quad \text{on } \Gamma_{2,+}; \tag{2.3}$$

$$\text{(N–D)} \quad [\partial_{n_1} u]^+_{\Gamma_{j,+}} = h_{1,N} \quad \text{on } \Gamma_{1,+} \quad \text{and} \quad [u]^+_{\Gamma_{2,+}} = h_{2,D} \quad \text{on } \Gamma_{2,+}; \tag{2.4}$$

here the Dirichlet and Neumann traces on $\Gamma_{j,+}$ are denoted by $[u]^+_{\Gamma_{j,+}}$ and $[\partial_{n_j} u]^+_{\Gamma_{j,+}}$, respectively. Note that the Dirichlet type conditions can be understood in the trace sense, while the Neumann type conditions are understood in the distributional sense (cf. [14]). The elements $h_{j,D} \in H^{1/2+\varepsilon}(\Gamma_{j,+})$, $h_{j,N} \in H^{-1/2+\varepsilon}(\Gamma_{j,+})$, $j = 1, 2$, are arbitrarily given since the dependence on the data is to be studied for well-posedness.

In addition, in the above first two cases (D–D and N–N), compatibility conditions are necessary to be considered in view of the combination of the same kind data. Taking this into account, from now on we will refer to:

- Problem $\mathcal{P}_{\text{D–D}}$ as the one characterized by (2.1) and (2.2), and the compatibility condition $h_{1,D}(x) - h_{2,D}(\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} x) \in r_{\Gamma_{1,+}} \widetilde{H}^{1/2+\varepsilon}(\Gamma_{1,+})$;

- Problem $\mathcal{P}_{\text{N–N}}$ as the one characterized by (2.1) and (2.3), and the compatibility condition $h_{1,N}(x)+h_{2,N}(e^{i\frac{\pi}{4}}x) \in r_{\Gamma_{1,+}}\widetilde{H}^{-1/2+\varepsilon}(\Gamma_{1,+})$;
- Problem $\mathcal{P}_{\text{N–D}}$ as the problem characterized by (2.1) and (2.4).

Here, and in what follows, r_Σ denotes the restriction operator to $\Sigma = \Gamma_{1,+}, \Gamma_{2,+}, \Gamma_3, \mathbb{R}_\pm$ (defined in corresponding Bessel potential spaces).

Remark 1. Due to the fact that $r_{\Gamma_{1,+}}\widetilde{H}^s(\Gamma_{1,+}) = H^s(\Gamma_{1,+})$ if and only if $-1/2 < s < 1/2$, it follows that the just introduced compatibility condition $h_{1,N}(x) + h_{2,N}(e^{i\frac{\pi}{4}}x) \in r_{\Gamma_{1,+}}\widetilde{H}^{-1/2+\varepsilon}(\Gamma_{1,+})$ in Problem $\mathcal{P}_{\text{N–N}}$ is automatically fulfilled for $0 < \varepsilon < 1$.

3. Uniqueness of solution for the homogeneous problems in H^1 spaces

In this section we will present conditions which will guarantee the uniqueness of the solution for each of the homogeneous problems under consideration (in the case of $\varepsilon = 0$).

Theorem 2. *Let $\varepsilon = 0$. Then the homogeneous problems $\mathcal{P}_{\text{D–D}}$, $\mathcal{P}_{\text{N–N}}$ and $\mathcal{P}_{\text{N–D}}$ have only the trivial solution $u = 0$ in the space $H^1(\Omega)$.*

Proof. Let R be a sufficiently large positive number and $B(R)$ be the disk centered at the origin with radius R. Set $\Omega_R := \Omega \cap B(R)$. Note that the domain Ω_R has piecewise smooth boundary

$$S_R := (\partial B(R) \cap \Omega) \cup (\Gamma_{1,+} \cap B(R)) \cup (\Gamma_{2,+} \cap B(R)) \cup \{(0,0)\},$$

and denote by $n(x)$ the outward unit normal vector at the non-singular point $x \in S_R$.

Let u be a solution of the homogeneous problem. Then the first Green's identity (see, e.g., [15]) for u and its complex conjugate $\bar{u}$ in the domain Ω_R yields

$$\int_{\Omega_R} \left[|\nabla u|^2 - k^2|u|^2\right] \mathrm{d}x = \int_{S_R} \partial_n u\, \bar{u}\, \mathrm{d}S_R. \tag{3.1}$$

From (3.1) we obtain

$$\begin{aligned}\int_{\Omega_R} \left[|\nabla u|^2 - k^2|u|^2\right] \mathrm{d}x &= \int_{\Gamma_{1,+}\cap B(R)} \partial_n u\bar{u}\, \mathrm{d}x + \int_{\Gamma_{2,+}\cap B(R)} \partial_n u\bar{u}\, \mathrm{d}x + \int_{\partial B(R)\cap\Omega} \partial_n u\bar{u}\, \mathrm{d}S \\ &= \int_{\partial B(R)\cap\Omega} \partial_n u\bar{u}\, \mathrm{d}S,\end{aligned} \tag{3.2}$$

in any of the cases (2.2)–(2.4) with trivial data. Note that, since $\Im\mathrm{m}\, k \neq 0$ the integral $\int_{\partial B(R)\cap\Omega} \partial_n u\, \bar{u}\, \mathrm{d}S$ tends to 0 as $R \to \infty$. Indeed, in (R, ϕ) polar coordinates we have

$$\int_{\partial B(R)\cap\Omega} \partial_n u\, \bar{u}\, \mathrm{d}S = R\int_0^{\frac{\pi}{4}} \partial_n u\, \bar{u}\, \mathrm{d}\phi = R \lim_{\delta_1,\delta_2\to 0+} \int_{\delta_1}^{\frac{\pi}{4}-\delta_2} \partial_n u\, \bar{u}\, \mathrm{d}\phi$$

and we take into account that the solution $u \in H^1(\Omega)$ of the Helmholtz equation exponentially vanishes at infinity in the sector $\phi \in (\delta_1, \frac{\pi}{4} - \delta_2)$ (which follows from the representation formula of a solution of the Helmholtz equation; see, for instance, [16]). Therefore passing to the limit as $R \to \infty$ in (3.2) it follows

$$\int_\Omega \left[|\nabla u|^2 - k^2|u|^2\right] \mathrm{d}x = 0.$$

From to the real and imaginary parts of the last identity, we obtain

$$\int_\Omega \left[|\nabla u|^2 + \left((\Im\mathrm{m}\, k)^2 - (\Re\mathrm{e}\, k)^2\right)|u|^2\right] \mathrm{d}x = 0 \quad \text{and} \quad -2(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k)\int_\Omega |u|^2\, \mathrm{d}x = 0.$$

Thus, it follows from the last two identities that $u = 0$ in Ω. $\square$

4. The fundamental solution and potentials

Let us denote the standard fundamental solution of the Helmholtz equation (in two dimensions) by

$$\mathcal{K}(x) := -\frac{i}{4} H_0^{(1)}(k|x|),$$

where $H_0^{(1)}(k|x|)$ is the Hankel function of the first kind of order zero (cf. [17, Section 3.4]). Furthermore, we introduce the single and double layer potentials on Γ_j

$$V_j(\psi)(x) = \int_{\Gamma_j} \mathcal{K}(x-w)\psi(w) d_w \Gamma_j, \quad x \notin \Gamma_j,$$

$$W_j(\varphi)(x) = \int_{\Gamma_j} [\partial_{n_j(y)} \mathcal{K}(x-w)]\varphi(w) d_w \Gamma_j, \quad x \notin \Gamma_j \ ,$$

where $j = 1, 2, 3$ and ψ, φ are density functions. Note that for $j = 1$ sometimes we will write $\mathbb{R}$ instead of Γ_1 and n instead of the unit normal n_1. In this case, for example, the single layer potential defined above has the form

$$V_1(\psi)(x_1, x_2) = \int_{\mathbb{R}} \mathcal{K}(x_1 - y, x_2)\psi(y) \mathrm{d}y, \quad x_2 \neq 0.$$

Set $\mathbb{R}^2_\pm := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \gtrless 0\}$ and let us first consider the operators $V := V_1$ and $W := W_1$.

Theorem 3 (*[18]*). *The single and double layer potentials V and W are continuous operators*

$$V : H^s(\mathbb{R}) \to H^{s+1+\frac{1}{2}}(\mathbb{R}^2_\pm), \qquad W : H^{s+1}(\mathbb{R}) \to H^{s+1+\frac{1}{2}}(\mathbb{R}^2_\pm) \tag{4.1}$$

for all $s \in \mathbb{R}$.

Clearly, a similar result holds true for the operators V_2, W_2, V_3 and W_3.

Let us now recall some properties of the above introduced potentials. Namely, the following jump relations are well-known (cf. [18]):

$$\begin{aligned} &[V(\psi)]^+_{\mathbb{R}} = [V(\psi)]^-_{\mathbb{R}} =: \mathcal{H}(\psi), \qquad [\partial_n V(\psi)]^\pm_{\mathbb{R}} =: \left[\mp\frac{1}{2} I\right](\psi), \\ &[W(\varphi)]^\pm_{\mathbb{R}} =: \left[\pm\frac{1}{2} I\right](\varphi), \qquad [\partial_n W(\varphi)]^+_{\mathbb{R}} = [\partial_n W(\varphi)]^-_{\mathbb{R}} =: \mathcal{L}(\varphi), \end{aligned} \tag{4.2}$$

where

$$\mathcal{H}(\psi)(z) := \int_{\mathbb{R}} \mathcal{K}(z - y, 0)\psi(y) \mathrm{d}y, \quad z \in \mathbb{R}, \tag{4.3}$$

$$\mathcal{L}(\varphi)(z) := \lim_{\mathbb{R}^2_+ \ni x \to z \in \mathbb{R}} \partial_{n(x)} \int_{\mathbb{R}} [\partial_{n(y)} \mathcal{K}(y - x_1, -x_2)]\varphi(y) \mathrm{d}y, \quad z \in \mathbb{R}, \tag{4.4}$$

and I denotes the identity operator.

In our further reasoning we will make a convenient use of the even and odd extension operators defined by

$$\ell^e \varphi(y) = \begin{cases} \varphi(y), & y \in \mathbb{R}_\pm \\ \varphi(-y), & y \in \mathbb{R}_\mp \end{cases} \quad \text{and} \quad \ell^o \varphi(y) = \begin{cases} \varphi(y), & y \in \mathbb{R}_\pm \\ -\varphi(-y), & y \in \mathbb{R}_\mp \end{cases},$$

respectively.

Remark 4 (*cf. [19]*). The following operators

$$\begin{aligned} &\ell^e : H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_\pm) \longrightarrow H^{\frac{1}{2}+\varepsilon}(\mathbb{R}), \qquad \ell^o : r_{\mathbb{R}_\pm} \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_\pm) \longrightarrow H^{\frac{1}{2}+\varepsilon}(\mathbb{R}), \\ &\ell^o : H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_\pm) \longrightarrow H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}), \qquad \ell^e : r_{\mathbb{R}_\pm} \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_\pm) \longrightarrow H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}), \end{aligned}$$

are continuous for all $\varepsilon \in [0, 1/2)$.

Lemma 5 (*cf.* [20]). *If* $0 \le \varepsilon < 1/2$, *then*

$$r_{\Gamma_3} \circ V \circ \ell^o \psi = 0, \quad r_{\Gamma_3} \circ W \circ \ell^o \tilde{\varphi} = 0, \quad r_{\Gamma_3} \circ \partial_{n_3} V \circ \ell^e \tilde{\psi} = 0, \quad r_{\Gamma_3} \circ \partial_{n_3} W \circ \ell^e \varphi = 0$$

for all $\psi \in H^{-\frac{1}{2}+\varepsilon}(\Gamma_{1,+})$, $\tilde{\varphi} \in r_{\Gamma_{1,+}} \widetilde{H}^{\frac{1}{2}+\varepsilon}(\Gamma_{1,+})$, $\tilde{\psi} \in r_{\Gamma_{1,+}} \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\Gamma_{1,+})$, *and* $\varphi \in H^{\frac{1}{2}+\varepsilon}(\Gamma_{1,+})$.

Note that analogous results are valid for the operators V_3 and W_3.

5. Solutions of the problems within a set of smoothness space orders

In the present section we will provide the solutions for the three problems in the non-homogeneous case. In addition, it will be obtained as an improvement of the solutions regularity. This will be done for a new set of smoothness index Bessel potential spaces above the case of $\varepsilon = 0$.

We will use the notation $g|_{\Gamma_{j,+}}$ $(j = 1, 2)$ to the value of g on $\Gamma_{j,+}$ in the case where no discontinuities arise in the values of g (from both sides of $\Gamma_{j,+}$). To distinguish from this situation, we will continue to use the notation $[g]^+_{\Gamma_{j,+}}$ for the value of g on $\Gamma_{j,+}$ when computed from the plus/right part of $\Gamma_{j,+}$ and when g has no value on the other side of $\Gamma_{j,+}$ or simply when it may arise a discontinuity on the other side of $\Gamma_{j,+}$.

Theorem 6. *If the boundary data satisfy the conditions*

$$(h_{1,D}, h_{2,D}) \in H^{\frac{1}{2}+\varepsilon}(\Gamma_{1,+}) \times H^{\frac{1}{2}+\varepsilon}(\Gamma_{2,+})$$

for $0 \le \varepsilon < \frac{1}{2}$, *then the Problem* $\mathcal{P}_{\mathrm{D–D}}$ *has a unique solution* $u \in H^{1+\varepsilon}(\Omega)$, *which is representable in the form*

$$u = 2W_1(\ell^o g_1) - 2W_3(\ell^o g_3) + 2W_2(\ell h_{2,D}), \tag{5.1}$$

where ℓ *is any extension of* $h_{2,D}$ *such that* $\ell h_{2,D} \in H^{\frac{1}{2}+\varepsilon}(\Gamma_2)$ *and*

$$g_1 := h_{1,D} - 2W_2(\ell h_{2,D})|_{\Gamma_{1,+}} \quad \text{and} \quad g_3(0, x_2) := g_1(x_2, 0).$$

Proof. First note that the compatibility condition yields $g_1 \in r_{\Gamma_{1,+}} \widetilde{H}^{\frac{1}{2}+\varepsilon}(\Gamma_{1,+})$ and therefore $\ell^o g_1 \in H^{\frac{1}{2}+\varepsilon}(\Gamma_1)$, cf. Remark 4. It is clear that (5.1) satisfies the Helmholtz equation in Ω and due to Theorems 2 and 3 it only remains to check the boundary conditions. We have

$$\begin{aligned}
[u]^+_{\Gamma_{1,+}} &= 2[W_1(\ell^o g_1)]^+_{\Gamma_{1,+}} - 2W_3(\ell^o g_3)|_{\Gamma_{1,+}} + 2W_2(\ell h_{2,D})|_{\Gamma_{1,+}} \\
&= g_1 + 2W_2(\ell h_{2,D})|_{\Gamma_{1,+}} = h_{1,D} - 2W_2(\ell h_{2,D})|_{\Gamma_{1,+}} + 2W_2(\ell h_{2,D})|_{\Gamma_{1,+}} = h_{1,D}.
\end{aligned}$$

Here we used Lemma 5 and the jump relations (4.2). Further,

$$\begin{aligned}
[u]^+_{\Gamma_{2,+}} &= 2W_1(\ell^o g_1)|_{\Gamma_{2,+}} - 2W_3(\ell^o g_3)|_{\Gamma_{2,+}} + 2[W_2(\ell h_{2,D})]^+_{\Gamma_{2,+}} \\
&= -2\, r_{\Gamma_{2,+}} \int_{\mathbb{R}} \partial_{x_2} \mathcal{K}(x_1 - y, x_2)(\ell^o g_1)(y)\mathrm{d}y + 2 r_{\Gamma_{2,+}} \int_{\Gamma_3} \partial_{x_1} \mathcal{K}(x_1, x_2 - y)(\ell^o g_3)(y)\mathrm{d}_y \Gamma_3 + h_{2,D} \\
&= -2\, r_{\Gamma_{2,+}} \int_{\mathbb{R}} \partial_{x_2} \mathcal{K}(x_1 - y, x_2)(\ell^o g_1)(y)\mathrm{d}y + 2 r_{\Gamma_{2,+}} \int_{\mathbb{R}} \partial_{x_1} \mathcal{K}(x_1, x_2 - y)(\ell^o g_1)(y)\mathrm{d}y + h_{2,D}.
\end{aligned}$$

Since

$$\partial_{x_2} \mathcal{K}(x_1 - y, x_2) = -\frac{i}{4} \dot{H}^{(1)}_0(k|(x_1 - y, x_2)|) \frac{x_2}{\sqrt{(x_1 - y)^2 + x_2^2}}$$

and

$$\partial_{x_1} \mathcal{K}(x_1, x_2 - y) = -\frac{i}{4} \dot{H}^{(1)}_0(k|(x_1, x_2 - y)|) \frac{x_1}{\sqrt{x_1^2 + (x_2 - y)^2}}$$

(here $\dot{H}_0^{(1)}$ denotes the ordinary derivative of the Hankel function, which equals to $-H_1^{(1)}$, cf. [21, 8.473(6)]), then on $\Gamma_{2,+}$, i.e., for $x_1 = x_2$, we obtain

$$-\partial_{x_2}\mathcal{K}(x_1 - y, x_1) + \partial_{x_1}\mathcal{K}(x_1, x_1 - y) = 0.$$

Therefore

$$-2r_{\Gamma_{2,+}}\int_{\mathbb{R}} \partial_{x_2}\mathcal{K}(x_1 - y, x_2)(\ell^o g_1)(y)\mathrm{d}y + 2r_{\Gamma_{2,+}}\int_{\mathbb{R}} \partial_{x_1}\mathcal{K}(x_1, x_2 - y)(\ell^o g_1)(y)\mathrm{d}y = 0.$$

Thus $[u]^+_{\Gamma_{2,+}} = h_{2,D}$. □

Theorem 7. *If the boundary data satisfy the conditions*

$$(h_{1,N}, h_{2,N}) \in H^{-\frac{1}{2}+\varepsilon}(\Gamma_{1,+}) \times H^{-\frac{1}{2}+\varepsilon}(\Gamma_{2,+})$$

for $0 \le \varepsilon < \frac{1}{2}$, *then the Problem* $\mathcal{P}_{\text{N–N}}$ *has a unique solution* $u \in H^{1+\varepsilon}(\Omega)$, *which is representable in the form*

$$u = -2V_1(\ell^e g_1) - 2V_3(\ell^e g_3) - 2V_2(\ell h_{2,N}), \tag{5.2}$$

where ℓ *is any extension of* $h_{2,N}$ *such that* $\ell h_{2,N} \in H^{-\frac{1}{2}+\varepsilon}(\Gamma_2)$ *and*

$$g_1 := h_{1,N} + [2\,\partial_n V_2(\ell h_{2,N})]_{\Gamma_{1,+}} \quad \textit{and} \quad g_3(0, x_2) := g_1(x_2, 0).$$

Proof. Analogously as in the proof of Theorem 6 we have that $\ell^e g_1 \in H^{-\frac{1}{2}+\varepsilon}(\Gamma_1)$ (due to compatibility condition and Remark 4), and we only need to check the boundary conditions. For these, we have:

$$\begin{aligned}[\partial_n u]^+_{\Gamma_{1,+}} &= -2\,[\partial_n V_1(\ell^e g_1)]^+_{\Gamma_{1,+}} - 2\,\partial_n V_3(\ell^e g_3)|_{\Gamma_{1,+}} - 2\,\partial_n V_2(\ell h_{2,N})|_{\Gamma_{1,+}}\\ &= g_1 - 2\,\partial_n V_2(\ell h_{2,N})|_{\Gamma_{1,+}} = h_{1,N} + 2\,\partial_n V_2(\ell h_{2,N})|_{\Gamma_{1,+}} - 2\,\partial_n V_2(\ell h_{2,N})|_{\Gamma_{1,+}} = h_{1,N}.\end{aligned}$$

Further,

$$\begin{aligned}[\partial_n u]^+_{\Gamma_{2,+}} &= -2\,\partial_n V_1(\ell^e g_1)|_{\Gamma_{2,+}} - 2\,\partial_n V_3(\ell^e g_3)|_{\Gamma_{2,+}} - 2\,[\partial_n V_2(\ell h_{2,N})]^+_{\Gamma_{2,+}}\\ &= -2\,\partial_n V_1(\ell^e g_1)|_{\Gamma_{2,+}} - 2\,\partial_n V_3(\ell^e g_3)|_{\Gamma_{2,+}} + h_{2,N} = h_{2,N}.\end{aligned}$$

Indeed, in here $n = n_2$ and therefore $\partial_n := -\frac{1}{\sqrt{2}}\partial_{x_1} + \frac{1}{\sqrt{2}}\partial_{x_2}$. Moreover,

$$(-\partial_{x_1} + \partial_{x_2})\mathcal{K}(x_1 - y, x_2) = -\frac{i}{4}\dot{H}_0^{(1)}(k|(x_1 - y, x_2)|)\frac{x_2 - x_1 + y}{\sqrt{(x_1 - y)^2 + x_2^2}}$$

and

$$(-\partial_{x_1} + \partial_{x_2})\mathcal{K}(x_1, x_2 - y) = -\frac{i}{4}\dot{H}_0^{(1)}(k|(x_1, x_2 - y)|)\frac{x_2 - x_1 - y}{\sqrt{x_1^2 + (x_2 - y)^2}}.$$

Thus, on $\Gamma_{2,+}$ (i.e., for $x_1 = x_2$), we have

$$\partial_{n_2}\mathcal{K}(x_1 - y, x_2) + \partial_{n_2}\mathcal{K}(x_1, x_2 - y) = 0$$

which yields

$$-2\,\partial_n V_1(\ell^e g_1)|_{\Gamma_{2,+}} - 2\partial_n V_3(\ell^e g_3)|_{\Gamma_{2,+}} = 0. \quad \square$$

Theorem 8. *If*

$$(h_{1,N}, h_{2,D}) \in H^{-\frac{1}{2}+\varepsilon}(\Gamma_{1,+}) \times H^{\frac{1}{2}+\varepsilon}(\Gamma_{2,+})$$

for $0 \le \varepsilon < \frac{1}{2}$ *and the boundary data satisfy the compatibility condition*

$$g_1 := h_{1,N} - 2\partial_n W_2(\ell h_{2,D})|_{\Gamma_{1,+}} \in r_{\Gamma_{1,+}}\widetilde{H}^{-\frac{1}{2}+\varepsilon}(\Gamma_{1,+}) \tag{5.3}$$

for some extension ℓ *of* $h_{2,D}$ *such that* $\ell h_{2,D} \in H^{\frac{1}{2}+\varepsilon}(\Gamma_2)$, *then the Problem* $\mathcal{P}_{\text{N–D}}$ *has a unique solution* $u \in H^{1+\varepsilon}(\Omega)$, *which is representable in the form*

$$u = -2V_1(\ell^e g_1) + 2V_3(\ell^e g_3) + 2W_2(\ell h_{2,D}),$$

where $g_3(0, x_2) := g_1(x_2, 0)$.

Proof. Due to compatibility condition we have that $\ell^e g_1 \in H^{-\frac{1}{2}+\varepsilon}(\Gamma_1)$ and arguing as above we need to check the boundary conditions. We have

$$[\partial_n u]^+_{\Gamma_{1,+}} = g_1 + 2\partial_n W_2(\ell h_{2,D})|_{\Gamma_{1,+}} = h_{1,N},$$

and

$$\begin{aligned}[u]^+_{\Gamma_{2,+}} &= -2\, r_{\Gamma_{2,+}} \int_{\mathbb{R}} \mathcal{K}(x_1 - y, x_2)(\ell^e g_1)(y)\mathrm{d}y + 2\, r_{\Gamma_{2,+}} \int_{\mathbb{R}} \mathcal{K}(x_1, x_2 - y)(\ell^e g_1)(y)\mathrm{d}y + h_{2,D} \\ &= h_{2,D}. \quad \square\end{aligned}$$

Note that the compatibility condition (5.3) is only a restriction for $\varepsilon = 0$ (in the case of $0 < \varepsilon < 1/2$ the condition is automatically fulfilled due to the same reason as in Remark 1). Anyway, it should also be pointed out that this is not a necessary condition as the ones introduced in the end of Section 2 but just a condition which appears due to the proposed technique.

Finally, we would like to point out that the above conclusions for the three problems (where the solutions are presented as sums of different potentials) open the possibility to eventual studies about corresponding efficient computations on these combinations of potentials — in accordance with Galerkin procedures which are known to be applied to isolated single or double potential operators (cf., e.g., [22,23]).

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