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Wave diffraction by a half-plane with an obstacle perpendicular to the boundary

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ABSTRACT

We prove the unique existence of solutions for different types of boundary value problems of wave diffraction by a half-plane with a screen or a crack perpendicular to the boundary. Representations of the solutions are also obtained upon the consideration of some associated operators. This is done in a Bessel potential spaces framework and for complex (non-real) wave numbers. The investigation is mostly based on the construction of explicit operator relations, the potential method, and a factorization technique for a certain class of oscillating Fourier symbols which naturally arise from the problems.

1. Introduction

The physical motivations behind the present study arise from the problem of acoustic and electromagnetic time-harmonic plane wave diffraction by a strip interacted with the boundary. In particular, we deal with boundary value problems for the Helmholtz equation, where the strip is located in the Oxz-plane (when adopting the Cartesian axes $Oxyz$) and perpendicular to y-axis – which may be viewed as a boundary of an obstacle. Throughout this work we assume that the material is invariant in the z-direction. Thus, in effect, the geometry of the problem is two-dimensional, which leads us

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from the strip to a finite interval $[0, a[$, $a > 0$. The problem is formulated for the complex (non-real) wave number case and worked out in a framework of Bessel potential spaces.

Boundary value problems for the Helmholtz equation in singular configurations have been studied by several authors, partly focused on special cases, under extra assumptions on the geometry or the underlying dimensions. Other concrete wave diffraction problems which have been studied in the literature and have some common points with our problem can be found, for instance, in the works of Castro, Kapanadze, Krutitskii, Malyuzhinets, Meister, Merzon, Moura Santos, Penzel, Rottbrand, dos Santos, Speck, Teixeira and others (cf. [2,6,7,13,21,24–32]). In special, Dirichlet and Neumann problems for the dissipative Helmholtz equation in exterior planar domains bounded by several closed curves and several open arcs (or cracks) have been studied in [17–19], and boundary value problems for the 2D Laplace equation in exterior domains bounded by several closed curves and several double-sided open arcs with a Dirichlet boundary condition on the whole boundary or with setting either Dirichlet or Neumann boundary conditions on different parts of the boundary have been studied in [20,22,23].

To treat the problems (mathematically formulated in the next section) we start by applying the so-called potential method (in Section 3), which allows us to equivalently reduce the original problems to the integro-differential equations on the boundary. It turns out that these equations are equivalent to some others characterized by Wiener–Hopf plus and minus Hankel operators. Moreover, these operators have oscillating Fourier symbols (see Section 4), which are additionally investigated. Namely, explicit appropriate factorizations of the representatives at infinity of those Fourier symbols are obtained and, therefore, uniqueness and existence results are concluded (in the last section). For all this the use of operator relations in Section 5 revealed to be fundamental, which allows us to associate a certain matrix Wiener–Hopf operator with Wiener–Hopf plus and minus Hankel operators. Additionally, we represent solutions of the wave diffraction problems with a screen or a crack perpendicular to the boundary by single and double layer potentials within Bessel potential spaces, and an improvement of the smoothness space parameters is exhibited for which the existence and uniqueness of solution (and continuous dependence on the data) is still guaranteed.

2. Formulation of the problems

In this section we establish the general notation which will already allow the mathematical formulation of the problem.

As usual, $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of all rapidly vanishing functions and $\mathcal{S}'(\mathbb{R}^n)$ the dual space of tempered distributions on $\mathbb{R}^n$. The Bessel potential space $H^s(\mathbb{R}^n)$, with $s \in \mathbb{R}$, is formed by the elements $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|\varphi\|_{H^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2} \cdot \mathcal{F}\varphi\|_{L_2(\mathbb{R}^n)}$ is finite. As the notation indicates, $\|\cdot\|_{H^s(\mathbb{R}^n)}$ is a norm for the space $H^s(\mathbb{R}^n)$ which makes it a Banach space. Here, $\mathcal{F} = \mathcal{F}_{x\mapsto\xi}$ denotes the Fourier transformation in $\mathbb{R}^n$.

For a given Lipshitz domain $\mathcal{D}$, on $\mathbb{R}^n$, we denote by $\widetilde{H}^s(\mathcal{D})$ the closed subspace of $H^s(\mathbb{R}^n)$ whose elements have supports in $\overline{\mathcal{D}}$, and $H^s(\mathcal{D})$ denotes the space of generalized functions on $\mathcal{D}$ which have extensions into $\mathbb{R}^n$ that belong to $H^s(\mathbb{R}^n)$. The space $\widetilde{H}^s(\mathcal{D})$ is endowed with the subspace topology, and on $H^s(\mathcal{D})$ we introduce the norm of the quotient space $H^s(\mathbb{R}^n)/\widetilde{H}^s(\mathbb{R}^n \setminus \overline{\mathcal{D}})$. Throughout the paper we will use the notation $\mathbb{R}^n_\pm := \{x = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n\colon \pm x_n > 0\}$. Note that the spaces $H^0(\mathbb{R}^n_+)$ and $\widetilde{H}^0(\mathbb{R}^n_+)$ can be identified, and we will denote them by $L_2(\mathbb{R}^n_+)$.

Let $\Omega := \{(x_1, x_2) \in \mathbb{R}^2\colon x_1 > 0,\ x_2 \in \mathbb{R}\}$, $\Gamma_1 := \{(x_1, 0)\colon x_1 \in \mathbb{R}\}$, and $\Gamma_2 := \{(0, x_2)\colon x_2 \in \mathbb{R}\}$. Let further $\mathcal{C} := \{(x_1, 0)\colon 0 < x_1 < a\} \subset \Gamma_1$ for a certain positive number a and $\Omega_{\mathcal{C}} := \Omega \setminus \overline{\mathcal{C}}$. Clearly, $\partial\Omega = \Gamma_2$ and $\partial\Omega_{\mathcal{C}} = \Gamma_2 \cup \mathcal{C}$.

For our purposes below we introduce further notations: $\Omega_1 := \{(x_1, x_2) \in \mathbb{R}^2\colon x_1 > 0,\ x_2 > 0\}$ and $\Omega_2 := \{(x_1, x_2) \in \mathbb{R}^2\colon x_1 > 0,\ x_2 < 0\}$, then $\partial\Omega_j = \mathcal{S}_j \cup \mathcal{S}$, for $j = 1, 2$, where $\mathcal{S} := \{(x_1, 0)\colon x_1 \geqslant 0\} \subset \Gamma_1$, $\mathcal{S}_1 := \{(0, x_2)\colon x_2 \geqslant 0\} \subset \Gamma_2$, and $\mathcal{S}_2 := \{(0, x_2)\colon x_2 \leqslant 0\} \subset \Gamma_2$. Finally, we introduce the following unit normal vectors $n_1 = \overrightarrow{(0, -1)}$ on Γ_1 and $n_2 = \overrightarrow{(-1, 0)}$ on Γ_2.

Let $\varepsilon \in [0, \frac{1}{2})$. We are interested in studying the problem of existence and uniqueness of an element $u \in H^{1+\varepsilon}(\Omega_{\mathcal{C}})$, such that

$$\left(\Delta + k^2\right)u = 0 \quad \text{in } \Omega_{\mathcal{C}}, \tag{1}$$

and u satisfies one of the following four representative boundary conditions:

$$[u]^{\pm}_{\mathcal{C}} = g_0^{\pm} \quad \text{on } \mathcal{C}, \quad \text{and} \quad [u]^{+}_{\mathcal{S}_j} = h_j \quad \text{on } \mathcal{S}_j, \tag{2}$$

$$[u]^{\pm}_{\mathcal{C}} = g_0^{\pm} \quad \text{on } \mathcal{C}, \quad \text{and} \quad [\partial_{n_2} u]^{+}_{\mathcal{S}_j} = f_j \quad \text{on } \mathcal{S}_j, \tag{3}$$

$$[\partial_{n_1} u]^{\pm}_{\mathcal{C}} = g_1^{\pm} \quad \text{on } \mathcal{C}, \quad \text{and} \quad [u]^{+}_{\mathcal{S}_j} = h_j \quad \text{on } \mathcal{S}_j, \tag{4}$$

$$[\partial_{n_1} u]^{\pm}_{\mathcal{C}} = g_1^{\pm} \quad \text{on } \mathcal{C}, \quad \text{and} \quad [\partial_{n_2} u]^{+}_{\mathcal{S}_j} = f_j \quad \text{on } \mathcal{S}_j, \tag{5}$$

for $j = 1, 2$. Here the wave number $k \in \mathbb{C} \setminus \mathbb{R}$ is given. The elements $[u]^{+}_{\mathcal{S}_j}$ and $[\partial_{n_2} u]^{+}_{\mathcal{S}_j}$ denote the Dirichlet and the Neumann traces on $\mathcal{S}_j$, respectively, while by $[u]^{\pm}_{\mathcal{C}}$ we denote the Dirichlet traces on $\mathcal{C}$ from both sides of the screen and by $[\partial_{n_1} u]^{\pm}_{\mathcal{C}}$ we denote the Neumann traces on $\mathcal{C}$ from both sides of the crack.

Throughout the paper on the given data we assume that $h_j \in H^{1/2+\varepsilon}(\mathcal{S}_j)$, $f_j \in H^{-1/2+\varepsilon}(\mathcal{S}_j)$, for $j = 1, 2$, and $g_i^{\pm} \in H^{1/2-i+\varepsilon}(\mathcal{C})$, for $i = 0, 1$. Furthermore, we suppose that they satisfy the following compatibility conditions:

$$\chi_a\left(g_0^{+} - g_0^{-}\right) \in r_{\mathcal{C}} \widetilde{H}^{1/2+\varepsilon}(\mathcal{C}), \tag{6}$$

$$\chi_a\left(g_1^{+} - g_1^{-}\right) \in r_{\mathcal{C}} \widetilde{H}^{-1/2+\varepsilon}(\mathcal{C}), \tag{7}$$

and

$$\chi_0\left(g_0^{+} - r_{\mathcal{C}} h_1 \circ e^{i\frac{\pi}{2}}\right), \chi_0\left(g_0^{-} - r_{\mathcal{C}} h_2 \circ e^{-i\frac{\pi}{2}}\right) \in r_{\mathcal{C}} \widetilde{H}^{1/2+\varepsilon}(\mathcal{C}), \tag{8}$$

$$\chi_0\left(g_1^{+} + r_{\mathcal{C}} f_1 \circ e^{i\frac{\pi}{2}}\right), \chi_0\left(g_1^{-} - r_{\mathcal{C}} f_2 \circ e^{-i\frac{\pi}{2}}\right) \in r_{\mathcal{C}} \widetilde{H}^{-1/2+\varepsilon}(\mathcal{C}). \tag{9}$$

Here, $r_{\mathcal{C}}$ denotes the restriction operator to $\mathcal{C}$ and $\chi_a(x) := \chi_0(a - x)$, where $\chi_0 \in C^{\infty}([0, a])$, such that $\chi_0(x) \equiv 1$ for $x \in [0, a/3]$ and $\chi_0(x) \equiv 0$ for $x \in [2a/3, a]$.

From now on we will refer to:

- Problem $\mathcal{P}_{D-D}$ as the problem characterized by (1), (2), (6), and (8);
- Problem $\mathcal{P}_{D-N}$ as the one characterized by (1), (3), (6);
- Problem $\mathcal{P}_{N-D}$ as the one characterized by (1), (4), (7);
- Problem $\mathcal{P}_{N-N}$ as the one characterized by (1), (5), (7), and (9).

As about the just stated compatibility conditions, note that they are necessary conditions to the well-posedness of the corresponding problems. Note also that, the compatibility conditions (7) and (9) included in Problems $\mathcal{P}_{N-D}$ and $\mathcal{P}_{N-N}$ are additional restrictions only for $\varepsilon = 0$.

3. The fundamental solution and potentials

We start this section by proving the uniqueness result for the problems in consideration.

Theorem 3.1. *The problems $\mathcal{P}_{D-D}$, $\mathcal{P}_{N-D}$, $\mathcal{P}_{D-N}$, and $\mathcal{P}_{N-N}$ have at most one solution.*

Proof. The proof is standard and uses the Green formula (being sufficient to consider the case $\varepsilon = 0$). Let R be a sufficiently large positive number and $B(R)$ be the disk centered at the origin with radius R. Set $\Omega_R := \Omega_{\mathcal{C}} \cap B(R)$. Note that the domain Ω_R has a piecewise smooth boundary S_R

including both sides of $\mathcal{C}$ and denote by $n(x)$ the outward unit normal vector at the non-singular points $x \in S_R$.

Let u be a solution of the homogeneous problem. Then the first Green identity for u and its complex conjugate $\bar{u}$ in the domain Ω_R, together with zero boundary conditions on S_R yields

$$\int_{\Omega_R} \left[|\nabla u|^2 - k^2|u|^2\right] dx = \int_{\partial B(R)\cap\Omega} (\partial_n u)\bar{u}\, dS_R. \tag{10}$$

Note that, since $\Im\mathrm{m}\, k \neq 0$, the integral $\int_{\partial B(R)\cap\Omega}(\partial_n u)\bar{u}\, dS$ tends to zero as $R \to \infty$. Indeed, in (R,ϕ) polar coordinates we have

$$\int_{\partial B(R)\cap\Omega} (\partial_n u)\bar{u}\, dS = R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\partial_n u)\bar{u}\, d\phi = R \lim_{\delta_1,\delta_2\to 0+} \int_{\delta_1-\frac{\pi}{2}}^{\frac{\pi}{2}-\delta_2} (\partial_n u)\bar{u}\, d\phi$$

and we take into account that the solution $u \in H^1(\Omega)$ of the Helmholtz equation exponentially decays at infinity. Therefore passing to the limit as $R \to \infty$ in (10) it follows

$$\int_{\Omega_{\mathcal{C}}} \left[|\nabla u|^2 - k^2|u|^2\right] dx = 0.$$

From the real and imaginary parts of the last identity, we obtain

$$\int_{\Omega_{\mathcal{C}}} \left[|\nabla u|^2 + \left((\Im\mathrm{m}\, k)^2 - (\Re\mathrm{e}\, k)^2\right)|u|^2\right] dx = 0,$$

$$-2(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k)\int_{\Omega_{\mathcal{C}}} |u|^2\, dx = 0.$$

Thus, for the condition $\Im\mathrm{m}\, k \neq 0$, it follows from the last two identities that $u = 0$ in $\Omega_{\mathcal{C}}$. $\square$

Now, without lost of generality we assume that $\Im\mathrm{m}\, k > 0$; the complementary case $\Im\mathrm{m}\, k < 0$ runs with obvious changes. Let us denote the standard fundamental solution of the Helmholtz equation (in two dimensions) by

$$\mathcal{K}(x) := -\frac{i}{4} H_0^{(1)}\left(k|x|\right),$$

where $H_0^{(1)}(k|x|)$ is the Hankel function of the first kind of order zero (cf. [14, §3.4]). Furthermore, we introduce the single and double layer potentials on Γ_j:

$$V_j(\psi)(x) = \int_{\Gamma_j} \mathcal{K}(x-y)\psi(y)\, d_y\Gamma_j, \quad x \notin \Gamma_j,$$

$$W_j(\varphi)(x) = \int_{\Gamma_j} \left[\partial_{n_j(y)}\mathcal{K}(x-y)\right]\varphi(y)\, d_y\Gamma_j, \quad x \notin \Gamma_j,$$

where $j = 1, 2$ and ψ, φ are density functions. Note that for $j = 1$ sometimes we will write $\mathbb{R}$ instead of Γ_1. In this case, for example, the single layer potential defined above has the form

$$V_1(\psi)(x_1, x_2) = \int_{\mathbb{R}} \mathcal{K}(x_1 - y, x_2)\psi(y)\,dy, \quad x_2 \neq 0.$$

Let us first consider the operators $V := V_1$ and $W := W_1$.

Theorem 3.2. *(See [7].) The single and double layer potentials V and W are continuous operators*

$$V : H^s(\mathbb{R}) \to H^{s+1+\frac{1}{2}}\big(\mathbb{R}^2_\pm\big), \qquad W : H^{s+1}(\mathbb{R}) \to H^{s+1+\frac{1}{2}}\big(\mathbb{R}^2_\pm\big) \tag{11}$$

for all $s \in \mathbb{R}$.

Clearly, a similar result holds true for the operators V_2 and W_2.

Let us now recall some properties of the above introduced potentials. The following limit relations are well known (cf. [7]):

$$\begin{aligned} \big[V(\psi)\big]^+_{\mathbb{R}} = \big[V(\psi)\big]^-_{\mathbb{R}} =: \mathcal{H}(\psi), \qquad \big[\partial_n V(\psi)\big]^\pm_{\mathbb{R}} =: \left[\mp\frac{1}{2}I\right](\psi), \\ \big[W(\varphi)\big]^\pm_{\mathbb{R}} =: \left[\pm\frac{1}{2}I\right](\varphi), \qquad \big[\partial_n W(\varphi)\big]^+_{\mathbb{R}} = \big[\partial_n W(\varphi)\big]^-_{\mathbb{R}} =: \mathcal{L}(\varphi), \end{aligned} \tag{12}$$

where

$$\mathcal{H}(\psi)(z) := \int_{\mathbb{R}} \mathcal{K}(z - y)\psi(y)\,dy, \quad z \in \mathbb{R}, \tag{13}$$

$$\mathcal{L}(\varphi)(z) := \lim_{\mathbb{R}^2_+ \ni x \to z \in \mathbb{R}} \partial_{n(x)} \int_{\mathbb{R}} \big[\partial_{n(y)}\mathcal{K}(y - x)\big]\varphi(y)\,dy, \quad z \in \mathbb{R}, \tag{14}$$

and I denotes the identity operator.

In our further reasoning we will make a convenient use of the even and odd extension operators defined by

$$\ell^e\varphi(y) = \begin{cases} \varphi(y), & y \in \mathbb{R}_\pm, \\ \varphi(-y), & y \in \mathbb{R}_\mp, \end{cases} \quad \text{and} \quad \ell^o\varphi(y) = \begin{cases} \varphi(y), & y \in \mathbb{R}_\pm, \\ -\varphi(-y), & y \in \mathbb{R}_\mp, \end{cases}$$

respectively.

Remark 3.3. (See [13].) The following operators

$$\begin{aligned} \ell^e : H^{\varepsilon+\frac{1}{2}}(\mathbb{R}_\pm) \longrightarrow H^{\varepsilon+\frac{1}{2}}(\mathbb{R}), \qquad \ell^o : r_{\mathbb{R}_\pm}\widetilde{H}^{\varepsilon+\frac{1}{2}}(\mathbb{R}_\pm) \longrightarrow H^{\varepsilon+\frac{1}{2}}(\mathbb{R}), \\ \ell^o : H^{\varepsilon-\frac{1}{2}}(\mathbb{R}_\pm) \longrightarrow H^{\varepsilon-\frac{1}{2}}(\mathbb{R}), \qquad \ell^e : r_{\mathbb{R}_\pm}\widetilde{H}^{\varepsilon-\frac{1}{2}}(\mathbb{R}_\pm) \longrightarrow H^{\varepsilon-\frac{1}{2}}(\mathbb{R}) \end{aligned}$$

are continuous for all $\varepsilon \in [0, 1/2)$.

Lemma 3.4. *(See [7].) If $0 \leqslant \varepsilon < 1/2$, then*

$$r_{\Gamma_2} \circ V \circ \ell^o \psi = 0, \qquad r_{\Gamma_2} \circ W \circ \ell^o \tilde{\varphi} = 0,$$
$$r_{\Gamma_2} \circ \partial_{n_2} V \circ \ell^e \tilde{\psi} = 0, \qquad r_{\Gamma_2} \circ \partial_{n_2} W \circ \ell^e \varphi = 0$$

for all $\psi \in H^{\varepsilon-\frac{1}{2}}(\mathcal{S})$, $\tilde{\psi} \in r_{\mathcal{S}} \widetilde{H}^{\varepsilon-\frac{1}{2}}(\mathcal{S})$, $\varphi \in H^{\varepsilon+\frac{1}{2}}(\mathcal{S})$, and $\tilde{\varphi} \in r_{\mathcal{S}} \widetilde{H}^{\varepsilon+\frac{1}{2}}(\mathcal{S})$.

Note that analogous results are valid for the operators V_2 and W_2.

4. The problems in the form of Wiener–Hopf plus Hankel equations

In the present section, we will equivalently write our problems in the form of single equations characterized by Wiener–Hopf plus Hankel operators. In view of this, the use of the pseudodifferential operators introduced in the last section together with an appropriate use of odd and even extension operators will be crucial. In addition, the reflection operator J given by the rule

$$J\psi(y) = \psi(-y) \quad \text{for all } y \in \mathbb{R},$$

will also play an important role here.

The boundary value problem $\mathcal{P}_{D-D}$ can equivalently be rewritten in the following form: Find $u_j \in H^{1+\varepsilon}(\Omega_j)$, $j = 1, 2$, such that

$$(\Delta + k^2) u_j = 0 \quad \text{in } \Omega_j, \tag{15}$$
$$[u_j]^+_{\mathcal{S}_j} = h_j \quad \text{on } \mathcal{S}_j, \tag{16}$$
$$[u_1]^+_{\mathcal{C}} = g_0^+, \qquad [u_2]^-_{\mathcal{C}} = g_0^- \quad \text{on } \mathcal{C}, \tag{17}$$

and

$$[u_1]^+_{\mathcal{C}^c} - [u_2]^-_{\mathcal{C}^c} = 0, \qquad [\partial_{n_1} u_1]^+_{\mathcal{C}^c} - [\partial_{n_1} u_2]^-_{\mathcal{C}^c} = 0 \quad \text{on } \mathcal{C}^c, \tag{18}$$

where $\mathcal{C}^c = \mathcal{S} \setminus \overline{\mathcal{C}}$.

Let us consider the following functions

$$u_1 = 2W_2\big(\ell^e h_1\big) + 2W_1\big(\ell^o\big(\ell_+ g_0^+ - \big[2W_2\big(\ell^e h_1\big)\big]^+_{\mathcal{S}}\big) + \ell^o(r_{\mathcal{S}}\varphi)\big) \quad \text{in } \Omega_1 \tag{19}$$

and

$$u_2 = 2W_2\big(\ell^e h_2\big) - 2W_1\big(\ell^o\big(\ell_+ g_0^- - \big[2W_2\big(\ell^e h_2\big)\big]^-_{\mathcal{S}}\big) + \ell^o(r_{\mathcal{S}}\varphi)\big) \quad \text{in } \Omega_2, \tag{20}$$

where φ is an arbitrary element of the space $\widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c)$ and $\ell_+ g_0^+ \in H^{\frac{1}{2}+\varepsilon}(\mathcal{S})$ is any fixed extension of $g_0^+ \in H^{\frac{1}{2}+\varepsilon}(\mathcal{C})$, while $\ell_+ g_0^- \in H^{\frac{1}{2}+\varepsilon}(\mathcal{S})$ denotes the extension of $g_0^- \in H^{\frac{1}{2}+\varepsilon}(\mathcal{C})$ which satisfies the condition $r_{\mathcal{C}^c}(\ell_+ g_0^+ - \ell_+ g_0^-) = 0$. Note that such extension exists due to the compatibility condition (6). Note also that, the compatibility conditions (8) ensure us that $\ell_+ g_0^+ - [2W_2(\ell^e h_1)]^+_{\mathcal{S}}$ and $\ell_+ g_0^- - [2W_2(\ell^e h_2)]^-_{\mathcal{S}}$ are elements of $r_{\mathcal{S}} \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathcal{S})$ and therefore we may apply the extension operator ℓ^o.

Using the results from Section 3 it is easy to verify that u_j belong to the spaces $H^{1+\varepsilon}(\Omega_j)$ and satisfy Eqs. (15)–(17). Moreover, on $\mathcal{C}^c$ we have

$$[u_1]^+_{\mathcal{C}^c} - [u_2]^-_{\mathcal{C}^c} = 0.$$

Therefore it remains to satisfy the condition

$$[\partial_{n_1} u_1]^+_{\mathcal{C}^c} - [\partial_{n_1} u_2]^-_{\mathcal{C}^c} = 0,$$

which together with (19) and (20) leads us to the following equation

$$r_{\mathcal{C}^c}\mathcal{L}\big(\ell^o r_{\mathcal{S}}\varphi\big) = \Phi, \tag{21}$$

where

$$\Phi = \frac{1}{2} r_{\mathcal{C}^c}\big(\partial_{n_1} W_2\big(\ell^e h_2 - \ell^e h_1\big) \\ - \mathcal{L}\big(\ell^o\big(\ell_+ g_0^+ + \ell_+ g_0^- - \big[2W_2\big(\ell^e h_1\big)\big]^+_{\mathcal{S}} - \big[2W_2\big(\ell^e h_2\big)\big]^-_{\mathcal{S}}\big)\big)\big).$$

Thus we need to investigate the invertibility of the operator

$$r_{\mathcal{C}^c}\mathcal{L}\ell^o r_{\mathcal{S}} : \widetilde{H}^{\frac{1}{2}+\varepsilon}\big(\mathcal{C}^c\big) \longrightarrow H^{-\frac{1}{2}+\varepsilon}\big(\mathcal{C}^c\big).$$

With the help of the operator J and the shift convolution operators $\mathcal{F}^{-1}\tau_{\pm a}\cdot\mathcal{F}$ (where we recall that $\mathcal{F}$ denotes the Fourier transformation and $\tau_b(\xi) := e^{ib\xi}$, $\xi \in \mathbb{R}$), we equivalently reduce the problem to the invertibility of the operator

$$r_{\mathbb{R}_+}\big(\mathcal{L} - \mathcal{L}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J\big) = r_{\mathbb{R}_+}\mathcal{F}^{-1}\tau_{-a}\cdot\mathcal{F}\mathcal{L}\ell^o r_{\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F} : \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \longrightarrow H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+).$$

Let us note here that due to Theorem 3.1 and having in mind the exhibited limit relations of the potentials, we already know that $\operatorname{Ker} r_{\mathbb{R}_+}(\mathcal{L} - \mathcal{L}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J) = \{0\}$.

The boundary value problem $\mathcal{P}_{D-N}$ can equivalently be rewritten in the following form: Find $u_j \in H^{1+\varepsilon}(\Omega_j)$, $j = 1, 2$, such that

$$\big(\Delta + k^2\big)u_j = 0 \quad \text{in } \Omega_j,$$
$$[\partial_{n_2} u_j]^+_{\mathcal{S}_j} = f_j \quad \text{on } \mathcal{S}_j,$$
$$[u_1]^+_{\mathcal{C}} = g_0^+, \qquad [u_2]^-_{\mathcal{C}} = g_0^- \quad \text{on } \mathcal{C},$$

and

$$[u_1]^+_{\mathcal{C}^c} - [u_2]^-_{\mathcal{C}^c} = 0, \qquad [\partial_{n_1} u_1]^+_{\mathcal{C}^c} - [\partial_{n_1} u_2]^-_{\mathcal{C}^c} = 0 \quad \text{on } \mathcal{C}^c.$$

For this problem let us consider the following functions

$$u_1 = -2V_2\big(\ell^o f_1\big) + 2W_1\big(\ell^e\big(\ell_+ g_0^+ + r_{\mathcal{S}}\varphi\big)\big) \quad \text{in } \Omega_1 \tag{22}$$

and

$$u_2 = -2V_2\big(\ell^o f_2\big) - 2W_1\big(\ell^e\big(\ell_+ g_0^- + r_{\mathcal{S}}\varphi\big)\big) \quad \text{in } \Omega_2, \tag{23}$$

where φ is an arbitrary element of the space $\widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c)$, while $\ell_+ g_0^{\pm}$ are above introduced extensions of $g_0^{\pm}$.

Similarly as above, the boundary conditions on $\mathcal{C}^c$ leads us to the following equation

$$r_{\mathcal{C}^c}\mathcal{L}\big(\ell^e r_{\mathcal{S}}\varphi\big) = \Phi, \tag{24}$$

where

$$\Phi = \frac{1}{2} r_{\mathcal{C}^c}\big(V_2\big(\ell^o f_1 - \ell^o f_2\big) - \mathcal{L}\big(\ell^e\big(\ell_+ g^+ + \ell_+ g^-\big)\big)\big).$$

Thus we need to investigate the invertibility of the operator

$$r_{\mathcal{C}^c}\mathcal{L}\ell^e r_{\mathcal{S}} : \widetilde{H}^{\frac{1}{2}+\varepsilon}\big(\mathcal{C}^c\big) \longrightarrow H^{-\frac{1}{2}+\varepsilon}\big(\mathcal{C}^c\big).$$

As previously, with the help of the operators J and $\mathcal{F}^{-1}\tau_{\pm a}\cdot\mathcal{F}$, we are able to equivalently transform this second problem into the invertibility of the operator

$$r_{\mathbb{R}_+}\big(\mathcal{L} + \mathcal{L}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J\big) = r_{\mathbb{R}_+}\mathcal{F}^{-1}\tau_{-a}\cdot\mathcal{F}\mathcal{L}\ell^e r_{\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F} : \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \longrightarrow H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+).$$

Again, let us note here that due to Theorem 3.1 and having in mind the exhibited limit relations of the potentials, we already know that $\operatorname{Ker} r_{\mathbb{R}_+}(\mathcal{L} + \mathcal{L}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J) = \{0\}$.

The boundary value problem $\mathcal{P}_{N-D}$ can equivalently be rewritten in the following form: Find $u_j \in H^{1+\varepsilon}(\Omega_j)$, $j = 1, 2$, such that

$$\big(\Delta + k^2\big)u_j = 0 \quad \text{in } \Omega_j, \tag{25}$$

$$[u_j]^+_{\mathcal{S}_j} = h_j \quad \text{on } \mathcal{S}_j, \tag{26}$$

$$[\partial_{n_1} u_1]^+_{\mathcal{C}} = g_1^+, \qquad [\partial_{n_1} u_2]^-_{\mathcal{C}} = g_1^- \quad \text{on } \mathcal{C}, \tag{27}$$

and

$$[u_1]^+_{\mathcal{C}^c} - [u_2]^-_{\mathcal{C}^c} = 0, \qquad [\partial_{n_1} u_1]^+_{\mathcal{C}^c} - [\partial_{n_1} u_2]^-_{\mathcal{C}^c} = 0 \quad \text{on } \mathcal{C}^c,$$

where $\mathcal{C}^c = \mathcal{S} \setminus \overline{\mathcal{C}}$.

Let us consider the following functions

$$u_1 = 2W_2\big(\ell^e h_1\big) - 2V_1\big(\ell^o\big(\ell_+ g_1^+ + r_{\mathcal{S}}\psi\big)\big) \quad \text{in } \Omega_1 \tag{28}$$

and

$$u_2 = 2W_2\big(\ell^e h_2\big) + 2V_1\big(\ell^o\big(\ell_+ g_1^- + r_{\mathcal{S}}\psi\big)\big) \quad \text{in } \Omega_2, \tag{29}$$

where ψ is an arbitrary element of the space $\widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c)$ and $\ell_+ g_1^+ \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{S})$ is any fixed extension of $g_1^+ \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{C})$, while $\ell_+ g_1^- \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{S})$ denotes the extension of $g_1^- \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{C})$ which satisfies the condition $r_{\mathcal{C}^c}(\ell_+ g_1^+ - \ell_+ g_1^-) = 0$. Note that such extension exists due to the compatibility condition (7).

Using the results from Section 3 it is easy to verify that u_j belong to the spaces $H^{1+\varepsilon}(\Omega_j)$ and satisfy Eqs. (25)–(27). Moreover, on $\mathcal{C}^c$ we have

$$[\partial_{n_1} u_1]^+_{\mathcal{C}^c} - [\partial_{n_1} u_2]^-_{\mathcal{C}^c} = 0.$$

Therefore it remains to satisfy the condition

$$[u_1]^+_{\mathcal{C}^c} - [u_2]^-_{\mathcal{C}^c} = 0,$$

which together with (28) and (29) leads us to the following equation

$$r_{\mathcal{C}^c}\mathcal{H}\big(\ell^o r_{\mathcal{S}}\psi\big) = \Psi, \tag{30}$$

where

$$\Psi = \frac{1}{2} r_{\mathcal{C}^c}\big(W_2\big(\ell^e h_1 - \ell^e h_2\big) - \mathcal{H}\big(\ell^o\big(\ell_+ g_1^+ + \ell_+ g_1^-\big)\big)\big).$$

Thus we need to investigate the invertibility of the operator

$$r_{\mathcal{C}^c}\mathcal{H}\ell^o r_{\mathcal{S}} : \widetilde{H}^{-\frac{1}{2}+\varepsilon}\big(\mathcal{C}^c\big) \longrightarrow H^{\frac{1}{2}+\varepsilon}\big(\mathcal{C}^c\big).$$

With the help of the operator J and the shift convolution operators $\mathcal{F}^{-1}\tau_{\pm a}\cdot\mathcal{F}$, we equivalently reduce the problem to the invertibility of the operator

$$r_{\mathbb{R}_+}\big(\mathcal{H} - \mathcal{H}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J\big) = r_{\mathbb{R}_+}\mathcal{F}^{-1}\tau_{-a}\cdot\mathcal{F}\mathcal{H}\ell^o r_{\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F} : \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \longrightarrow H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+).$$

Let us note here that due to Theorem 3.1 and having in mind the exhibited limit relations of the potentials, we already know that $\operatorname{Ker} r_{\mathbb{R}_+}(\mathcal{H} - \mathcal{H}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J) = \{0\}$.

The boundary value problem $\mathcal{P}_{N-N}$ can equivalently be rewritten in the following form: Find $u_j \in H^{1+\varepsilon}(\Omega_j)$, $j = 1, 2$, such that

$$\begin{gathered}
\big(\Delta + k^2\big)u_j = 0 \quad \text{in } \Omega_j,\\
[\partial_{n_2} u_j]^+_{\mathcal{S}_j} = f_j \quad \text{on } \mathcal{S}_j,\\
[\partial_{n_1} u_1]^+_{\mathcal{C}} = g_1^+, \qquad [\partial_{n_1} u_2]^-_{\mathcal{C}} = g_1^- \quad \text{on } \mathcal{C},
\end{gathered}$$

and

$$[u_1]^+_{\mathcal{C}^c} - [u_2]^-_{\mathcal{C}^c} = 0, \qquad [\partial_{n_1} u_1]^+_{\mathcal{C}^c} - [\partial_{n_1} u_2]^-_{\mathcal{C}^c} = 0 \quad \text{on } \mathcal{C}^c.$$

For this problem let us consider the following functions

$$u_1 = -2V_2\big(\ell^o f_1\big) - 2V_1\big(\ell^e\big(\ell_+ g_1^+ + 2\big[\partial_{n_1} V_2\big(\ell^o f_1\big)\big]^+_{\mathcal{S}}\big) + \ell^e(r_{\mathcal{S}}\psi)\big) \quad \text{in } \Omega_1 \tag{31}$$

and

$$u_2 = -2V_2\big(\ell^o f_2\big) + 2V_1\big(\ell^e\big(\ell_+ g_1^- + 2\big[\partial_{n_1} V_2\big(\ell^o f_2\big)\big]^-_{\mathcal{S}}\big) + \ell^e(r_{\mathcal{S}}\psi)\big) \quad \text{in } \Omega_2, \tag{32}$$

where ψ is an arbitrary element of the space $\widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c)$, while $\ell_+ g_1^{\pm}$ are above introduced extensions of $g_1^{\pm}$. Note also that, the compatibility conditions (9) ensure us that $\ell_+ g_1^+ + 2[\partial_{n_1} V_2(\ell^o f_1)]_{\mathcal{S}}^+$ and $\ell_+ g_1^- + 2[\partial_{n_1} V_2(\ell^o f_2)]_{\mathcal{S}}^-$ are elements of $r_{\mathcal{S}}\widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathcal{S})$ and therefore we may apply the extension operator ℓ^o.

Similarly as above, the boundary conditions on $\mathcal{C}^c$ lead us to the following equation

$$r_{\mathcal{C}^c}\mathcal{H}\big(\ell^e r_{\mathcal{S}}\psi\big) = \Psi, \tag{33}$$

where

$$\Psi = \frac{1}{2} r_{\mathcal{C}^c}\big(V_2\big(\ell^o f_2 - \ell^o f_1\big) - \mathcal{H}\big(\ell^e\big(\ell_+ g^+ + \ell_+ g^- + \big[\partial_{n_1} V_2\big(\ell^o f_1\big)\big]_{\mathcal{S}}^+ + \big[\partial_{n_1} V_2\big(\ell^o f_2\big)\big]_{\mathcal{S}}^-\big)\big)\big).$$

Thus we need to investigate the invertibility of the operator

$$r_{\mathcal{C}^c}\mathcal{H}\ell^e r_{\mathcal{S}} : \widetilde{H}^{-\frac{1}{2}+\varepsilon}\big(\mathcal{C}^c\big) \longrightarrow H^{\frac{1}{2}+\varepsilon}\big(\mathcal{C}^c\big).$$

As previously, with the help of the operators J and $\mathcal{F}^{-1}\tau_{\pm a}\cdot\mathcal{F}$, we are able to equivalently transform this second problem into the invertibility of the operator

$$r_{\mathbb{R}_+}\big(\mathcal{H} + \mathcal{H}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J\big) = r_{\mathbb{R}_+}\mathcal{F}^{-1}\tau_{-a}\cdot\mathcal{F}\mathcal{H}\ell^e r_{\mathbb{R}_+}\mathcal{F}^{-1}\tau_a\cdot\mathcal{F} : \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \longrightarrow H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+).$$

We observe that due to Theorem 3.1 and having in mind the exhibited limit relations of the potentials, we already know that $\operatorname{Ker} r_{\mathbb{R}_+}(\mathcal{H} + \mathcal{H}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J) = \{0\}$.

5. Analysis of Wiener–Hopf plus and minus Hankel operators

In this section we will consider general operators with the global structure of Wiener–Hopf plus and minus Hankel operators, and we will recall – in an appropriate framework for our purposes – some known operator relations between these operators and Wiener–Hopf operators.

In view of this, let us also recall that two bounded linear operators T and S (acting between Banach spaces) are said to be *equivalent* if $T = ESF$ for some boundedly invertible operators E and F. In such a case we will write $T \sim S$. In addition, when the use of identity extension operators is needed in combination with the related operators T and S, such corresponding relations are denominated (*toplinear*) *equivalence after extension relations* (see [1,12] for a detailed description about such operator relations).

Let us define

$$\Lambda_{\pm}^{s}(\xi) := (\xi \pm i)^s = \big(1+\xi^2\big)^{\frac{s}{2}} \exp\big\{si \arg(\xi \pm i)\big\},$$

with a branch chosen in such a way that $\arg(\xi \pm i) \to 0$ as $\xi \to +\infty$, i.e., with a cut along the negative real axis (see Example 1.7 in [15] for additional information about the properties of these functions). In addition, we will also use the notation

$$\zeta(\xi) := \frac{\Lambda_-(\xi)}{\Lambda_+(\xi)} = \frac{\xi - i}{\xi + i}, \quad \xi \in \mathbb{R}.$$

Lemma 5.1. *(See [15, §4].) Let $s, r \in \mathbb{R}$, and consider the operators*

$$\Lambda_+^s(D) = (D+i)^s,$$
$$\Lambda_-^s(D) = r_{\mathbb{R}_+}(D-i)^s \ell^{(r)},$$

where $(D \pm i)^{\pm s} = \mathcal{F}^{-1}(\xi \pm i)^{\pm s} \cdot \mathcal{F}$, and $\ell^{(r)} : H^r(\mathbb{R}_+) \to H^r(\mathbb{R})$ is any bounded extension operator in these spaces (which particular choice does not change the definition of $\Lambda_-^s(D)$).

These operators arrange isomorphisms in the following space settings

$$\Lambda_+^s(D) : \widetilde{H}^r(\mathbb{R}_+) \to \widetilde{H}^{r-s}(\mathbb{R}_+),$$
$$\Lambda_-^s(D) : H^r(\mathbb{R}_+) \to H^{r-s}(\mathbb{R}_+)$$

(for any $s, r \in \mathbb{R}$).

Bearing in mind the purpose of this section, let $A = \mathrm{Op}(a) = \mathcal{F}^{-1} a \cdot \mathcal{F}$ and $B = \mathrm{Op}(b)$ be pseudodifferential operators of order $\mu \in \mathbb{R}$; thus, $\langle \cdot \rangle^{-\mu} a, \langle \cdot \rangle^{-\mu} b \in L^\infty(\mathbb{R})$, where $\langle \xi \rangle := (1+\xi^2)^{\frac{1}{2}}$. Then $C_\pm := A \pm BJ$ arrange continuous maps

$$r_{\mathbb{R}_+} C_\pm : \widetilde{H}^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+) \tag{34}$$

for all $s \in \mathbb{R}$. In addition, assume also that a^{-1} exists and so $\langle \cdot \rangle^\mu a^{-1} \in L^\infty(\mathbb{R})$.

Lemma 5.1 allows us to construct an equivalence relation between $r_{\mathbb{R}_+} C_\pm$ and

$$r_{\mathbb{R}_+} \mathcal{C}_\pm : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+), \tag{35}$$

which is explicitly given by the following identity

$$r_{\mathbb{R}_+} \mathcal{C}_\pm := \Lambda_-^{s-\mu} r_{\mathbb{R}_+} C_\pm \Lambda_+^{-s} = r_{\mathbb{R}_+}(\mathcal{A} \pm \mathcal{B} J), \tag{36}$$

where

$$\mathcal{A} := (D-i)^{s-\mu} A (D+i)^{-s} \quad \text{and} \quad \mathcal{B} := (D-i)^{s-\mu} BJ(D+i)^{-s} J. \tag{37}$$

Indeed, due to the fact that $\Lambda_-^{s-\mu} : H^{s-\mu}(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$ and $\Lambda_+^{-s} : L_2(\mathbb{R}_+) \to \widetilde{H}^s(\mathbb{R}_+)$ are invertible operators (cf. Lemma 5.1), the identity (36) shows that

$$r_{\mathbb{R}_+} C_\pm \sim r_{\mathbb{R}_+} \mathcal{C}_\pm. \tag{38}$$

Note that

$$\Lambda_+^s(-\xi) = \Lambda_-^s(\xi) e^{s\pi i}, \qquad \Lambda_-^s(-\xi) = \Lambda_+^s(\xi) e^{-s\pi i}$$

which in particular allow us to describe the operators $\mathcal{A}$ and $\mathcal{B}$ and their symbols in the following way

$$\mathcal{A} = \mathrm{Op}(\tilde{a}), \qquad \tilde{a}(\xi) = \Lambda_-^{s-\mu}(\xi) a(\xi) \Lambda_+^{-s}(\xi),$$
$$\mathcal{B} = \mathrm{Op}(\tilde{b}), \qquad \tilde{b}(\xi) = \Lambda_-^{s-\mu}(\xi) b(\xi) \Lambda_+^{-s}(-\xi) = \Lambda_-^{-\mu}(\xi) b(\xi) e^{-s\pi i}.$$

Further, let us consider a pseudodifferential operator $\mathrm{Op}(\Xi)$ with 2×2 matrix symbol

$$\Xi(\xi)=\begin{pmatrix}\alpha_{11}(\xi) & \alpha_{12}(\xi)\\ \alpha_{21}(\xi) & \alpha_{22}(\xi)\end{pmatrix} \tag{39}$$

with

$$\begin{aligned}\alpha_{11}(\xi)&=\tilde{a}(\xi)-\tilde{b}(\xi)\big(\tilde{a}(-\xi)\big)^{-1}\tilde{b}(-\xi)\\ &=\Lambda_-^{s-\mu}(\xi)a(\xi)\Lambda_+^{-s}(\xi)-\Lambda_-^{s-\mu}(\xi)b(\xi)\big(a(-\xi)\big)^{-1}b(-\xi)\Lambda_+^{-s}(\xi), \qquad (40)\\ \alpha_{12}(\xi)&=-\tilde{b}(\xi)\big(\tilde{a}(-\xi)\big)^{-1}=-\Lambda_-^{s-\mu}(\xi)b(\xi)\big(a(-\xi)\big)^{-1}\Lambda_+^{-s+\mu}(\xi)e^{(s-\mu)\pi i},\\ \alpha_{21}(\xi)&=\big(\tilde{a}(-\xi)\big)^{-1}\tilde{b}(-\xi)=e^{s\pi i}\Lambda_-^{s}(\xi)\big(a(-\xi)\big)^{-1}b(-\xi)\Lambda_+^{-s}(\xi),\\ \alpha_{22}(\xi)&=\big(\tilde{a}(-\xi)\big)^{-1}=\Lambda_-^{s}(\xi)\big(a(-\xi)\big)^{-1}e^{(2s-\mu)\pi i}\Lambda_+^{-s+\mu}(\xi). \qquad (41)\end{aligned}$$

Under the above conditions on a and b, it is straightforward to conclude that

$$r_{\mathbb{R}_+}\mathrm{Op}(\Xi):\big[L_2(\mathbb{R}_+)\big]^2\to\big[L_2(\mathbb{R}_+)\big]^2 \tag{42}$$

is a continuous operator. Moreover, the determinant of the symbol of this operator is always nonzero. Indeed, we have for the determinant of the corresponding 2×2 matrix symbol

$$\begin{aligned}\det\Xi(\xi)&=\alpha_{11}(\xi)\alpha_{22}(\xi)-\alpha_{21}(\xi)\alpha_{12}(\xi)\\ &=\zeta^{2s-\mu}(\xi)\frac{a(\xi)}{a(-\xi)}e^{(2s-\mu)\pi i}\neq 0\end{aligned} \tag{43}$$

for all $\xi\in\mathbb{R}$.

The importance of operator $r_{\mathbb{R}_+}\mathrm{Op}(\Xi)$ is clarified in the next result.

Theorem 5.2. *(See [8, §6].)*

(i) *The operators*

$$r_{\mathbb{R}_+}\mathcal{C}_\pm=r_{\mathbb{R}_+}\mathcal{A}\pm r_{\mathbb{R}_+}\mathcal{B}J:L_2(\mathbb{R}_+)\to L_2(\mathbb{R}_+)$$

(*defined in* (35)–(37)) *are both invertible if and only if the operator* $r_{\mathbb{R}_+}\mathrm{Op}(\Xi)$ (*given in* (42)) *is invertible.*

(ii) *The operators $r_{\mathbb{R}_+}\mathcal{C}_+$ and $r_{\mathbb{R}_+}\mathcal{C}_-$ have both the Fredholm property if and only if $r_{\mathbb{R}_+}\mathrm{Op}(\Xi)$ has the Fredholm property. In addition, when in the presence of the Fredholm property for these three operators, their Fredholm indices satisfy the identity*

$$\mathit{Ind}\, r_{\mathbb{R}_+}\mathcal{C}_+ + \mathit{Ind}\, r_{\mathbb{R}_+}\mathcal{C}_- = \mathit{Ind}\, r_{\mathbb{R}_+}\mathrm{Op}(\Xi). \tag{44}$$

In fact, this theorem is a consequence of a stronger fact which basically states that $r_{\mathbb{R}_+}\mathrm{Op}(\Xi)$ is (toplinear) equivalence after extension with a diagonal matrix operator whose diagonal entries are the operators $r_{\mathbb{R}_+}\mathcal{C}_+$ and $r_{\mathbb{R}_+}\mathcal{C}_-$. Moreover, it is interesting to clarify that all the necessary operators to identify such (toplinear) equivalence after extension relation can be built in an explicit way (see [9–12]).

6. Final conclusions

In this final section we would like to consider all the operators which we have equivalently associated with our problems in the above sense, and look for their invertibility.

Considering the general structure of the Wiener–Hopf operator in the last section, for the cases under study, we have

$$r_{\mathbb{R}_+}\operatorname{Op}(\Xi):\left[L_2(\mathbb{R}_+)\right]^2\to\left[L_2(\mathbb{R}_+)\right]^2 \tag{45}$$

so that $A=\mathcal{L}$ and $B=\mathcal{L}\operatorname{Op}(\tau_{-2a})$ or $A=\mathcal{H}$ and $B=\mathcal{H}\operatorname{Op}(\tau_{-2a})$. Therefore, having in mind that our goal is to analyze the invertibility of such operator, in a first stage we will start by studying its Fredholm property. In view of this, it is important to recall the complete symbol of the pseudodifferential operators $\mathcal{H}$ and $\mathcal{L}$ (cf. [7,8]):

$$\sigma(\mathcal{H})(\xi)=-\frac{i}{2w(\xi)}\quad\text{and}\quad\sigma(\mathcal{L})(\xi)=-\frac{iw(\xi)}{2}, \tag{46}$$

where $w=w(\xi):=(\varrho^2+\rho^2)^{\frac{1}{4}}(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2})$, with

$$\varrho=\varrho(\xi):=(\Re\mathrm{e}\,k)^2-(\Im\mathrm{m}\,k)^2-\xi^2,$$
$$\rho:=2(\Re\mathrm{e}\,k)(\Im\mathrm{m}\,k)$$

and

$$\alpha:=\begin{cases}\arctan\frac{\rho}{|\varrho|} & \text{if } \varrho>0,\ \rho>0,\\ \frac{\pi}{2} & \text{if } \varrho=0,\ \rho>0,\\ \pi-\arctan\frac{\rho}{|\varrho|} & \text{if } \varrho<0,\ \rho>0,\\ \pi & \text{if } \rho=0,\\ 2\pi-\arctan\frac{|\rho|}{|\varrho|} & \text{if } \varrho>0,\ \rho<0,\\ \frac{3\pi}{2} & \text{if } \varrho=0,\ \rho<0,\\ \pi+\arctan\frac{|\rho|}{|\varrho|} & \text{if } \varrho<0,\ \rho<0.\end{cases} \tag{47}$$

Now we provide detailed arguments for the operators $A=\mathcal{H}$ and $B=\mathcal{H}\operatorname{Op}(\tau_{-2a})$ with $s=-\frac{1}{2}+\varepsilon$, $\mu=-1$. The remaining case can be treated analogously with obvious changes in the corresponding places.

Computing formulas (40)–(41) for the present case (and using in particular the fact that $\omega(\xi)=\omega(-\xi)$), we obtain

$$\Xi(\xi)=\Xi_p(\xi)=\begin{pmatrix}0 & -i\zeta^{\frac{1}{2}+\varepsilon}(\xi)\tau_{-2a}(\xi)e^{\varepsilon\pi i}\\ -i\zeta^{-\frac{1}{2}+\varepsilon}(\xi)\tau_{2a}(\xi)e^{\varepsilon\pi i} & 2i\omega(\xi)\Lambda_-^{-\frac{1}{2}+\varepsilon}(\xi)\Lambda_+^{-\frac{1}{2}-\varepsilon}(\xi)e^{2\varepsilon\pi i}\end{pmatrix}. \tag{48}$$

We then realize that Ξ_p belongs to the very general C^*-algebra of the semi-almost periodic two by two matrix functions on the real line ($[SAP(\mathbb{R})]^{2\times 2}$); see [33]. We recall that $[SAP(\mathbb{R})]^{2\times 2}$ is the smallest closed subalgebra of $[L^\infty(\mathbb{R})]^{2\times 2}$ that contains the (classical) algebra of (two by two) *almost periodic elements* ($[AP]^{2\times 2}$) and the (two by two) continuous matrices with possible jumps at infinity.

Due to a known characterization of the structure of $[SAP(\mathbb{R})]^{2\times 2}$ (see [3,4,33]), we can choose a continuous function on the real line, say γ, such that $\gamma(-\infty)=0$, $\gamma(+\infty)=1$ and

$$\Xi_p=(1-\gamma)(\Xi_p)_l+\gamma(\Xi_p)_r+(\Xi_p)_0$$

where $(\Xi_p)_0$ is a continuous two by two matrix function with zero limit at infinity, and $(\Xi_p)_l$ and $(\Xi_p)_r$ are matrices with almost periodic elements, uniquely determined by Ξ_p, and that in our case have the following form (due to the behavior of Ξ_p at $\pm\infty$):

$$(\Xi_p)_l = \begin{pmatrix} 0 & i\tau_{-2a}e^{-\varepsilon\pi i} \\ i\tau_{2a}e^{-\varepsilon\pi i} & -2 \end{pmatrix},$$
$$(\Xi_p)_r = \begin{pmatrix} 0 & -i\tau_{-2a}e^{\varepsilon\pi i} \\ -i\tau_{2a}e^{\varepsilon\pi i} & -2e^{2\varepsilon\pi i} \end{pmatrix}.$$

In here, it is worth noting that we had in consideration that $\omega(\xi) \to i|\xi|$ as $\xi \to \pm\infty$, and

$$\zeta^{\nu}(\xi) \to 1 \quad \text{as } \xi \to \infty,$$

and

$$\zeta^{\nu}(\xi) \to e^{-2\pi\nu i} \quad \text{as } \xi \to -\infty.$$

For a given Banach algebra (with unit element) $\mathcal{M}$, by $\mathcal{GM}$ we will denote the collection of all invertible elements of $\mathcal{M}$.

Definition 6.1. (See, e.g., [16] or §6.3 in [5].) An invertible almost periodic matrix function $\Phi \in \mathcal{G}[AP]^{2\times 2}$ admits a *canonical right AP-factorization* if

$$\Phi = \Phi^{-}\Phi^{+}, \tag{49}$$

where $\Phi^{\pm} \in \mathcal{G}[AP^{\pm}]^{2\times 2}$, with $AP^{\pm}$ denoting the intersection of AP with the non-tangential limits of functions in $H^{\infty}(\mathbb{C}_{\pm})$ (the set of all bounded and analytic functions in $\mathbb{C}_{\pm}$).

Proposition 6.2. *(Cf., e.g., [5, Proposition 2.22].) Let $A \subset (0,\infty)$ be an unbounded set and let*

$$\{I_\alpha\}_{\alpha\in A} = \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$$

be a family of intervals $I_\alpha \subset \mathbb{R}$ such that $|I_\alpha| = y_\alpha - x_\alpha \to \infty$ as $\alpha \to \infty$. If $\varphi \in AP$, then the limit

$$M(\varphi) := \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \varphi(x)\, dx$$

exists, is finite, and is independent of the particular choice of the family $\{I_\alpha\}$.

Definition 6.3. (i) For any $\varphi \in AP$, the number that has just been introduced in Proposition 6.2, $M(\varphi)$, is called the *Bohr mean value* (or simply the *mean value*) of φ. In the matrix case the Bohr mean value is defined entry-wise.

(ii) For a matrix function $\Phi \in \mathcal{G}[AP]^{2\times 2}$ admitting a canonical right AP-factorization (49), we may define the new matrix

$$\mathbf{d}(\Phi) := M(\Phi^{-})M(\Phi^{+}), \tag{50}$$

which is known as the *geometric mean* of Φ.

It is worth mention that (50) is independent of the particular choice of the (canonical) right *AP*-factorization of Φ, and that this definition is consistent with the corresponding one for the scalar case (which can be defined in a somehow more global way).

Theorem 6.4. *For $0 \leqslant \varepsilon < 1/2$, the operator $r_{\mathbb{R}_+} \operatorname{Op}(\Xi_p) : [L_2(\mathbb{R}_+)]^2 \to [L_2(\mathbb{R}_+)]^2$, with Ξ_p given by (48), is a Fredholm operator with zero Fredholm index.*

Proof. The matrices $(\Xi_p)_l$ and $(\Xi_p)_r$ admit the following right canonical *AP*-factorizations:

$$(\Xi_p)_l = \begin{pmatrix} -\frac{1}{2}e^{-2\varepsilon\pi i} & -\frac{1}{2}i\tau_{-2a}e^{-\varepsilon\pi i} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i\tau_{2a}e^{-\varepsilon\pi i} & -2 \end{pmatrix}, \tag{51}$$

$$(\Xi_p)_r = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}i\tau_{-2a}e^{-\varepsilon\pi i} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i\tau_{2a}e^{\varepsilon\pi i} & -2e^{2\varepsilon\pi i} \end{pmatrix} \tag{52}$$

(in which the necessary factor properties are evident; cf. Definition 6.1).

Having built the factorizations (51)–(52), we can now apply Theorem 3.2 in [16] or Theorem 10.11 in [5] in view of evaluating about the Fredholm property for $r_{\mathbb{R}_+} \operatorname{Op}(\Xi_p)$. Indeed, within our case of $\Xi_p \in \mathcal{G}[SAP(\mathbb{R})]^{2\times 2}$ and whose local representatives at infinity admit canonical right *AP*-factorizations (51)–(52), applying that theorems we have that $r_{\mathbb{R}_+} \operatorname{Op}(\Xi_p)$ is a Fredholm operator if and only if

$$\operatorname{sp}\bigl[\mathbf{d}^{-1}\bigl((\Xi_p)_r\bigr)\mathbf{d}\bigl((\Xi_p)_l\bigr)\bigr] \cap (-\infty, 0] = \emptyset,$$

where $\operatorname{sp}[\mathbf{d}^{-1}((\Xi_p)_r)\mathbf{d}((\Xi_p)_l)]$ stands for the set of eigenvalues of the matrix $\mathbf{d}^{-1}((\Xi_p)_r)\mathbf{d}((\Xi_p)_l) := [\mathbf{d}((\Xi_p)_r)]^{-1}\mathbf{d}((\Xi_p)_l)$.

Noticing that directly from the definition of Bohr mean value we have $M(1/2) = 1/2$, $M(1) = 1$, $M(-2) = -2$, $M(\tau_{2a}) = 0$ and $M(\tau_{-2a}) = 0$, it follows

$$\begin{aligned} \mathbf{d}\bigl((\Xi_p)_l\bigr) &= M\left[\begin{pmatrix} -\frac{1}{2}e^{-2\varepsilon\pi i} & -\frac{1}{2}i\tau_{-2a}e^{-\varepsilon\pi i} \\ 0 & 1 \end{pmatrix}\right] M\left[\begin{pmatrix} 1 & 0 \\ i\tau_{2a}e^{-\varepsilon\pi i} & -2 \end{pmatrix}\right] \\ &= \begin{pmatrix} -\frac{1}{2}e^{-2\varepsilon\pi i} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2}e^{-2\varepsilon\pi i} & 0 \\ 0 & -2 \end{pmatrix}, \end{aligned} \tag{53}$$

$$\begin{aligned} \mathbf{d}\bigl((\Xi_p)_r\bigr) &= M\left[\begin{pmatrix} -\frac{1}{2} & \frac{1}{2}i\tau_{-2a}e^{-\varepsilon\pi i} \\ 0 & 1 \end{pmatrix}\right] M\left[\begin{pmatrix} 1 & 0 \\ -i\tau_{2a}e^{\varepsilon\pi i} & -2e^{2\varepsilon\pi i} \end{pmatrix}\right] \\ &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -2e^{2\varepsilon\pi i} \end{pmatrix}. \end{aligned} \tag{54}$$

As a consequence,

$$\mathbf{d}^{-1}\bigl((\Xi_p)_r\bigr)\mathbf{d}\bigl((\Xi_p)_l\bigr) = \begin{pmatrix} e^{-2\varepsilon\pi i} & 0 \\ 0 & e^{-2\varepsilon\pi i} \end{pmatrix}$$

and

$$\operatorname{sp}\bigl[\mathbf{d}^{-1}\bigl((\Xi_p)_r\bigr)\mathbf{d}\bigl((\Xi_p)_l\bigr)\bigr] \cap (-\infty, 0] = \bigl\{e^{-2\varepsilon\pi i}\bigr\} \cap (-\infty, 0] = \emptyset, \tag{55}$$

which allows us to conclude that $r_{\mathbb{R}_+}\operatorname{Op}(\Xi_p):[L_2(\mathbb{R}_+)]^2 \to [L_2(\mathbb{R}_+)]^2$ is a Fredholm operator (for the case in consideration of $0 \leqslant \varepsilon < 1/2$).

The zero Fredholm index is obtained from the formula (cf. Theorem 10.21 in [5])

$$\mathit{Ind}\, r_{\mathbb{R}_+}\operatorname{Op}(\Xi_p) = -\,\mathit{ind}[\det \Xi_p] - \sum_{k=1}^{2}\left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg \xi_k\right\}\right),$$

where $\mathit{ind}[\det \Xi_p]$ denotes the Cauchy index of the determinant of Ξ_p, the numbers $\xi_1, \xi_2 \in \mathbb{C} \setminus (-\infty, 0]$ are the eigenvalues of the matrix $\mathbf{d}^{-1}((\Xi_p)_r)\mathbf{d}((\Xi_p)_l)$ and $\{\cdot\}$ stands for the fractional part of a real number. $\square$

From the proof of this last result, and in particular from (55), we realize that if we would allow the case $\varepsilon = 1/2$ then our operators would not have the Fredholm property (and therefore would not be invertible operators).

Corollary 6.5. *Let $0 \leqslant \varepsilon < \frac{1}{2}$. The Wiener–Hopf plus and minus Hankel operators*

$$r_{\mathbb{R}_+}\big(\mathcal{L} \pm \mathcal{L}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J\big) : \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \longrightarrow H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+), \tag{56}$$

$$r_{\mathbb{R}_+}\big(\mathcal{H} \pm \mathcal{H}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J\big) : \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \longrightarrow H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \tag{57}$$

(*which characterize our four problems*) *are invertible operators.*

Proof. Bearing in mind the equivalence relation between the operators (34) and (35), we have for the operators associated with our two problems:

$$\dim\operatorname{CoKer} r_{\mathbb{R}_+}\big(\mathcal{H} \pm \mathcal{H}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J\big) = \dim\operatorname{CoKer} r_{\mathbb{R}_+}\mathcal{C}_{\pm}, \tag{58}$$

$$\dim\operatorname{Ker} r_{\mathbb{R}_+}\big(\mathcal{H} \pm \mathcal{H}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J\big) = \dim\operatorname{Ker} r_{\mathbb{R}_+}\mathcal{C}_{\pm}. \tag{59}$$

From Theorems 5.2 and 6.4, we obtain that $r_{\mathbb{R}_+}\mathcal{C}_+$ and $r_{\mathbb{R}_+}\mathcal{C}_-$ are Fredholm operators. Moreover, recalling that $\operatorname{Ker} r_{\mathbb{R}_+}(\mathcal{H} \pm \mathcal{H}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J) = \{0\}$, from identities (44), (58)–(59) and Theorem 6.4 it follows

$$\begin{aligned}
0 &= \mathit{Ind}\, r_{\mathbb{R}_+}\mathcal{C}_+ + \mathit{Ind}\, r_{\mathbb{R}_+}\mathcal{C}_- \\
&= \mathit{Ind}\, r_{\mathbb{R}_+}\big(\mathcal{H} + \mathcal{H}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J\big) + \mathit{Ind}\, r_{\mathbb{R}_+}\big(\mathcal{H} - \mathcal{H}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J\big) \\
&= \big(0 - \dim\operatorname{CoKer} r_{\mathbb{R}_+}\big(\mathcal{H} + \mathcal{H}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J\big)\big) \\
&\quad + \big(0 - \dim\operatorname{CoKer} r_{\mathbb{R}_+}\big(\mathcal{H} - \mathcal{H}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J\big)\big).
\end{aligned}$$

Thus, we have

$$\dim\operatorname{CoKer} r_{\mathbb{R}_+}\big(\mathcal{H} + \mathcal{H}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J\big) = -\dim\operatorname{CoKer} r_{\mathbb{R}_+}\big(\mathcal{H} - \mathcal{H}\mathcal{F}^{-1}\tau_{-2a}\cdot\mathcal{F}J\big) = 0$$

and so we reach to the conclusion that both operators in (56) are invertible.

Similarly, we obtain the invertibility results for both operators in (57). $\square$

Due to a direct combination of the results of Sections 3 and 4, and Corollary 6.5, we now obtain the main conclusion of the present work for the problems in consideration.

Theorem 6.6. *Let* $0 \leqslant \varepsilon < \frac{1}{2}$.

(i) *The problem* $\mathcal{P}_{D-D}$ *has a unique solution which is representable as a pair* (u_1, u_2) *defined by the formulas* (19), (20), *where* φ *is the unique solution of Eq.* (21).
(ii) *The problem* $\mathcal{P}_{D-N}$ *has a unique solution which is representable as a pair* (u_1, u_2) *defined by the formulas* (22), (23), *where* φ *is the unique solution of Eq.* (24).
(iii) *The problem* $\mathcal{P}_{N-D}$ *has a unique solution which is representable as a pair* (u_1, u_2) *defined by the formulas* (28), (29), *where* ψ *is the unique solution of Eq.* (30).
(iv) *The problem* $\mathcal{P}_{N-N}$ *has a unique solution which is representable as a pair* (u_1, u_2) *defined by the formulas* (31), (32), *where* ψ *is the unique solution of Eq.* (33).

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