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WAVE DIFFRACTION BY A 270 DEGREES WEDGE SECTOR WITH DIRICHLET, NEUMANN AND IMPEDANCE BOUNDARY CONDITIONS

1. Introduction

In the present work we will be concerned with the mathematical formulation and analysis of different types of wave diffraction by a wedge region.

From the historical point of view, the problem of wave diffraction by a wedge goes back to 1892 when H. Poincaré published his paper [10]. The study of this problem continued with A. Sommerfeld and his fundamental work [11], where the analytic solution of the problem was presented. Much later, in 1952, H.G. Garnir [5] gave the Green function of the metaharmonic operator in a wedge (for Dirichlet and Neumann boundary conditions). Then, G.D. Malyuzhinets generalized this work in 1958 to the more complex case of impedance boundary conditions on the faces of the wedge [8].

In the meantime, the so-called factorization technique was also more developed in the second half of the twentieth century, providing therefore several other possibilities of finding corresponding solutions.

Regardless of these developments (and several other more recent ones like in [2, 6]), for several cases a complete space setting description for those problems with a consequent analysis of solvability, and the eventual obtainment of more regular solutions is missing. In fact, the pertinent question of what are the most appropriate spaces to be considered in such problems should be regarded as a good example for justifying innovative (and sometimes incompatible) approaches.

The present paper is devoted to the analysis of the boundary value problem originated by the problem of wave diffraction by a wedge with a 270° angle, and for the three different cases of Dirichlet-Dirichlet, Impedance-Dirichlet and Impedance-Neumann data in Bessel potential spaces.

We observe that for some exterior wedge diffraction problems certain partial results are presently known. This is the case for Dirichlet-Dirichlet,

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Neumann-Neumann and mixed Dirichlet-Neumann boundary conditions. Namely, in [9] these cases were analyzed in H^1 spaces by using the so-called *operators around the corner*.

2. Formulation of the Problems and Main Results

As usual, $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of all rapidly vanishing functions and $\mathcal{S}'(\mathbb{R}^n)$ the dual space of tempered distributions on $\mathbb{R}^n$. The Bessel potential space $H^s(\mathbb{R}^n)$, with $s \in \mathbb{R}$, is formed by the elements $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|\varphi\|_{H^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2} \cdot \mathcal{F}\varphi\|_{L_2(\mathbb{R}^n)}$ is finite. As the notation indicates, $\|\cdot\|_{H^s(\mathbb{R}^n)}$ is a norm for the space $H^s(\mathbb{R}^n)$ which makes it a Banach space. Here, $\mathcal{F} = \mathcal{F}_{x\mapsto\xi}$ denotes the Fourier transformation in $\mathbb{R}^n$.

For a given domain $\mathcal{D}$, on $\mathbb{R}^n$, we denote by $\widetilde{H}^s(\mathcal{D})$ the closed subspace of $H^s(\mathbb{R}^n)$ whose elements have supports in $\overline{\mathcal{D}}$, and $H^s(\mathcal{D})$ denotes the space of generalized functions on $\mathcal{D}$ which have extensions into $\mathbb{R}^n$ that belong to $H^s(\mathbb{R}^n)$. The space $\widetilde{H}^s(\mathcal{D})$ is endowed with the subspace topology, and on $H^s(\mathcal{D})$ we introduce the norm of the quotient space $H^s(\mathbb{R}^n)/\widetilde{H}^s(\mathbb{R}^n\backslash\overline{\mathcal{D}})$. Throughout the paper we will use the notation $\mathbb{R}^n_\pm := \{x=(x_1,\ldots,x_{n-1},x_n) \in \mathbb{R}^n : \pm x_n > 0\}$. Note that the spaces $H^0(\mathbb{R}^n_+)$ and $\widetilde{H}^0(\mathbb{R}^n_+)$ can be identified, and we will denote them by $L_2(\mathbb{R}^n_+)$.

Let $\Omega := \mathbb{R}^2\backslash\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \le 0, x_2 \le 0\}$, $\Gamma_1 := \{(x_1,0) : x_1 \in \mathbb{R}\}$ and $\Gamma_2 := \{(0,x_2) : x_2 \in \mathbb{R}\}$. Let further $\Gamma_{1,\pm} := \{(x_1,0) : x_1 \in \mathbb{R}_\pm\}$, $\Gamma_{2,\pm} := \{(0,x_2) : x_2 \in \mathbb{R}_\pm\}$ and $\partial\Omega := \Gamma_{1,-}\cup\Gamma_{2,-}\cup\{(0,0)\}$. Denote by $n_1 = \overrightarrow{(0,-1)}$, $n_2 = \overrightarrow{(-1,0)}$ the unit normal vectors on Γ_1 and Γ_2, respectively.

Let $\varepsilon \in [0,\frac{1}{2})$. We are interested in studying the problem of existence and uniqueness of an element $u \in H^{1+\varepsilon}(\Omega)$, such that

$$\left(\Delta + k^2\right) u = 0 \qquad \text{in} \qquad \Omega, \tag{1}$$

$$c_j[\partial_{n_j} u]^+_{\Gamma_{j,-}} - d_j[u]^+_{\Gamma_{j,-}} = h_j \qquad \text{on} \qquad \Gamma_{j,-}, \tag{2}$$

where the wave number $k \in \mathbb{C}\backslash\mathbb{R}$ is given, as well as the constants $c_j, d_j \in \mathbb{C}$ ($j = 1,2$). The elements $h_j \in H^{-1/2+\varepsilon}(\Gamma_{j,-})$ (in case that $c_j \neq 0$) or $h_j \in H^{1/2+\varepsilon}(\Gamma_{j,-})$ (in case that $c_j = 0$) are arbitrarily given, since the dependence on the data is to be studied for well-posedness ($j = 1,2$).

In (2), the elements $[u]^+_{\Gamma_{j,-}}$ and $[\partial_{n_j} u]^+_{\Gamma_{j,-}}$ denote the Dirichlet and Neumann traces on $\Gamma_{j,-}$, respectively.

From (2) let us single out the following three representative boundary conditions

$$[\partial_n u]^+_{\Gamma_{1,-}} - p\,[u]^+_{\Gamma_{1,-}} = h_1 \text{ on } \Gamma_{1,-}, \text{ and } [u]^+_{\Gamma_{2,-}} = h_2 \text{ on } \Gamma_{2,-}, \tag{3}$$

$$[\partial_n u]^+_{\Gamma_{1,-}} - p\,[u]^+_{\Gamma_{1,-}} = h_1 \text{ on } \Gamma_{1,-}, \text{ and } [\partial_n u]^+_{\Gamma_{2,-}} = h_2 \text{ on } \Gamma_{2,-}, \tag{4}$$

$$[u]^+_{\Gamma_{1,-}} = h_1 \text{ on } \Gamma_{1,-}, \text{ and } [u]^+_{\Gamma_{2,-}} = h_2 \text{ on } \Gamma_{2,-}, \tag{5}$$

where $p \in \mathbb{C}$. Note that the conditions on $\Gamma_{1,-}$ and $\Gamma_{2,-}$ can be interchanged, leading therefore to the corresponding problems which can be treated in a corresponding way as those ones. Therefore, a detailed exposition of these "dual cases" will be omitted here.

From now on we will refer to:

- Problem $\mathcal{P}_{I\text{-}D}$ as the problem characterized by (1) and (3);
- Problem $\mathcal{P}_{I\text{-}N}$ as the one characterized by (1), (4), and the compatibility condition $h_1 + h_2 \in r_{\mathbb{R}_-}\widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_-)$;
- Problem $\mathcal{P}_{D\text{-}D}$ as the one characterized by (1), (5), and the compatibility condition $h_1 - h_2 \in r_{\mathbb{R}_-}\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_-)$.

To treat these problems, we use an operator theoretical machinery and translate $\mathcal{P}_{I\text{-}D}$, $\mathcal{P}_{I\text{-}N}$, and $\mathcal{P}_{D\text{-}D}$ to the study of properties of well-known types of operators. Besides providing the unique solution (and well-posedness) in the natural order Bessel potential spaces, it is also proved that the same solution can be interpreted in higher regularity Bessel potential spaces. The involved methods of the present work combine operator and function theoretic features in a constructive way. For instance, several explicit toplinear [7, Chapter IV, §1] equivalence (after extension) relations [1, 4] will be built for appropriate pseudodifferential operators. These will allow us a transparent transfer of Fredholm and invertibility properties between the obtained operators and, therefore, lead us to a complete solution of the problems in study (including the above-mentioned regularity results). In particular, the present methods are centered in the construction of explicit operator matrix identities. This leads to an identification of special classes of operators which allow a complete characterization of the boundary value problems in study. Among those, the so-called Wiener-Hopf plus Hankel type operators revealed to have a central role in such characterization.

The following theorem is true (for more detailed formulation of the theorem and corresponding proof, cf. [3]).

Theorem 2.1. *Let $0 \le \varepsilon < \frac{1}{2}$ and let one of the following conditions be satisfied:*

(a) $(\mathrm{Re}\, k)(\mathrm{Im}\, k) > 0$, $\mathrm{Im}\, p \ge 0$, $\mathrm{Re}\, p \ge 0$,
(b) $(\mathrm{Re}\, k)(\mathrm{Im}\, k) < 0$, $\mathrm{Im}\, p \le 0$, $\mathrm{Re}\, p \ge 0$,
(c) $|\,\mathrm{Im}\, k| \ge |\,\mathrm{Re}\, k|$, $0 \le -\mathrm{Re}\, p \le |\,\mathrm{Im}\, p\,|$,
(d) $\mathrm{Re}\, k = 0, \mathrm{Im}\, p \ne 0$,
(e) $\mathrm{Re}\, k \ne 0\,, \mathrm{Im}\, p \ne 0\,, (\mathrm{Im}\, k)^2 - (\mathrm{Re}\, k)^2 + 2(\mathrm{Re}\, k)(\mathrm{Im}\, k)\frac{\mathrm{Re}\, p}{\mathrm{Im}\, p} \ge 0.$

Then

(i) *Problem $\mathcal{P}_{I\text{-}D}$ has a unique solution which is representable by a single- and a double-layer potentials.*
(ii) *Problem $\mathcal{P}_{I\text{-}N}$ has a unique solution which is representable by a single- and a double-layer potentials.*
(iii) *Problem $\mathcal{P}_{D\text{-}D}$ has a unique solution which is representable by a single- and a double-layer potentials.*

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