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LOCALIZATION OF A HELMHOLTZ BOUNDARY VALUE PROBLEM IN A DOMAIN WITH PIECEWISE-SMOOTH BOUNDARY

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Abstract. In the recent papers by A. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet, Jr, X. Claeys, M. Dauge and some others spectral properties of BVPs for the "anisotropic" Helmholtz equation with sign-different constants in two neighbouring domains were investigated with vanishing boundary conditions. They obtained sufficient conditions for the solvability by using some refinement of Lax-Milgram Lemma for T-coercive operators. In the present paper we describe the reduction of the problem to model problems. The model problems will be investigated in forthcoming papers.

რეზიუმე. ბონეტ-ბენ დჰაის, ჩესნელის, სიარლეს, კლეისის, დოჟის და ზოგიერთი სხვა ავტორის ახლახან გამოსულ სტატიებში შესწავლილია ორ მოსაზღვრე არეში ნიშან-მონაცვლე კოეფიციენტებიანი "ანიზოტროპული" ჰელმჰოლცის განტოლებისათვის დასმული ნულოვანი სასაზღვრო პირობების მქონე სასაზღვრო ამოცანების სპექტრალური თვისებები. T-კოერციტიული ოპერატორებისათვის მისადაგებული ლაქს-მილგრამის ლემის გამოყენებით მათ მიიღეს ამოხსნადობის საკმარისი პირობები. წარმოდგენილ სტატიაში ჩვენ აღვწერთ ამ სახის ამოცანის დაყვანას მოდელურ ამოცანებამდე, რომლებიც ჩვენს მომდევნო სტატიებში იქნება შესწავლილი.

Introduction

In recent years there is a substantial interest to investigate the following problem: look for a vector-function $u(x) = (u_1(x), u_2(x), u_3(x))^\top$ in two neighbouring domains Ω_1 and Ω_2 which solves an "anisotropic" Helmholtz

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equations

$$\begin{cases} \operatorname{div} \mathcal{E}_1 \operatorname{grad} u + k_1^2 u = 0 \quad \text{in} \quad \Omega_1, \\ \operatorname{div} \mathcal{E}_2 \operatorname{grad} u + k_2^2 u = 0 \quad \text{in} \quad \Omega_2, \\ [\partial_{\boldsymbol{\nu}} u]^+ = h \quad \text{or} \quad u^+ = g \quad \text{on} \quad \Gamma := \partial(\overline{\Omega_1 \cup \Omega_2}), \\ u^-(t) = u^+(t), \quad [\partial_{\boldsymbol{\nu}} u]^-(t) = [\partial_{\boldsymbol{\nu}} u]^+(t) \quad \text{on} \quad \mathcal{L} := \partial\Omega_1 \cap \partial\Omega_2, \end{cases} \tag{0.1}$$

where $\mathcal{E}_1$ is a negative definite 3×3 matrix, while $\mathcal{E}_2$ is a positive definite 3×3 matrix; $\Gamma = \partial(\overline{\Omega_1 \cup \Omega_2})$ is the boundary of the unified domain $\Omega := \Omega_1 \cup \Omega_2$, while $\mathcal{L} := \partial\Omega_1 \cap \partial\Omega_2$ is the interface.

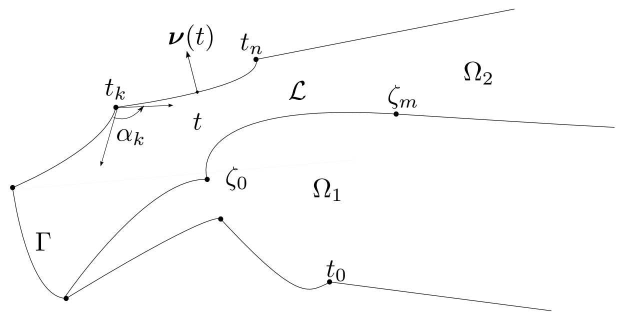

Figure 1

If $\mathcal{E}_j = \text{const} = c_j I$, where I is the three dimensional unit matrix, then

$$\operatorname{div} \mathcal{E}_j \operatorname{grad} u(x) + k_j^2 u(x) = c_j \Delta\, u(x) + k_j^2 u(x) \quad \text{in} \quad \Omega_j,$$

and we have usual "isotropic" Helmholtz equation. But even if $\pm\mathcal{E}_j \neq \text{const}$ are positive definite matrices, there exist square roots $\mathcal{E}_j^{1/2}$, $\left(\mathcal{E}_j^{1/2}\right)^2 = \pm\mathcal{E}_j$ and the function $v(x) := u\left(\mathcal{E}_j^{1/2} x\right)$ solves the system of Helmholtz equations

$$\Delta\, v(x) - (-1)^j k_j^2 v(x) = 0 \quad \text{in} \quad \Omega_j^0.$$

A strong interest to such BVPs is motivated by the rapid expansion of research into nanophotonics based on Surface Plasmon–Polaritons (SPP). These electromagnetic waves propagate along metal–dielectric interfaces dielectric material in Ω_1 and a metamaterial in Ω_2 and can be guided by metallic nanostructures beyond the diffraction limit. This remarkable capability has unique prospects for the design of highly integrated photonic signal-processing systems, nanoresolution optical imaging techniques and sensors.

In the recent papers by A. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet, Jr, X. Claeys, M. Dauge and some others (see [1, 2] and the references cited therein) spectral properties of BVPs type (0.1) were investigated in cases, when $\mathcal{E}_1$, $\mathcal{E}_2$ are scalar, but variable functions and boundary conditions-zero. The conditions obtained are sufficient and for the proofs was used some refinement of Lax-Milgram Lemma for T-coercive operators.

The purpose of the research is to find a criterion of unique solvability of BVPs (0.1). For this we will apply boundary integral equation method and need some preliminaries.

We will implement this program step by step. First we will study unique solvability of the BVP (0.1) in the Sobolev space $\mathbb{H}^1(\Omega_1 \cup \Omega_2)$. Then we will apply localization and reduce the investigation of BVP (0.1) to six model problems.

Let, as usual, $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of all rapidly vanishing functions and $\mathcal{S}'(\mathbb{R}^n)$ the dual space of tempered distributions on $\mathbb{R}^n$. The Bessel potential space $\mathbb{H}^s_p(\mathbb{R}^n)$, with $s \in \mathbb{R}$, is formed by the elements $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ such that the norm

$$\left\|\varphi|\mathbb{H}^s_p(\mathbb{R}^n)\right\| = \left\|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}\varphi|\mathbb{L}_p(\mathbb{R}^n)\right\| \tag{0.2}$$

is finite [9]. As the notation indicates, (0.2) is a norm for the space $\mathbb{H}^s_p(\mathbb{R}^n)$ which makes it a Banach space. Here, $\mathcal{F} = \mathcal{F}_{x\mapsto\xi}$ denotes the Fourier transformation in $\mathbb{R}^n$.

For a given domain, $\mathcal{D}$, on $\mathbb{R}^n$ we denote by $\widetilde{\mathbb{H}}^s_p(\mathcal{D})$ the closed subspace of $\mathbb{H}^s_p(\mathbb{R}^n)$ whose elements have supports in $\overline{\mathcal{D}}$, and $\mathbb{H}^s_p(\mathcal{D})$ denotes the space of distributions on $\mathcal{D}$ which have extensions into $\mathbb{R}^n$ belonging to $\mathbb{H}^s_p(\mathbb{R}^n)$. The space $\widetilde{\mathbb{H}}^s_p(\mathcal{D})$ is endowed with the subspace topology, and on $\mathbb{H}^s_p(\mathcal{D})$ we introduce the norm of the quotient space $\mathbb{H}^s_p(\mathbb{R})/\widetilde{\mathbb{H}}^s_p(\mathbb{R}^n\setminus\overline{\mathcal{D}})$. Obviously, these definitions are valid for $\mathbb{L}_p$ spaces. Note that the spaces $\mathbb{H}^0_p(\mathbb{R}^n_+)$ and $\widetilde{\mathbb{H}}^0_p(\mathbb{R}^n_+)$, where $\mathbb{R}^n_+ := \mathbb{R}^{n-1}\times\mathbb{R}^+$ can be identified, and we denote them by $\mathbb{L}_p(\mathbb{R}^n_+)$.

1. Localization and the Model Boundary Value Problems

We study the existence and uniqueness of a function $u \in H^1(\Omega)$ which solves the BVP for "anisotropic" Helmholtz equations

$$\begin{cases} \operatorname{div} \mathcal{E}_1 \operatorname{grad} u + k_1^2 u = 0 \quad \text{in} \quad \Omega_1, \\ \operatorname{div} \mathcal{E}_2 \operatorname{grad} u + k_2^2 u = 0 \quad \text{in} \quad \Omega_2, \\ a_j [\partial_{\boldsymbol{\nu}} u]^+ - b_j u^+ = h_j \quad \text{on} \quad \Gamma_j, \quad j = 0, \ldots, n, \\ c_{j,k}^1 [\partial_{\boldsymbol{\nu}} u]^{(1)} - d_{j,k}^1 u^{(1)} = c_{j,k}^2 [\partial_{\boldsymbol{\nu}} u]^{(2)} - d_{j,k}^2 u^{(2)} \\ \qquad \text{on} \quad \mathcal{L}_j, \quad k = 1, 2, \quad j = 1, \ldots, m, \end{cases} \tag{1.1}$$

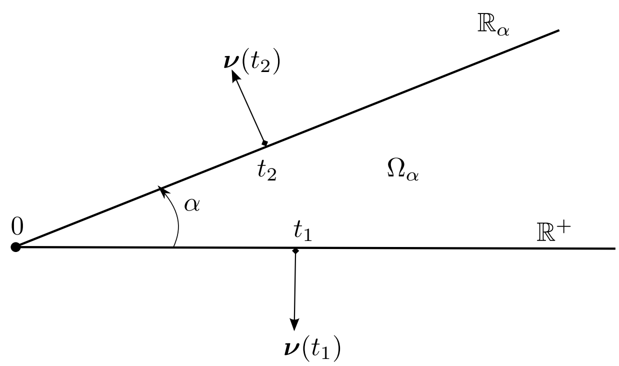

Figure 2

where $\partial_{\boldsymbol{\nu}}$ is the normal derivative $\partial_{\boldsymbol{\nu}} u := \boldsymbol{\nu}_1 \partial_1 u + \boldsymbol{\nu}_2 \partial_2 u$. $\mathcal{E}_1$ is a negative definite 3×3 matrix, while $\mathcal{E}_2$ is a positive definite 3×3 matrix; $\Gamma = \partial(\overline{\Omega_1 \cup \Omega_2}) = \bigcup_{j=1}^{n} \Gamma_j$ is the outer boundary of the unified domain $\Omega := \Omega_1 \cup \Omega_2$, while $\mathcal{L} := \partial\Omega_1 \cap \partial\Omega_2 = \bigcup_{j=1}^{m} \mathcal{L}_j$ is the interface.

To BVP (1.1) we apply a quasi-localization. Details of the localization technique is described in the literature (see, for example, the monographs by I. Gohberg & N. Krupnik [7], B. Silberman & V. Didenko [5], T. Buchukuri, O. Chkadua, R. Duduchava, D. Natroshvili [3], the paper by I. Simonenko [8] for different local principles. For applications of the local principle to BVPs see the monograph T. Buchukuri, O. Chkadua, R. Duduchava, D. Natroshvili [3], the papers by R. Duduchava [6], R. Duduchava & F. Speck [4, 6]. Here the localization program is implemented in several steps:

Step I. We identify a quasi-local representative of BVP (1.1) at all points $t \in \overline{\Omega} \cup \{\infty\}$, including infinity and angular points; these local representatives are obtained by freezing coefficients and rectifying the curves (see Section 1);

Step II. The main theorem on quasi-localization ensures that if local representatives (model BVPs) are Fredholm for all $t \in \overline{\Omega} \cup \{\infty\}$, the original BVP (1.1) is Fredholm as well.

Six localized (model) problems (see below) will be investigated in forthcoming papers:

Step III. Under certain conditions will be proved that the homogeneous model BVP has a trivial solution only;

Step IV. We will apply the representation formulae of a solution to the model BVP with the help of single and double layer potentials, in which one density function is known from boundary data, while another is unknown. The model BVP is then reduced to a certain equivalent boundary singular integral equation (BSIE);

Step V. It can be proved that the equivalent BSIE is Fredholm for all model BSIEs;

Step VI. We will prove that the index of equivalent BSIE is 0. Since the equivalent model BVP has a unique solution for all proper data (see Step III), we conclude unique solvability of the model BVP;

Step VII. The initial BVP (1.1) is Fredholm under proper constraints on coefficients and data (see Step II–Step IV). We will prove that the original BVP has a unique solution under certain conditions and equivalent BSIE has index 0. Then BVP (1.1) has a unique solution, which is given by a representation formula with the densities already known from the data or representing a solution to BSIE.

Next we describe those model problems which are obtained by quasilocalization from the initial BVP (1.1) at different points. For details of such localization we refer to [3].

I model problem. A local representative of the BVP (1.1) at an inner point $t \in \Omega_1 \cup \Omega_2$ is a model problem in the entire $\mathbb{R}^2$:

$$\operatorname{div} \mathcal{E}_j \operatorname{grad} u + k_j^2 u = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{1.2}$$

where $j = 1, 2$ is fixed. The fundamental solution is the inverse to the model differential equation and the invertibility is granted. In this case we do not need even ellipticity of the operator.

II model problem. A local representative of the BVP (1.1) at a boundary point $t \in \Gamma$ different from vertexes $t \neq t_1, \dots, t_n$, is a model problem in a half plane $\mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}^+$

$$\begin{cases} \operatorname{div} \mathcal{E}_j \operatorname{grad} u + k_j^2 u = 0 \quad \text{in} \quad \mathbb{R}^2_+, \\ a[\partial_{\boldsymbol{\nu}} u]^+ - b u^+ = h \quad \text{on} \quad \mathbb{R} := \partial \mathbb{R}^2_+, \end{cases} \tag{1.3}$$

where $j = 1, 2$ is fixed, a and b are known constants and $h \in \mathbb{H}^{1/2}(\mathbb{R})$ is a known function. Only the ellipticity of the symbol ensures the unique solvability of BVP (1.3) and we drop the details again.

III model problem. A local representative of the BVP (1.1) at a boundary vertex $t = t_k$, different from the one where the boundary curve Γ meets the interface curve $\mathcal{L}$ is the model problem in an angular domain Ω_α (see Figure 2).

$$\begin{cases} \Delta\, u + k^2 u = 0 \quad \text{in} \quad \Omega_\alpha, \\ a_\ell[\partial_{\boldsymbol{\nu}} u]^+ - b_\ell u^+ = h \quad \text{on} \quad R_\ell, \quad \ell = 1, 2. \end{cases} \tag{1.4}$$

Here Ω_α is the angle of magnitude α between the half axes $R_1 := \mathbb{R}^+$ and the beam $R_2 := \mathbb{R}_\alpha$ turned by the angle $\alpha = \alpha_k$ from $\mathbb{R}^+$

IV model problem. A local representative of the BVP (1.1) at the boundary vertex where the boundary Γ and the interface curves $\mathcal{L}$ meet, is the model problem in a double angular domain (see Figure 3)

$$\begin{cases} \operatorname{div} \mathcal{E}_1 \operatorname{grad} u + k_1^2 u = 0 \quad \text{in} \quad \Omega_\beta, \\ \operatorname{div} \mathcal{E}_2 \operatorname{grad} u + k_2^2 u = 0 \quad \text{in} \quad \Omega_\alpha, \\ a_1[\partial_{\boldsymbol{\nu}} u]^+ - b_1 u^+ = h \quad \text{on} \quad \mathbb{R}^+, \\ a_2[\partial_{\boldsymbol{\nu}} u]^+ - b_2 u^+ = h \quad \text{on} \quad \mathbb{R}_\beta, \\ c_\ell^1[\partial_{\boldsymbol{\nu}} u]^+ - d_\ell^1 u^+ = c_\ell^2[\partial_{\boldsymbol{\nu}} u]^- - d_\ell^2 u^- \quad \text{on} \quad \mathbb{R}_\alpha, \quad \ell = 1, 2. \end{cases} \tag{1.5}$$

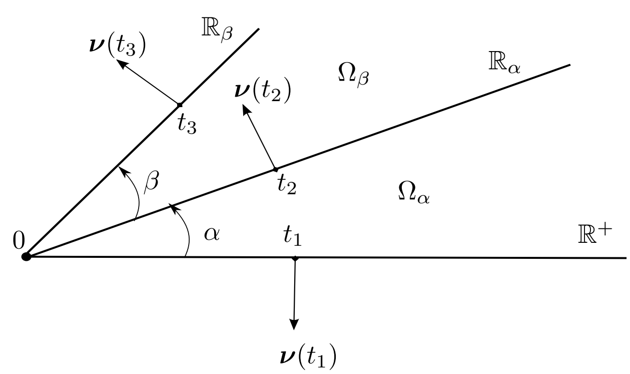

Figure 3

V model problem. A local representative of the BVP (1.1) at an interface non-vertex point $\mathcal{L} \ni t \neq \zeta_1, \ldots, \zeta_m$ is the model problem in the entire plane, divided by the real axes $\mathbb{R}$

$$\begin{cases} \operatorname{div}\mathcal{E}_1 \operatorname{grad} u + k_1^2 u = 0 \quad \text{in} \quad \mathbb{R}^2_+ := \mathbb{R}\times\mathbb{R}^+, \\ \operatorname{div}\mathcal{E}_2 \operatorname{grad} u + k_2^2 u = 0 \quad \text{in} \quad \mathbb{R}^2_- := \mathbb{R}\times\mathbb{R}^-, \\ c^1_\ell[\partial_2 u]^+ - d^1_\ell u^+ = c^2_\ell[\partial_2 u]^- - d^2_\ell u^- \quad \text{on} \quad \mathbb{R} \quad \ell = 1,2. \end{cases} \tag{1.6}$$

because $\partial_{\boldsymbol{\nu}} = \partial_2$. This problem is easily solvable and we drop the details again.

VI model problem. A local representative of the BVP (1.1) at an interface vertex $t = \zeta_k$, $k = 1, \ldots, m$, is the model interface problem in the union of the angular domain Ω_α and it's complementary domain $\Omega_{2\pi-\alpha}$

$$\begin{cases} \operatorname{div}\mathcal{E}_1 \operatorname{grad} u + k_1^2 u = 0 \quad \text{in} \quad \Omega_\alpha, \\ \operatorname{div}\mathcal{E}_2 \operatorname{grad} u + k_2^2 u = 0 \quad \text{in} \quad \Omega_{2\pi-\alpha}, \\ c^1_\ell[\partial_{\boldsymbol{\nu}} u]^+ - d^1_\ell u^+ = c^2_\ell[\partial_{\boldsymbol{\nu}} u]^- - d^2_\ell u^- \quad \text{on} \quad \partial\Omega_\alpha \quad \ell = 1,2. \end{cases} \tag{1.7}$$

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References

1. A.-S. Bonnet-Ben Dhia, L. Chesnel and P. Ciarlet, Jr., T-coercivity for scalar interface problems between dielectrics and metamaterials. *ESAIM Math. Model. Numer. Anal.* **46** (2012) 1363–1387.
2. A.-S. Bonnet-Ben Dhia, L. Chesnel and X. Claeys, Radiation condition for a non-smooth interface between a dielectric and a metamaterial. *Published online*: http://hal.inria.fr/hal-00651008/
3. T. Buchukuri, O. Chkadua, R. Duduchava and D. Natroshvili, Interface crack problems for metallic-piezoelectric. *Memoirs Differential Equations Math. Phys.* **55**, 2012, 1–150. Free access: http://www.rmi.ge/jeomj/memoirs/vol55/contents.htm
4. L. P. Castro, R. Duduchava and F.-O. Speck, Localization and minimal normalization of some basic mixed boundary value problems. *Factorization, singular operators and related problems, Funchal*, (2002), 73–100.
5. V. D. Didenko and B. Silbermann, Approximation of additive convolution-like operators. Real C^*-algebra approach. *Frontiers in Mathematics. Birkhäuser Verlag, Basel*, 2008.
6. R. Duduchava, On multidimensional singular integral operators I-II. *J. Operator Theory* **11** (1984), No. 2, 199–214.
7. I. Gohberg and N. Krupnik, One-Dimensional Linear Singular Integral Equations, I-II. *Oper. Theory Adv. Appl.* 53–54, Birkhäuser, Basel, 1979.
8. I. Simonenko, A new general method of investigating linear operator equations of singular integral equation type. I. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **29** (1965), 567–586.
9. H. Triebel, Interpolation theory, function spaces, differential operators, 2-nd edition. *Johann Ambrosius Barth, Heidelberg*, 1995.

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