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ON ELECTROMAGNETIC SCATTERING PROBLEMS FOR SCREENS

1. Introduction and Formulation of the Problems

Abboud and Starling [1] studied the scattering of time–harmonic electromagnetic waves by a smooth screen type perfect conductor using techniques from the theory of pseudo–differential operators. The case for Lipschitz screens was investigated by Buffa and Christiansen [2]. They obtained results allowing us to reduce requirements from [1] on the given boundary data. The purpose of the present paper is to continue this type of research and study the screen type boundary value problems for Maxwell's equations with different boundary conditions on both sides of the screen.

The screen S is considered as a connected part of a smooth boundary Γ of a bounded and connected domain $\Omega \subset \mathbb{R}^3$ and denote by ν the unit normal exterior with respect to Ω.

Let $H^1_{\rm loc}(\mathbb{R}^3)$, $H^{\frac{1}{2}}(\Gamma)$, and $H^{-\frac{1}{2}}(\Gamma)$ be the usual Sobolev spaces (cf. [7]) and recall the definition of the following spaces

$$\begin{aligned}
\mathbb{H}_{\rm loc}(\mathrm{curl};\mathbb{R}\backslash\overline{S}) &:= \{u \in (L^2_{\rm loc}(\mathbb{R}\backslash\overline{\Gamma}))^3 : \mathrm{curl}\, u \in (L^2_{\rm loc}(\mathbb{R}\backslash\overline{\Gamma}))^3\},\\
\mathbb{H}_t^{-\frac{1}{2}}(\Gamma) &:= \{u \in (H^{-\frac{1}{2}}(\Gamma))^3 : \nu\cdot u = 0 \text{ on } \Gamma\},\\
\mathbb{H}_{\rm div}^{-\frac{1}{2}}(\Gamma) &:= \{u \in \mathbb{H}_t^{-\frac{1}{2}}(\Gamma),\ \mathrm{div}_\Gamma\, u \in H^{-\frac{1}{2}}(\Gamma)\},\\
\mathbb{H}_{\rm curl}^{-\frac{1}{2}}(\Gamma) &:= \{u \in \mathbb{H}_t^{-\frac{1}{2}}(\Gamma),\ \mathrm{curl}_\Gamma\, u \in H^{-\frac{1}{2}}(\Gamma)\},
\end{aligned}$$

where div_Γ and curl_Γ denote the surface divergence and the surface scalar curl operators, respectively. For detailed introduction of these spaces we refer to [6], [8].

Note that $\mathbb{H}_{\rm div}^{-\frac{1}{2}}(\Gamma)$ and $\mathbb{H}_{\rm curl}^{-\frac{1}{2}}(\Gamma)$ are Hilbert spaces with respect to the norms

$$||u||_{1/2,\mathrm{div}} = \big(||u||_{-1/2,\Gamma} + ||\,\mathrm{div}_\Gamma\, u||_{-1/2,\Gamma}\big)^{1/2}$$

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and

$$||u||_{1/2,\mathrm{curl}} = \left(||u||_{-1/2,\Gamma} + ||\operatorname{curl}_\Gamma u||_{-1/2,\Gamma}\right)^{1/2},$$

respectively. Moreover, we have following duality and trace results (cf. [9], [1]):

$$\mathbb{H}^{-\frac{1}{2}}_{\mathrm{curl}}(\Gamma) = \left(\mathbb{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma)\right)' = \mathbb{H}^{-\frac{1}{2}}_t(\Gamma) + \operatorname{grad}_\Gamma\left(\mathbb{H}^{-\frac{1}{2}}_t(\Gamma)\right)$$

and

$$\mathbb{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma) = \left(\mathbb{H}^{-\frac{1}{2}}_{\mathrm{curl}}(\Gamma)\right)' = \mathbb{H}^{-\frac{1}{2}}_t(\Gamma) + \vec{\operatorname{curl}}_\Gamma\left(\mathbb{H}^{-\frac{1}{2}}_t(\Gamma)\right),$$

where $\operatorname{grad}_\Gamma$ and $\vec{\operatorname{curl}}_\Gamma$ are the surface gradient and the surface vector curl operators, respectively.

The mapping $u \in C^\infty(\Omega) \mapsto u \times \nu|_\Gamma$ can be uniquely extended to a surjective continuous operator $\gamma_\tau : \mathbb{H}(\mathrm{curl};\Omega) \to \mathbb{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma)$, while the mapping $u \in C^\infty(\Omega) \mapsto (u - u\cdot\nu)|_\Gamma$ can be uniquely extended to a surjective continuous operator $\gamma_t : \mathbb{H}(\mathrm{curl};\Omega) \to \mathbb{H}^{-\frac{1}{2}}_{\mathrm{curl}}(\Gamma)$.

To define the traces on S we need the following spaces: $\mathbb{H}^{-\frac{1}{2}}_{\mathrm{curl}}(S)$, which is the range of the space $\mathbb{H}^{-\frac{1}{2}}_{\mathrm{curl}}(\Gamma) \subset \mathbb{H}^{-\frac{1}{2}}_t(\Gamma)$ by the mapping r_S, and $\mathbb{H}^{-\frac{1}{2}}_{\mathrm{div}}(S)$, which is the range of the space $\mathbb{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma) \subset \mathbb{H}^{-\frac{1}{2}}_t(\Gamma)$ by the mapping r_S. Here r_S is the canonical surjection from $H^s(\Gamma)$ onto $H^s(S)$.

The duality result stated above allow us to introduce $\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\mathrm{div}}(S)$ the dual space of $\mathbb{H}^{-\frac{1}{2}}_{\mathrm{curl}}(\Gamma)$. For more details cf. [1], [5], here we additionally note that the normal trace of $u \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\mathrm{div}}(S)$ at ∂S is is well defined and is zero, also u can be extended by zero to a function in $\mathbb{H}^{-\frac{1}{2}}_{\mathrm{div}}(\Gamma)$.

The scattering of the electromagnetic waves by the open surface S leads us to the following boundary value problem for the scattered electric field $E \in \mathbb{H}_{\mathrm{loc}}(\mathrm{curl};\mathbb{R}\backslash\overline{S})$ and magnetic field $H \in \mathbb{H}_{\mathrm{loc}}(\mathrm{curl};\mathbb{R}\backslash\overline{S})$:

$$\operatorname{curl} E - ikH = 0 \quad \text{in} \quad \mathbb{R}\backslash\overline{S}, \tag{1.1}$$

$$\operatorname{curl} H + ikE = 0 \quad \text{in} \quad \mathbb{R}\backslash\overline{S}, \tag{1.2}$$

$$\gamma_\tau E^\pm = c^\pm \quad \text{on} \quad S, \tag{1.3}$$

where $k > 0$ is a the wave number and $c^\pm \in \mathbb{H}^{-\frac{1}{2}}_{\mathrm{div}}(S)$ are given data, such that they satisfy the following compatibility condition

$$c^+ - c^- \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\mathrm{div}}(S). \tag{1.4}$$

Additionally, it is required that the scattered field E, H satisfies the Silver Müller radiation condition

$$\lim_{|x|\to\infty} |x|(H \times \hat{x} - E) = 0 \tag{1.5}$$

uniformly in $\hat{x} = x/|x|$.

2. The Uniqueness and Existence Results

Theorem 2.1. *Maxwell's boundary value problem* (1.1)–(1.5) *has at most one solution.*

The proof is standard, for details cf. [1].

For the existence result let us recall that the rotation operator r from [3], which corresponds to the geometric operation $\nu \times \cdot$, establishes connection of γ_τ and γ_τ traces. Namely, for any $E \in \mathbb{H}(\mathrm{curl};\Omega)$ we have $\gamma_t E = r(\gamma_\tau E)$. This allow us to rewrite (1.3) in the following equivalent form

$$\gamma_t E^{\pm} = r(c^{\pm}) \in \mathbb{H}^{-\frac{1}{2}}_{\mathrm{curl}}(S). \tag{2.6}$$

Furthermore, an application of Stratton–Chu's formula in Ω and $\mathbb{R}^3\backslash\overline{\Omega}$ which holds true also for $\mathbb{H}(\mathrm{curl};\cdot)$ spaces (cf. [4]), and using the continuity of the tangential components of E and $\mathrm{curl}\, E$ across $\Gamma\backslash\overline{S}$ gives us that a solution $E \in \mathbb{H}_{\mathrm{loc}}(\mathrm{curl};\mathbb{R}\backslash\overline{S})$ can be represented as

$$\begin{aligned} E(x) = {} & \mathrm{curl}\int_S [E(y)\times\nu(y)]\Phi(x,y)ds_y - \int_S [\nu(y)\times \mathrm{curl}\, E(y)]\Phi(x,y)ds_y - \\ & - \frac{1}{k^2}\,\mathrm{grad}_x \int_S \mathrm{div}_S[\nu(y)\times \mathrm{curl}\, E(y)]\Phi(x,y)ds_y \end{aligned} \tag{2.7}$$

where $\Phi(x,y) := \frac{e^{ik|x-y|}}{|x-y|}$ and $[\cdot]$ denotes the jump across the screen S, thus we have

$$[E\times\nu] = E^+ \times \nu|_S - E^- \times \nu|_S = c^+ - c^-$$

and therefore

$$g := \mathrm{curl}\int_S [c^+(y) - c^-(y)]\Phi(x,y)ds_y \in \mathbb{H}_{\mathrm{loc}}(\mathrm{curl};\mathbb{R}\backslash\overline{S})$$

is a known datum, with well defined traces $\gamma_t^{\pm} g \in \mathbb{H}^{-\frac{1}{2}}_{\mathrm{curl}}(S)$. Note that, in general, $\gamma_t^+ g \neq \gamma_t^- g$. Then from the representation formula (2.7) and the boundary condition (2.6) we obtain

$$A(\nu\times \mathrm{curl}\, E) = \gamma_t^+ g - r(c^+) \tag{2.8}$$

where the operator A defined by

$$A\varphi(x) = \gamma_t\left(\int_S \varphi(y)\Phi(x,y)ds_y + \frac{1}{k^2}\,\mathrm{grad}_x\int_S \mathrm{div}_S\,\varphi(y)\Phi(x,y)ds_y\right)$$

and $\gamma_t^+ - r(c^+) \in \mathbb{H}^{-\frac{1}{2}}_{\mathrm{curl}}(S)$. Now using the known result [2, Corollary 3.6] we have that the operator $A : \widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\mathrm{div}}(S) \to \mathbb{H}^{-\frac{1}{2}}_{\mathrm{curl}}(S)$ is an isomophism and therefore the equation (2.8) is uniquely solvable.

Summing up we have the following result:

Theorem 2.2. *The electromagnetic scattering problem for screens* (1.1)–(1.5) *has a unique solution* $E, H \in \mathbb{H}_{\mathrm{loc}}(\mathrm{curl}; \mathbb{R}\backslash\overline{S})$*, where* E *is given by* (2.7)*,* $\nu \times \mathrm{curl}\, E$ *is a unique solution of* (2.8) *and* $H = \frac{1}{ik}\,\mathrm{curl}\, E$*.*

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