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Exact solution of a nonlinear heat conduction problem in a doubly periodic 2D composite material

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An analytic solution of a stationary heat conduction problem in an unbounded doubly periodic 2D composite whose matrix and inclusions consist of isotropic temperature-dependent materials is given. Each unit cell of the composite contains a finite number of circular non-overlapping inclusions. The corresponding nonlinear boundary value problem is reduced to a Laplace equation with nonlinear interface conditions. In the case when the conductive properties of the inclusions are proportional to that of the matrix, the problem is transformed into a fully linear boundary value problem for doubly periodic analytic functions. This allows one to solve the original nonlinear problem and reconstruct temperature and heat flux throughout the entire plane. The solution makes it possible to calculate the average properties over the unit cell and discuss the effective conductivity of the composite. We compare the outcomes of the present paper with a few results from literature and present numerical examples to indicate some peculiarities of the solution.

Key words: nonlinear doubly periodic composite material, conductivity problem, effective properties of the composite.

1. Introduction

The present paper is devoted to analysis of a steady-state heat conduction problem in 2D unbounded doubly periodic composite materials with temperature dependent conductivities. Our primary goal is to find an exact solution to the problem. Then we try to utilize this solution to make some conclusions on the effective properties of the nonlinear composite. Note that two problems, i.e., reconstruction of the exact solution (temperature and flux) at each point of the composite material and the evaluation of its effective properties, are mutually related but completely different problems. It is clear that, knowing an exact solution for periodic composite, one can provide the standard averaging procedure over the unit cell and thus obtain some estimate for the effective properties of the entire composite. On the other hand, having effective properties of a composite, one can solve the respective boundary value problem for that averaged material and then compute the respective solution. However this information is not sufficient to reconstruct the mechanical/physical fields at each specific point of the original composite.

Moreover, depending on the assumptions made during an averaging process (which effectively means searching for different basic cell solutions), different approximate formulae for the effective properties can be delivered. There are several justified methods for doing so in the case of linear composite [1]. In those cases the relation between the two problems (solving the problem for composite and estimation of its effective properties) are well-developed. Unfortunately, in the case of composites with components depending on the solution it still remains a challenging problem. In this paper we deliver not only an exact solution to the specific nonlinear 2D unbounded doubly periodic composite but also indicate some open questions related to the evaluation of the effective properties of such composites.

The theory and technique of finding the solution to linear boundary value problems for 2D unbounded doubly periodic composite materials with constant conductivities of their components are well-developed. The multipole expansion method provides an efficient analysis of properties of different complex heterogeneous structures (see, for example, [1] where a history of the multipole expansion method development is also given in detail). This method is efficient in both 2D and 3D cases and an arbitrary shape of inclusions. Another method utilising complex analysis techniques and functional equations was proposed in [2] and further developed in [3]–[5]. It allows the finding of temperature and flux distributions in composites with an arbitrary number of circular non-overlapping inclusions of different size in the periodicity cell and to determine in an explicit form the effective conductivity of such composite materials. Composites with a rectangular checkerboard structure were analytically investigated using the methods of complex analysis in [6].

General methods to deduce approximate formulae for effective properties of heterogeneous media were presented in [7] and [8]. Effective properties of materials with complicated macrostructure are usually studied on the basis of the asymptotic homogenization method stated in [9]–[13] and others. Solutions to the problems with multicracks (which is different in comparison with the inclusions) were discussed in [14]. Important cross-section relationships between elastic and conductive properties of heterogeneous materials were given in [15]. Essential progress in obtaining properties of composites has already been achieved utilizing numerical analysis. We refer the prospective reader to the papers (and the literature therein) for finite element method (cf. [16], [17]) and boundary element method (cf. [18]).

Problems involving nonlinear heat conduction can be divided into two major classes. The first one is when the material parameters depend on the gradient of the temperature, and the second one is when the parameters are functions of the temperature itself. The problems of the first class are close to the problems for nonlinear dielectrics and discussed in the series of works [19]–[25], and others. The respective theory is well-developed.

In contrast, the theory of composite materials with temperature-dependent properties is still under development. Homogenization theory for random composites is studied in [26]. For periodic media, there are several attempts to evaluate the thermal conductivity of thermo-sensitive heterogeneous materials. The asymptotic homogenization technique for periodic microstructure with temperature independent thermal conductivities is used and extended in [27]. Authors also derived Hashin-Shtrikman type bounds for the effective conductivity of certain types of nonlinear composites. In [28] and [29], the authors revisited the homogenization problem for a nonlinear composite in terms of Padé approximation evaluating the effective conductivity of a square array of densely packed cylinders. Recently, based upon classical approaches, the homogenization procedure for a random composite with conductivities dependent on temperature in a partial case was developed in [30]. The authors proved that the Eshelby inclusion approach is not valid when the material parameters are functions of temperature and explained why problems for nonlinear composite materials from the second class are particularly difficult, as this drastically reduces the number of methods (discussed above for linear case) which researchers could use.

In the present paper, we construct an exact solution for the unbounded doubly periodic nonlinear composite under specific assumptions on material properties of the components. Namely, we consider the static thermal conductivity problem of unbounded 2D anisotropic composite materials with circular non-overlapping inclusions in the square unit periodicity cell geometrically forming a doubly periodic structure. We suppose that each component of the composite is perfectly embedded in the matrix. Conductivities of the matrix and the inclusions depend on the temperature. The key assumption is that ratios of the component conductivities are independent of the temperature. The external flux is assumed to be arbitrarily oriented with respect to the composite symmetry. We determine the temperature and flux distributions and derive the effective conductivity of such composites.

The paper is organized as follows. An accurate formulation of the problem is given in Section 2. In Section 3, we reduce the given nonlinear boundary value problem defined by nonlinear partial differential equations and linear interface conditions to an equivalent, generally speaking, nonlinear boundary problem for Laplace equations with nonlinear interface conditions. Then we formulate conditions for which the transformed problem becomes linear and thus can be effectively solved using the technique developed by the authors elsewhere. Numerical calculations are performed and discussed in Section 5. In this section, we present effective properties of the composite and discuss the obtained results. Comparison of the results for periodic and random composites from [30] is performed in Section 6. The paper is finished by discussions and conclusions.

2. Statement of the problem

We consider a lattice L which is defined by the two fundamental translation vectors 1 and $\imath$ (where $\imath^2 = -1$) in the complex plane $\mathbb{C} \cong \mathbb{R}^2$ of the complex variable $z = x + \imath y$. Here, the representative cell is the unit square

$$Q_{(0,0)} := \left\{ z = t_1 + \imath t_2 \in \mathbb{C} : -\frac{1}{2} < t_p < \frac{1}{2},\, p = 1, 2 \right\}.$$

Let $\mathcal{E} := \bigcup_{m_1,m_2}\{m_1 + \imath m_2\}$ be the set of the lattice points, where $m_1, m_2 \in \mathbb{Z}$. The cells corresponding to the points of the lattice $\mathcal{E}$ are denoted by

$$Q_{(m_1,m_2)} = Q_{(0,0)} + m_1 + \imath m_2 := \{z \in \mathbb{C} : z - m_1 - \imath m_2 \in Q_{(0,0)}\}.$$

The situation under consideration is when mutually non-overlapping disks (inclusions) of different radii $D_k := \{z \in \mathbb{C} : |z - a_k| < r_k\}$ with boundaries $\partial D_k := \{z \in \mathbb{C} : |z - a_k| = r_k\}$ $(k = 1, 2, \ldots, N)$ are located inside the cell $Q_{(0,0)}$ and periodically repeated in all cells $Q_{(m_1,m_2)}$. We denote by

$$D_0 := Q_{(0,0)} \setminus \Big(\bigcup_{k=1}^{N} D_k \cup \partial D_k \Big)$$

the connected domain obtained by removing of the inclusions from the cell $Q_{(0,0)}$.

Discussing the entire infinite composite, the matrix and inclusions occupy domains

$$D_{\text{matrix}} = \bigcup_{m_1,m_2} ((D_0 \cup \partial Q_{(0,0)}) + m_1 + \imath m_2)$$

and

$$D_{inc} = \bigcup_{m_1,m_2} \bigcup_{k=1}^{N} (D_k + m_1 + \imath m_2)$$

with thermal conductivities $\lambda = \lambda(T)$ and $\lambda_k = \lambda_k(T)$, respectively. Here, temperature T is defined in the whole $\mathbb{R}^2$. We assume that the conductivities λ, λ_k $(k = 1, \ldots, N)$ are continuous, bounded, positive functions on $\mathbb{R}$.

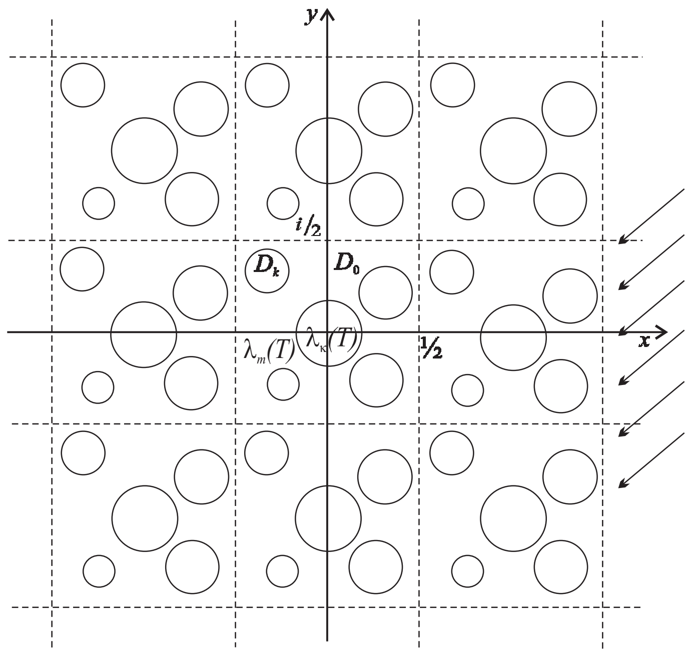

FIG. 1. 2D doubly periodic composite with inclusions.

We search for the steady-state distribution of the temperature and heat flux within such a composite. The problem is equivalent to determining the function $T = T(x, y)$ satisfying the nonlinear differential equations

$$\nabla(\lambda(T)\nabla T) = 0, \qquad (x, y) \in D_{\text{matrix}}, \tag{2.1}$$

$$\nabla(\lambda_k(T)\nabla T) = 0, \qquad (x, y) \in D_{\text{inc}}. \tag{2.2}$$

We assume that the perfect (ideal) contact conditions on the boundaries between the matrix and inclusions are satisfied:

$$T(s) = T_k(s), \qquad s \in \bigcup_{m_1,m_2} (\partial D_k + m_1 + \imath m_2), \tag{2.3}$$

$$\lambda(T(s))\frac{\partial T(s)}{\partial n} = \lambda_k(T_k(s))\frac{\partial T_k(s)}{\partial n}, \qquad s \in \bigcup_{m_1,m_2} (\partial D_k + m_1 + \imath m_2). \tag{2.4}$$

Here, the vector n is the outward unit normal vector to ∂D_k. According to the formulation, the flux and the temperature are continuous functions throughout the entire structure.

We assume that the average flux vector of intensity A is directed at an angle θ to axis Ox (see Fig. 1) which does not coincide, in general, with the orientation of the periodic cell. This gives the following conditions

$$\int\limits_{\partial Q^{(\mathrm{top})}_{(m_1,m_2)}} \lambda(T)T_y\, ds = -A\sin\theta, \tag{2.5}$$

$$\int\limits_{\partial Q^{(\mathrm{right})}_{(m_1,m_2)}} \lambda(T)T_x\, ds = -A\cos\theta. \tag{2.6}$$

Note that, in general, the flux is not periodic. However, since there are no sources and sinks in the composite, the energy conservation law dictates

$$\int\limits_{\partial Q_{(m_1,m_2)}} \lambda(T)\frac{\partial T}{\partial n}\, ds = 0. \tag{2.7}$$

This, in turns, allows us to replace conditions (2.5) and (2.6) with those defined on the opposite sides of the cell.

3. Reformulation of the problem

To solve the problem, we use the Kirchhoff transformation (cf. [31]) and introduce new continuous functions f and f_k $(k = 1, \ldots, N)$

$$f(T) = \int\limits_0^T \lambda(\xi)\, d\xi, \qquad f_k(T) = \int\limits_0^T \lambda_k(\xi)\, d\xi. \tag{3.1}$$

Then, using representations (3.1) and changing the dependent variables in the following manner:

$$u(x,y) = f(T(x,y)), \qquad u_k(x,y) = f_k(T_k(x,y)), \tag{3.2}$$

we transform the original equations (2.1) and (2.2) into the Laplace equations

$$\Delta u = 0, \qquad (x,y) \in D_{\mathrm{matrix}}, \tag{3.3}$$

$$\Delta u_k = 0, \qquad (x,y) \in D_{\mathrm{inc}}. \tag{3.4}$$

Note that f and f_k are monotonic increasing functions of temperature and, therefore, there exist their inverses f^{-1} and f_k^{-1}. The contact conditions (2.3) and (2.4) can be rewritten now as follows:

$$u = F_k(u_k), \qquad (x,y) \in \bigcup_{m_1,m_2} (\partial D_k + m_1 + \imath m_2), \tag{3.5}$$

$$\frac{\partial u}{\partial n} = \frac{\partial u_k}{\partial n}, \qquad (x,y) \in \bigcup_{m_1,m_2} (\partial D_k + m_1 + \imath m_2), \tag{3.6}$$

where the functions

$$F_k(\xi) := f(f_k^{-1}(\xi)) \tag{3.7}$$

are defined for all $\xi \in \mathbb{R}$. Note that in general the functions u and u_k may have different values on the interface ∂D_k. The derivative of F_k can be computed as follows:

$$F_k'(\xi) = \frac{f'(f_k^{-1}(\xi))}{f_k'(f_k^{-1}(\xi))} = \frac{\lambda(T_k)}{\lambda_k(T_k)}, \tag{3.8}$$

where $\xi = f_k(T_k)$. Now we use the basic assumption of the paper on the nonlinear conduction coefficients

$$\lambda(T) = C_k \lambda_k(T). \tag{3.9}$$

This property is satisfied for any $T \in \mathbb{R}$ by some positive real constants C_k. Then, one can immediately conclude that all functions F_k are linear:

$$F_k(\xi) = D_k + C_k \xi. \tag{3.10}$$

From (3.1) we have $f(0) = 0$ and $f_k(0) = 0$, and, therefore, $D_k = 0$. Note that

$$\int\limits_{\Gamma} \frac{\partial u}{\partial n} ds = 0, \quad \Gamma \subset D_{\text{matrix}}, \tag{3.11}$$

for any closed curve Γ in the matrix. Moreover, since there is no source (sink) inside the composite (neither in the matrix nor in any inclusion), the same condition is satisfied for any closed simply connected curve within the inclusion

$$\int\limits_{\Gamma_k} \frac{\partial u_k}{\partial n} ds = 0, \quad \Gamma_k \subset D_k. \tag{3.12}$$

Finally, the conditions (2.5) and (2.6) transform into the following:

$$\int\limits_{\partial Q^{(\text{top})}_{(m_1,m_2)}} u_y \, ds = -A \sin\theta, \tag{3.13}$$

$$\int\limits_{\partial Q^{(\text{right})}_{(m_1,m_2)}} u_x \, ds = -A \cos\theta. \tag{3.14}$$

Let us introduce inside the inclusions new harmonic functions:

$$\tilde{u}_k(x, y) = C_k u_k(x, y). \tag{3.15}$$

Then the transmission conditions (3.5) and (3.6) become

$$u = \tilde{u}_k, \qquad (x,y) \in \bigcup_{m_1,m_2} (\partial D_k + m_1 + \imath m_2), \tag{3.16}$$

$$\frac{\partial u}{\partial n} = \frac{1}{C_k} \frac{\partial \tilde{u}_k}{\partial n}, \qquad (x,y) \in \bigcup_{m_1,m_2} (\partial D_k + m_1 + \imath m_2). \tag{3.17}$$

A new improved algorithm for solving such a linear boundary value problem is developed and described in detail in [5]. We use this approach in our computations.

4. Effective properties of the composite

This section is devoted to evaluation of the effective properties of a nonlinear composite. We assume that the effective conductivity tensor Λ_e depends on average temperature $\langle T \rangle$ and is defined in the following way:

$$\langle \lambda(T) \nabla T \rangle = \Lambda_e(\langle T \rangle) \langle \nabla T \rangle \qquad \text{or} \qquad R_e(\langle T \rangle) \langle \lambda(T) \nabla T \rangle = \langle \nabla T \rangle, \tag{4.1}$$

where $R_e = \Lambda_e^{-1}$ is the effective resistance tensor. A similar definition to (4.1) has been used in [25]. Here, the operator $\langle \cdot \rangle$ is defined as

$$\langle f \rangle = \iint\limits_{Q_{(m_1,m_2)}} f(x,y)\, dx\, dy.$$

Note that definition (4.1) needs further justification as the question arises whether the approach is invariant with respect to the averaging cell. We will discuss this issue later during the computations.

We represent all elements involved in (4.1) in terms of a solution u and u_k of the problem (3.3)–(3.6). For the total flux in the x-direction, we have

$$\begin{aligned} &\iint\limits_{Q_{(m_1,m_2)}} \lambda(T) \frac{\partial T}{\partial x}\, dx\, dy \\ &= \iint\limits_{D_0+m_1+\imath m_2} \lambda(T) \frac{\partial T}{\partial x}\, dx\, dy + \sum_{k=1}^{N} \iint\limits_{D_k+m_1+\imath m_2} \lambda_k(T_k) \frac{\partial T_k}{\partial x}\, dx\, dy \\ &= \iint\limits_{Q_{(m_1,m_2)}} (f(T))_x\, dx\, dy + \sum_{k=1}^{N} \iint\limits_{D_k+m_1+\imath m_2} (f_k(T_k))_x\, dx\, dy \\ &= \iint\limits_{D_0+m_1+\imath m_2} \frac{\partial u}{\partial x}\, dx\, dy + \sum_{k=1}^{N} \iint\limits_{D_k+m_1+\imath m_2} \frac{\partial u_k}{\partial x}\, dx\, dy. \end{aligned} \tag{4.2}$$

Using the first Green's formula and formulas (3.3), (3.4) and (3.6), we obtain

$$\iint\limits_{Q_{(m_1,m_2)}} \lambda(T)\frac{\partial T}{\partial x}\,dx\,dy = -A\cos\theta,$$

and similarly

$$\iint\limits_{Q_{(m_1,m_2)}} \lambda(T)\frac{\partial T}{\partial y}\,dx\,dy = -A\sin\theta.$$

Thus, one can write

$$\langle\lambda(T)\,\nabla T\rangle = -A[\cos\theta,\sin\theta]^\top. \tag{4.3}$$

Note that this relationship is a direct consequence of the absence of sources or sinks inside the composite.

Due to Gauss–Ostrogradsky formula and the boundary condition (2.3), the components of the term $\langle\nabla T\rangle$ in (4.1) are defined as

$$\begin{aligned}\iint\limits_{Q_{(m_1,m_2)}} \frac{\partial T}{\partial x}\,dx\,dy &= \iint\limits_{D_0+m_1+\imath m_2} \frac{\partial T}{\partial x}\,dx\,dy + \sum_{k=1}^{N}\iint\limits_{D_k+m_1+\imath m_2}\frac{\partial T_k}{\partial x}\,dx\,dy\\ &= \oint\limits_{\partial D_0+m_1+\imath m_2} T(s)\cos(n_s,e_i)\,ds + \sum_{k=1}^{N}\oint\limits_{\partial D_k+m_1+\imath m_2}[T_k(s)-T(s)]\cos(n_s^k,e_i)\,ds\\ &= \oint\limits_{\partial D_0+m_1+\imath m_2} T(s)\cos(n_s,e_i)\,ds = \oint\limits_{\partial D_0+m_1+\imath m_2} f^{-1}(u(x,y))\cos(n_s,e_i)\,ds,\end{aligned}$$

where n_s and n_s^k are the outward unit normal vectors to $\partial D_0+m_1+\imath m_2$ and $\partial D_k+m_1+\imath m_2$, respectively, and e_i is the basis vector. Analogously,

$$\iint\limits_{Q_{(m_1,m_2)}} \frac{\partial T}{\partial y}\,dx\,dy = \oint\limits_{\partial D_0+m_1+\imath m_2} f^{-1}(u(x,y))\cos(n_s,e_j)\,ds.$$

Finally, the average temperature is

$$\begin{aligned}\langle T\rangle &= \iint\limits_{Q_{(m_1,m_2)}} T(x,y)\,dx\,dy\\ &= \iint\limits_{D_0+m_1+\imath m_2} f^{-1}(u(x,y))\,dx\,dy + \sum_{k=1}^{N}\iint\limits_{D_k+m_1+\imath m_2} f_k^{-1}(u_k(x,y))\,dx\,dy.\end{aligned} \tag{4.4}$$

It is more convenient to first compute the components of the effective resistance tensor R_e from the second formula in (4.1) and then find the effective conductivity tensor $\Lambda_e = R_e^{-1}$.

5. Numerical examples

5.1. Description of the periodic composite

In our computations we consider a composite where four inclusions are situated inside the cell $Q_{(0,0)}$ with the centers (defined in the notations of complex variables): $a_1 = -0.18 + 0.2\imath$, $a_2 = 0.33 - 0.34\imath$, $a_3 = 0.33 + 0.35\imath$, $a_4 = -0.18 - 0.2\imath$. The radii of the inclusions are the same $r_k = R = 0.145$, thus the volume fraction of the inclusions for such composite is $\nu = 4\pi R^2 = 0.2642$. With such a choice of the positions of the inclusions, one can expect that the composite exhibits anisotropic properties and we will observe this fact. Moreover, apart from the fact that the volume fracture is not particularly high, the inclusion boundaries are situated very close to each other (the minimal distance 0.02). Thus, the far field approach in defining the effective properties of such composite is rather problematic.

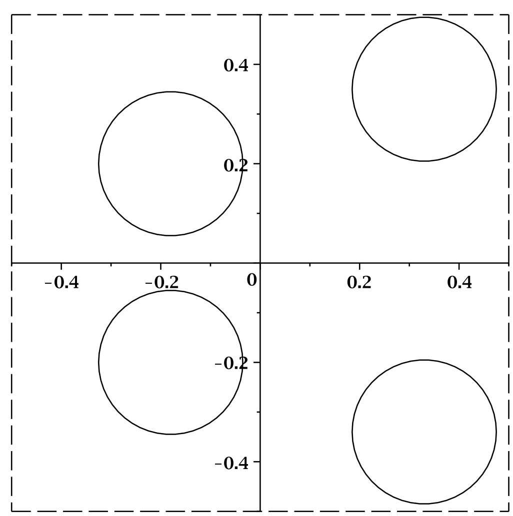

Fig. 2. Configuration of the unit cell with four inclusions considered in computation.

Further, we suppose that the heat flows in x-direction ($\theta = 0$) with the intensity $A = -1$. We choose the conductivity of the matrix $\lambda(T)$ to be defined in the following form:

$$\lambda(T) = \begin{cases} y_1, & T < x_1, \\ y_2 + \dfrac{y_1 - y_2}{x_1} T, & x_1 \le T \le 0, \\ y_2 + \dfrac{y_1 - y_2}{x_2} T, & 0 \le T \le x_2, \\ y_1, & T > x_2, \end{cases} \tag{5.1}$$

where y_1, y_2 are positive constants, and $x_1 < x_2$. We take for the calculations $x_1 = -2$, $x_2 = 2$, $y_1 = 4.5$, $y_2 = 13.5$, and define the conductivities of the matrix λ from the condition (3.9) with $C_k = 0.09$ $(k = 1, ..., 4)$. In Fig. 3, we represent the conductivity function λ. The function λ_k has the identical shape with the pike taking value $\lambda_k(0) = \lambda(0)/C_k = 150$.

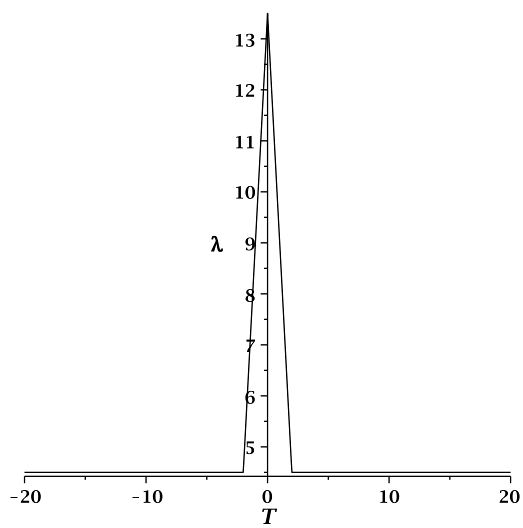

FIG. 3. The function λ.

For computations, we use the algorithm described in [5]. For the chosen configuration it guarantees a computational error less than 10^{-6}.

5.2. Evaluation of the effective conductivity tensor

Note that in the linear case temperature is defined up to an arbitrary additive constant, and this constant is not involved in the determination of the effective conductivity of a composite material. This is not generally speaking the case for nonlinear problems, and one needs to clarify how the additive constant, appearing during the stage of solving the auxiliary linear problem (3.3)–(3.14), influences (or not) the computation of the effective conductivity tensor of the equivalent nonlinear composite. Two procedures can be suggested to evaluate the effective conductivity.

- First, one can solve the auxiliary linear boundary value problem in a doubly periodic domain preserving its uniqueness by any appropriately chosen condition (for example, here we impose that the function $u = u_*$ satisfies the condition $u_*(0) = 0$). Then, to evaluate the properties of the composite material, one can compute the average temperature and the effective resistivity for each particular unit cell presenting the data as the functional relationship $R_e = R_e(\langle T \rangle)$.

It is clear from the character of the chosen nonlinearity of the conductivity coefficients that the domain where the nonlinear behavior manifests itself lies only inside an infinite strip of unknown *finite thickness* which depends on the flux characteristics: intensity A and angle θ. Thus, it is obvious that the effective conductivity tensor demonstrates nonlinear behavior within a finite interval of temperatures, and thus, one does not need to trace all cells. On the other hand, there is still an infinite number of the cells belonging to the strip, therefore one can expect that the result of such a procedure is representative enough to reconstruct a continuous function using some smoothing procedure.

- One can suggest also another method for evaluation of the effective properties. Namely, we consider an arbitrary cell in the original domain and build a set of solutions to the auxiliary problem in the form $u = u_* + C$, where C is an arbitrary constant. Then, for every constant C, the components of the effective resistance tensor, R_e, and the average temperature, $\langle T \rangle$, are functions of the parameter C. Changing the value of C continuously from $-\infty$ to ∞, one receives the sought for effective conductivity tensor of the composite as a continuous function of the average temperature. Naturally, for the conductivities of the composite components analyzed in this example, the nonlinear character of the relationship will be observed only within the finite interval of the parameter C. It is clear that this procedure does not depend on the chosen cell.

Note that the both methods allow one to determine two components of the effective resistance tensor R_e for each of the two orthogonal flux directions. Thus considering $\theta = 0$, we define $R_e[1, j] = R_e[1, j](\langle T \rangle)$ $(j = 1, 2)$, and choosing $\theta = \pi/2$ we find $R_e[2, j] = R_e[2, j](\langle T \rangle)$. As a result, the entire tensor $R_e(\langle T \rangle)$ is defined.

To check whether and when the two aforementioned procedures are equivalent, we use both of them in our computations. The respective components of the effective resistance tensor are represented in Figs. 4 and 5. Dots on the curves correspond to the second approach, while the continuous lines correspond to values computed for consecutive unit cells. These continuous lines were obtained by spline interpolation. Discrepancy between the methods is on the level 10^{-5} while the computational accuracy of the solution itself is 10^{-6}. Taking into account the fact that one needs to integrate and interpolate the data to compute the effective properties, the revealed discrepancy can be considered as a perfect evidence that the both methods provide the same result. However, an accurate mathematical proof is still to be delivered.

Note also that due to the chosen functions determining the conductivities, one can expect the effective conductivity to be an even function of the average temperature. This fact could be also taken into account to reconstruct the

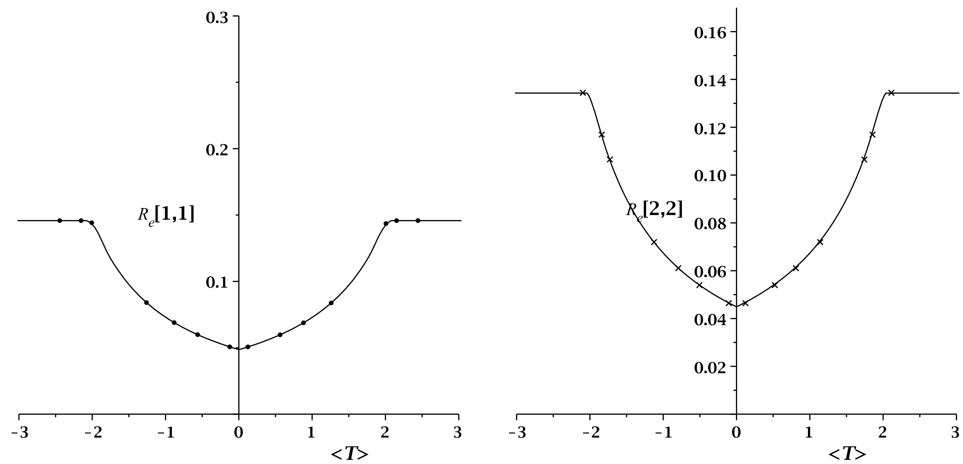

FIG. 4. Main diagonal elements of the effective resistance tensor R_e ($R_e[1,1]$ and $R_e[2,2]$) computed by each of the proposed methods.

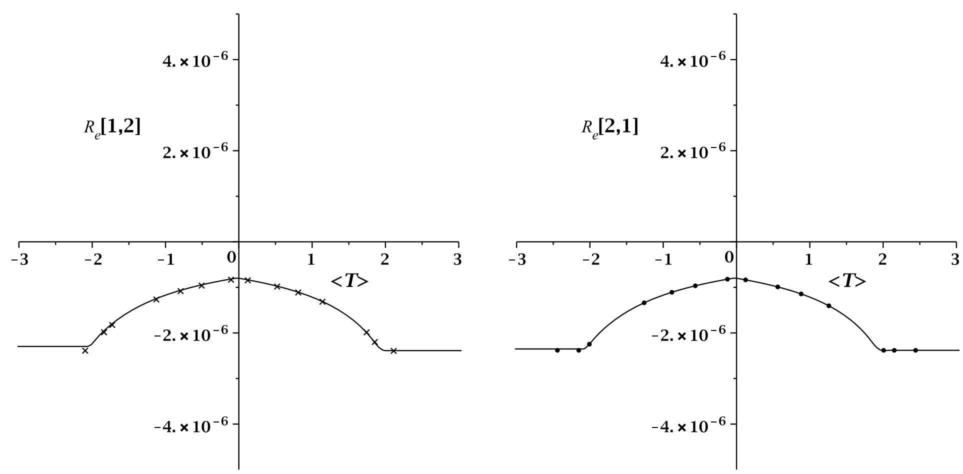

FIG. 5. Components $R_e[1,2]$ and $R_e[2,1]$ of the effective resistance tensor R_e computed by each of the proposed methods.

properties. However, we do not use this argument in this study to eliminate unnecessary assumptions.

Finally, having the effective resistance tensor $R_e(\langle T\rangle)$, we calculate the effective conductivity tensor $\Lambda_e(\langle T\rangle)$ defined in (4.1). The respective results are presented in Figs. 6 and 7, respectively. One can see that the shape of the functions are quite similar to that of the functions $\lambda(\langle T\rangle)$ and $\lambda_k(\langle T\rangle)$ for the composite components. Only some deviations can be observed near the points where the functions are not smooth.

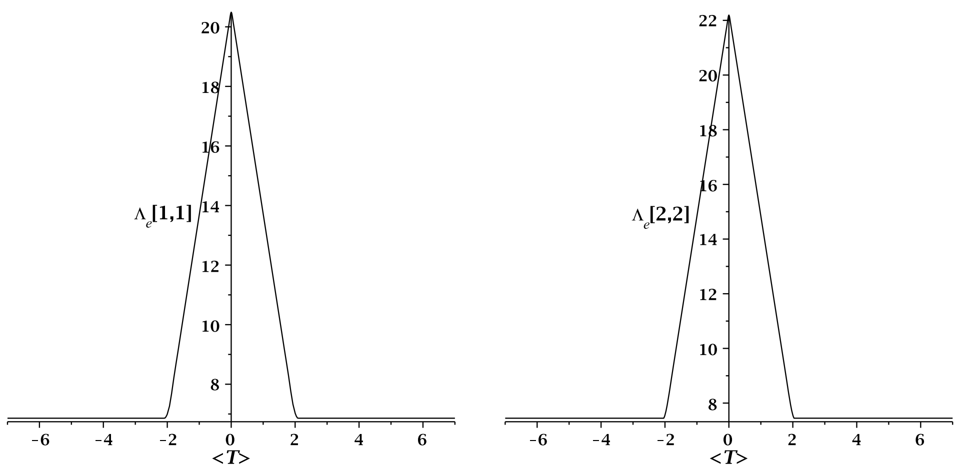

Fig. 6. Diagonal components $\Lambda_e[1,1]$, $\Lambda_e[2,2]$ of the effective conductivity tensor Λ_e.

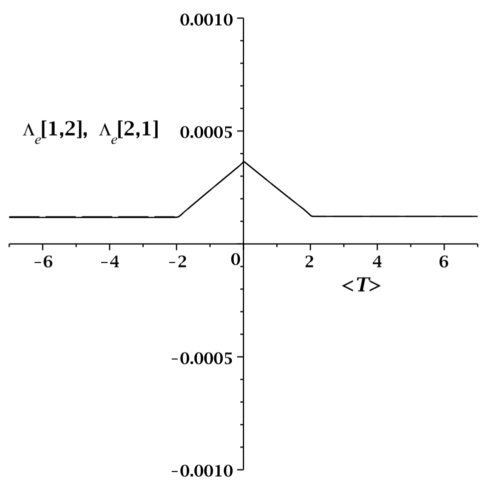

Fig. 7. Components $\Lambda_e[1,2]$ and $\Lambda_e[2,1]$ of the effective conductivity tensor.

As one can expect from the very beginning, the composite demonstrates anisotropic properties, that is the components on the main diagonals are not the same in spite of the fact that the components are fully isotropic materials. This follows from the irregular distribution of the inclusions in the cell. Additionally, the main axes of the effective conductivity tensor do not coincide with the initial x, y-axes. As a result, the resulted tensor is not a diagonal one. However, the corresponding components (anti-diagonal tensor elements) are small and may be comparable with the accuracy of the computations. This requires additional verification, which will be carried out later.

In the next section we evaluate known bounds for the effective properties of the nonlinear composite in question. At first glance there is not much logic in doing so. Indeed, since we constructed an analytical solution to the nonlinear problem and then determined the effective conductivity of the composite directly, it looks like there is no need for such estimate. On the other hand, since the question "how should the effective properties of such composites be determined" is still opened, we decided to demonstrate how our results fit into the general framework.

5.3. Hashin–Shtrikman bounds and other estimates

The general estimates for the effective properties have been evaluated for nonlinear composite in [27] and [32]. First, we start with more crude estimates, giving straightforward elementary bounds of the effective conductivity tensor

$$\mu_1(T)I \le \Lambda_e(T) \le \mu_2(T)I, \tag{5.2}$$

where I is the unit tensor and

$$\begin{aligned} \mu_1(T) &= \left(\frac{1-\nu}{\lambda(T)} + \frac{\nu}{\lambda_k(T)}\right)^{-1}, \\ \mu_2(T) &= (1-\nu)\lambda(T) + \nu\lambda_k(T). \end{aligned} \tag{5.3}$$

Here $\lambda(T)$ and $\lambda_k(T)$ are the conductivities of the matrix and the inclusions, while ν is the volume fraction of the inclusions.

Inequalities (5.2) are the Reuss-type and Voigt-type bounds on the effective coefficients (see [1], [27]).

Let us recall that the notation $A \ge B$ used in (5.2) for matrices means that the inequality $(Ax, x) \ge (Bx, x)$ holds true for an arbitrary vector $x \in \mathbb{R}^n$ ($n = 2$ in our case). In other words, one needs to show that the following inequalities are true:

$$m_{11}(T) = \mu_1 - \lambda^e_{11} \le 0, \qquad m_{21}(T) = \mu_2 - \lambda^e_{11} \ge 0, \tag{5.4}$$

$$m_{12}(T) = 4(\mu_1 - \lambda^e_{11})(\mu_1 - \lambda^e_{22}) - (\lambda^e_{12} + \lambda^e_{21})^2 \ge 0, \tag{5.5}$$

$$m_{22}(T) = 4(\mu_2 - \lambda^e_{11})(\mu_2 - \lambda^e_{22}) - (\lambda^e_{12} + \lambda^e_{21})^2 \ge 0. \tag{5.6}$$

In Figs. 8 and 9 we present the graphs of the minors from equalities $(5.4)_1$, (5.5), $(5.4)_2$ and (5.6), respectively. It is clear that the equalities hold true in a strict way.

Now we check feasibility of the Hashin-Shtrikman bounds extended in [27]. These estimates are more narrow than the elementary bounds (5.2) and can be

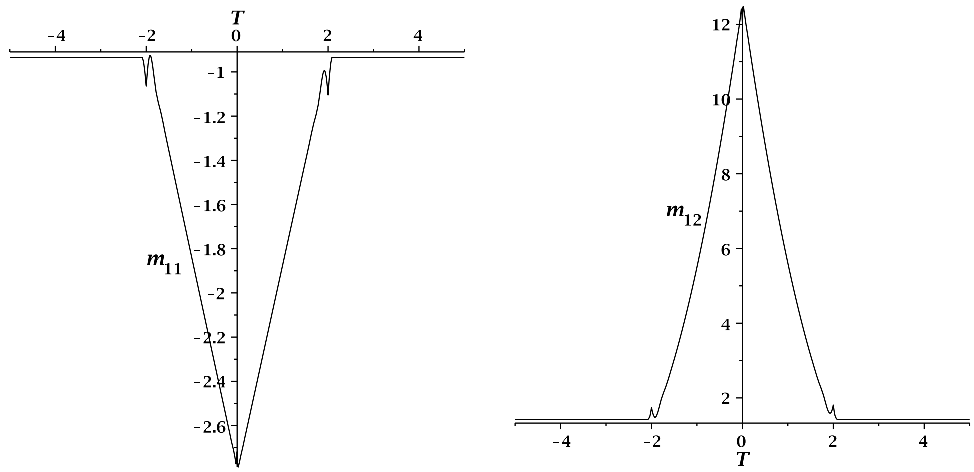

Fig. 8. Verification of the Reuss-type inequality for the effective properties of the nonlinear composite from (5.2).

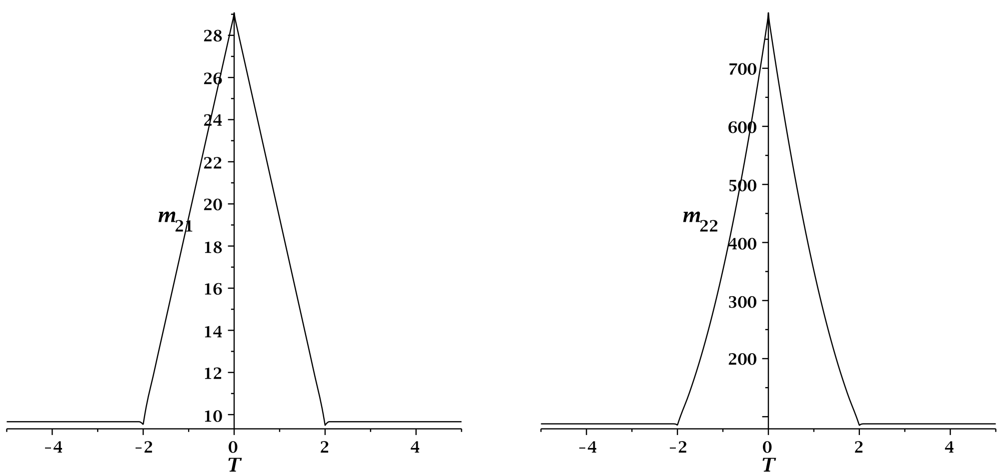

Fig. 9. Verification of the Voigt-type inequality for the effective properties of the nonlinear composite from (5.2).

written in our notations (compare with [27]):

$$tr\left[(\Lambda_e(T)-\lambda(T)I)^{-1}\right] \leq \frac{1}{\mu_2(T)-\lambda(T)} + \frac{1}{\mu_1(T)-\lambda(T)}, \tag{5.7}$$

and

$$tr\left[(\lambda_k(T)I-\Lambda_e(T))^{-1}\right] \leq \frac{1}{\lambda_k(T)-\mu_2(T)} + \frac{1}{\lambda_k(T)-\mu_1(T)}, \tag{5.8}$$

where $trA = A_{jj}$, $(j = 1, 2)$. The left- and right-hand sides of the inequalities

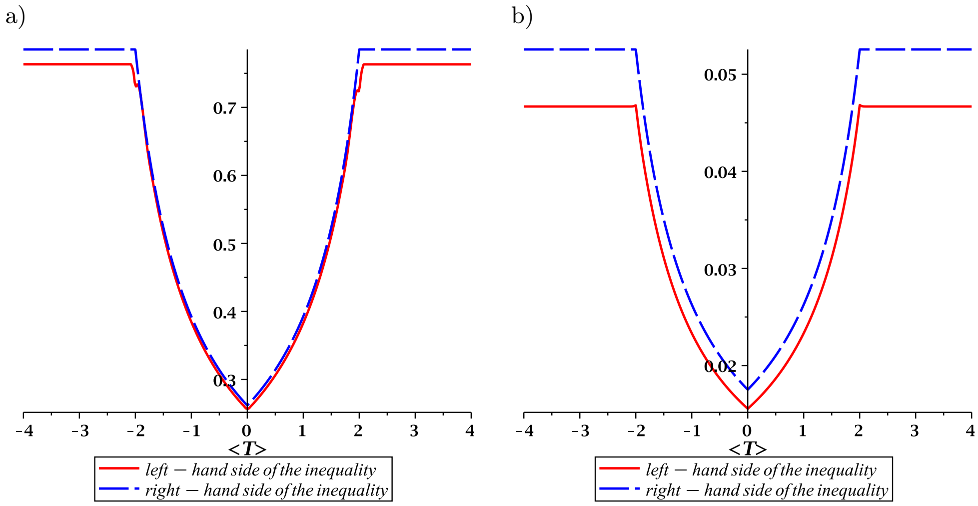

Fig. 10. Verification of the Hashin–Shtrikman bounds (5.7) – Fig 10a and (5.8) – Fig 10b. Solid (dash) lines correspond to the left (right)-hand sides of the inequalities.

(5.7) and (5.8) are presented in Fig. 10, where solid (dash) lines correspond to the left (right)-hand side of the inequalities.

Let us note that the differences between the left (right)-hand sides in the estimates are much smaller in comparison with the crude estimates (5.2). Besides, it turns out that the differences are smaller in the region where the material properties λ and λ_k depends on the temperature ($|\langle T\rangle| < 2$) than in the region where the properties take constant values.

6. Comparison of the results for periodic and random composites

According to [30], the effective conductivity λ_e of a random composite with temperature-dependent conductivities whose values are proportional to each other may be computed by the standard homogenization techniques in the following manner:

$$\Lambda_e(T) = \lambda(T) \cdot \Lambda_e, \tag{6.1}$$

where Λ_e is the effective conductivity tensor of the linear problem with the same constant ratios C_k between the conductivities of the matrix and the inclusions as for the nonlinear case (see (3.9)).

Note that, in general, the formulae (4.1) and (6.1) can give different results. However it turns out that, for the parameters chosen in Subsection 5.1, these formulae give similar results.

To estimate how close the formulae (4.1) and (6.1) are, we use the following two measures δ_l and δ_r:

$$
\delta_l = (\Lambda_e(T) - \lambda(T) \cdot \Lambda_e) \cdot (\Lambda_e(T))^{-1}, \tag{6.2}
$$
$$
\delta_r = (\Lambda_e(T))^{-1} \cdot (\Lambda_e(T) - \lambda(T) \cdot \Lambda_e),
$$

since the tensors given in (4.1) and (6.1) are not co-axial and do not necessarily commute.

For the composite under consideration, we found with the same accuracy as above (10^{-6}) the effective conductivity tensor of the linear problem:

$$
\Lambda_e = \begin{pmatrix} 1.524131 & 0.000027 \\ 0.000027 & 1.650632 \end{pmatrix}. \tag{6.3}
$$

The components of the tensors of error δ_l and δ_r from (6.2) are presented in Fig. 11. The curves are given using a logarithmic scale to clearly indicate the

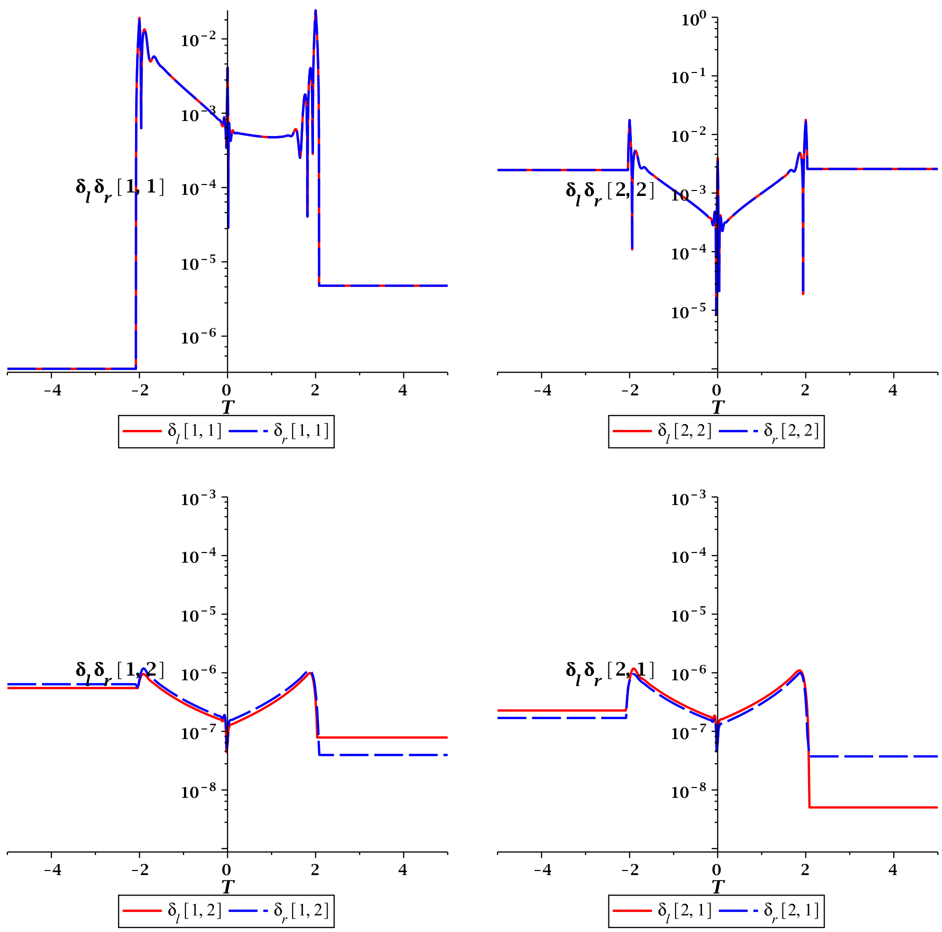

Fig. 11. Relative difference between components of the tensors δ_l and δ_r from (6.2) showing the perfect matching between the formulae for effective properties of the periodic and random composites.

order of component magnitudes. The dash line representees the components of δ_r while the solid line corresponds to the components of δ_l.

We note that, although the result obtained in [30] are related to different types of composite than that analyzed in this paper, our computations show perfect correlation between the models. The largest deviations (near 2%) take place near the points where the conductivities of the components as functions of temperature are not smooth. Moreover, this difference is only observed for the components of the main diagonal of the tensors (6.2). The other two components are almost identical (taking into account the computational accuracy). The latter allows us to conclude that the values of the non-diagonal elements (which are on the level of the predicted computational accuracy) have been computed with sufficient accuracy.

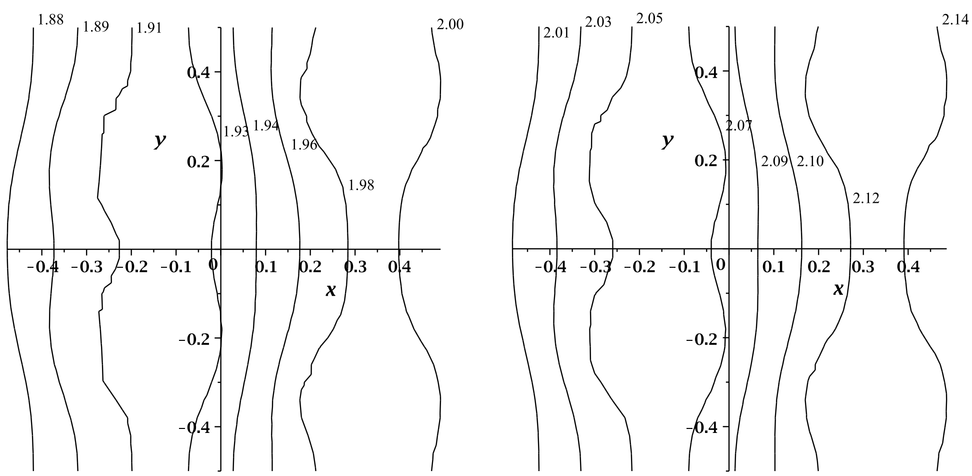

FIG. 12. Temperature distribution inside the cell $Q_{(0,0)}$ calculated during utilisation of the second method described in the Subsection 5.2. The results correspond to the constants $C = 27$ and $C = 28$, respectively.

Finally, we present in Fig. 12 the distribution of the temperature inside the representative cell $Q_{(0,0)}$ calculated during utilisation of the second method to determine the effective properties described in Subsection 5.2. One can observe that, for the value of constant $C = 27$, the jump of the temperature along the cell boundaries is not high, and the entire cell lies inside the region where the conductivity exhibits nonconstant behaviour. However, for $C = 28$, the range of the temperature inside the cell lies within the interval when both materials (matrix and inclusions) have constant magnitudes, and thus the average properties computed in this case coincide with the respective composite having constant properties of its constituencies.

7. Discussions and conclusions

As a result of our analysis the following conclusions can be formulated.

- An exact solution to the boundary value problem for the unbounded doubly periodic nonlinear composite, with a special assumption that the component conductivities are proportional, i.e., their ratios are independent of the temperature, can be effectively found by transforming the problem to the linear one.
- This allows, probably for the first time, the computation of components of the effective conductivity tensor in explicit form.
- We show that the average properties of the composite with nonlinear properties satisfy the Reuss-Voigt types and Hashin-Shtrikman estimations.
- We give one example for which we perform numerical calculations and obtain data for the effective conductivity, which allows us to compare formulae (4.1) for the effective conductivity tensor for a doubly periodic composite and (6.1) derived in [30] for random composites.

We should mention here that the problem of computation of the effective properties of a doubly periodic composite with conductivities depending on temperature in its general formulation remains open. It also refers to the accurate proof of the Hashin-Shtrikman estimations (the proof provided in [27] requires strong assumptions which cannot be verified directly from the problem formulation). We also have not discussed here how the flux intensity may impact on the results obtained in this paper. At first glance (as it follows from the linear formulation), the flux intensity should not influence the results. On the other hand, for a high flux intensity the temperature may change dramatically within one cell, and only a small part of the cell may have the temperature in the range where the conductivities exhibit nonlinear behaviour. All of these problems await accurate analysis.

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