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MEASURING THE NONCOMMUTATIVITY OF *DG*-ALGEBRAS

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Many constructions that successfully work for commutative DG-algebras fail in the noncommutative case. There exists a classical tool, namely the Steenrod $\cup_1$-product, which measures the noncommutativity of a DG-algebra $(A, d, \cdot)$. It satisfies the condition

$$d(a \cup_1 b) = da \cup_1 b + a \cup_1 db + a \cdot b - b \cdot a \tag{1}$$

(the signs are ignored in the entire text). The existence of $\cup_1$ guarantees the commutativity of $H(A)$. But this structure is too pure for most applications. A $\cup_1$-product satisfying only condition (1) cannot compensate the commutativity. In many constructions, we need some deeper properties of $\cup_1$, for example, the compatibility with the product of A

$$a \cup_1 (b \cdot c) = b \cdot (a \cup_1 c) + (a \cup_1 b) \cdot c \tag{2}$$

(the Hirsch formula) is needed.

In this article, we use a multiplication in the bar construction

$$\mu : BA \otimes BA \to BA,$$

which turns the DG-coalgebra BA into a DG-Hopf algebra, as a tool which compensates the commutativity of A. In fact, such a multiplication is uniquely determined by a collection of operations

$$\{E_{pq} : (\bigotimes^p A) \otimes (\bigotimes^q A) \to A,\ p, q = 0, 1, 2, 3, \dots\}$$

subject to certain compatibility conditions. In particular, the binary component $E_{11} : A \otimes A \to A$ satisfies condition (1); therefore, it can be considered as a $\cup_1$-product measuring the noncommutativity of A. For convenience, we call such an object $(A, \cdot, d, \{E_{pq}\})$ the *Hirsch algebra* since the defining properties of operations E_{pq} in fact generalize the classical Hirsch formula (2). Actually, this structure is a particular case of the notion of B_∞-algebra [2, 7], which is defined as a structure on A granting that BA becomes a DG-Hopf algebra. In fact, this structure consists of a new differential $\widetilde{d} : BA \to BA$ and new multiplication $\widetilde{\mu} : BA \otimes BA \to BA$. A Hirsch algebra is the case where the standard differential of bar construction remains unchanged.

The most important particular case of Hirsch algebra is the structure known as the *homotopy G-algebra* [6, 23]. In this case, all E_{pq}'s except for E_{01} and E_{1k}, $k = 0, 1, 2, 3, \dots$, are zero. Thus, it is a DG-algebra with $\cup_1$-product and some sequence of cochain operations

$$\left\{E_{1k} : A \otimes \left(\bigotimes^k A\right) \to A,\ k = 1, 2, 3, \dots,\ E_{11} = \cup_1\right\}$$

satisfying certain compatibility conditions. Some constructions and results that are valid for commutative DG-algebras are also valid for homotopy G-algebras. This structure arises in some important cases, namely, there exist *explicit* formulas for operations $E_{1,k}$ in the following cases:

(i) in the cochain complex of a topological space $C^*(X)$;

(ii) in the Hochschild cochain complex $C^*(A, A)$ of the algebra A;

(iii) in the cobar construction $\Omega\mathcal{H}$ of a DG-Hopf algebra $\mathcal{H}$, in particular, in the cobar construction of the bar construction ΩBA of the algebra A.

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We note that in all of these three cases the starting operation $E_{11} = \cup_1$ is classical: the Steenrod $\cup_1$-product in $C^*(X)$, the Gerstenhaber circle product in $C^*(A,A)$ [5], and the Adams $\cup_1$-product in $\Omega\mathcal{H}$ [1]. The suitable tails, i.e., the higher operations E_{1k}, were actually constructed in [2] for $C^*(X)$ and in [6, 7, 13, 23] for $C^*(A,A)$.

Below, we give some applications of these structures.

Section 1 starts with the study of the structure of a product in the bar construction, which motivates the notion of a Hirsch algebra. Then we give a comparison of this structure with the $B(\infty)$-algebra structure, DG-Lie algebra structure, homotopy G-algebra structure, and strong homotopy commutative (shc) algebra structure. In concluding Sec. 1, we describe two versions of the notion of twisting element (the first version controls deformations of algebras and the second version is related to the degeneracy of A_∞-algebra structures) and the suitable notion of their (gauge) transformation in a homotopy G-algebra.

In Sec. 2, the above-mentioned three examples of homotopy G-algebra are given.

In Sec. 3, we present some applications: multiplicative twisted tensor products, deformation of algebras, and degeneracy of $A(\infty)$-algebras.

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1. Hirsch Algebras

1.1. Products in the bar construction. Let $(A, d, \cdot)$ be a DG-algebra with the differential $d : A^* \to A^{*+1}$ (cochain algebra) and let

$$BA = T^c(s^{-1}A) = \Lambda + s^{-1}A + s^{-1}A \otimes s^{-1}A + s^{-1}A \otimes s^{-1}A \otimes s^{-1}A + \dots$$

be its bar construction (here $s^{-1}A$ is the desuspension of A, i.e., $(s^{-1}A)^n = A^{n+1}$, and T^c is the tensor coalgebra functor). By definition, BA is a DG-coalgebra with the differential

$$d(a_1 \otimes \dots \otimes a_n) = \sum_k \pm a_1 \otimes \dots \otimes da_k \otimes \dots \otimes a_n + \sum_k \pm a_1 \otimes \dots \otimes a_k \cdot a_{k+1} \otimes \dots \otimes a_n,$$

the coproduct $\nabla : BA \to BA \otimes BA$ defined by the formula

$$\nabla(a_1 \otimes \dots \otimes a_n) = \sum_{k=0}^{n} (a_1 \otimes \dots \otimes a_k) \otimes (a_{k+1} \otimes \dots \otimes a_n),$$

and the counit $1_\Lambda \in \Lambda \subset BA$.

We are interested in the structure of multiplications

$$\mu : BA \otimes BA \to BA$$

that turn BA into a DG-Hopf algebra, i.e., we require that μ must be an associative DG-coalgebra map, which has the unit element $1_\Lambda \in \Lambda \subset BA$.

Since the tensor coalgebra $BA = T^c(s^{-1}A)$ is cofree, each map of graded coalgebras

$$\mu : BA \otimes BA \to BA$$

is uniquely determined by the projection

$$E = \mathrm{pr} \cdot \mu : BA \otimes BA \to BA \to A.$$

Moreover, each homomorphism $E : BA \otimes BA \to A$ of degree $+1$ determines a graded coalgebra map $\mu_E : BA \otimes BA \to BA$ by the rule

$$\mu_E = \sum_{k=0}^{\infty} (E \otimes \dots \otimes E) \nabla^k_{BA \otimes BA},$$

where $\nabla^k_{BA\otimes BA} : BA \otimes BA \to \bigotimes^k (BA \otimes BA)$ is the k-fold iteration of the standard coproduct

$$\nabla_{BA\otimes BA} = (\mathrm{id} \otimes T \otimes \mathrm{id})(\nabla \otimes \nabla) : BA \otimes BA \to \bigotimes^2 (BA \otimes BA),$$

$T : BA \otimes BA \to BA \otimes BA$ is interchange map, and ∇^k is the k-fold iteration of a coproduct ∇:

$$\nabla^0 = \epsilon, \quad \nabla^1 = \mathrm{id}, \quad \nabla^2 = \nabla, \quad \nabla^k = (\nabla^{k-1} \otimes \mathrm{id})\nabla.$$

The map μ_E is a *chain map* (i.e., it is a map of DG-coalgebras) if and only if E is a twisting cochain in the sense of E. Brown, i.e., if it satisfies the condition

$$dE + Ed_{BA\otimes BA} = E \cup E,$$

where the $\cup$-product in $\mathrm{Hom}(BA \otimes BA, A)$ is given by

$$f \cup g = \mu(f \otimes g)\nabla;$$

again, since the tensor coalgebra $BA = T^c(s^{-1}A)$ is cofree, the condition $d_{BA}\mu_E = \mu_E d_{BA\otimes BA}$ holds if and only if it holds after the projection on A, i.e., if $\mathrm{pr} \cdot d_{BA}\mu_E = \mathrm{pr} \cdot \mu_E d_{BA\otimes BA}$, but this condition is nothing other than the Brown condition.

The same argument shows that the product μ_E is associative if and only if the condition

$$\mathrm{pr} \cdot \mu_E(\mu_E \otimes \mathrm{id}) = \mathrm{pr} \cdot \mu_E(\mathrm{id} \otimes \mu_E)$$

holds or, taking the relation $E = \mathrm{pr} \cdot \mu_E$ into account, we obtain

$$E(\mu_E \otimes \mathrm{id}) = E(\mathrm{id} \otimes \mu_E).$$

Thus, we can summarize that any multiplication $\mu : BA \otimes BA \to BA$, which specifies on BA the structure of a DG-Hopf algebra, is induced by a homomorphism

$$E : BA \otimes BA \to BA$$

of degree $+1$, which satisfies the followng conditions:

$$d_A E + E(d_{BA} \otimes \mathrm{id} + \mathrm{id} \otimes d_{BA}) = E \cup E \tag{3}$$

(i.e., E is a twisting cochain) and

$$E(\mu_E \otimes \mathrm{id}) = E(\mathrm{id} \otimes \mu_E) \tag{4}$$

(this implies the associativity of μ).

Each twisting cochain $E : BA \otimes BA \to BA$ has components

$$\begin{array}{ccccccc} & & E_{01} & & E_{10} & & \\ & E_{02} & & E_{11} & & E_{20} & \\ E_{03} & & E_{12} & & E_{21} & & E_{30} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array},$$

where E_{pq} is the restriction of E to $\left(\bigotimes^p s^{-1}A\right) \otimes \left(\bigotimes^q s^{-1}A\right)$. Thus, a twisting cochain can be considered as a collection of *multioperations*

$$E_{pq} : \left(\bigotimes^p s^{-1}A\right) \otimes \left(\bigotimes^q s^{-1}A\right) \to A.$$

The value of E_{pq} on the element $(s^{-1}a_1 \otimes \cdots \otimes s^{-1}a_p) \otimes (s^{-1}b_1 \otimes \cdots \otimes s^{-1}b_q) \in (\otimes^p s^{-1}A) \otimes (\otimes^q s^{-1}A)$ is denoted by $E_{pq}(a_1 \otimes \cdots \otimes a_p | b_1 \otimes \cdots \otimes b_q)$.

The above requirements on μ_E imply some restrictions on the collection $\{E_{pq}\}$.

First, it is easy to verify that the element $1_\Lambda \in \Lambda \subset BA$ is the unit for the multiplication μ_E if and only if

$$E_{01} = E_{10} = \mathrm{id}, \quad E_{0k} = E_{k0} = 0, \quad k > 1. \tag{5}$$

Thus, each multiplication on BA with the unit 1_Λ is induced by the collection of operations

$$\begin{array}{ccccccccccc} & & & E_{01}=\mathrm{id} & & & & E_{10}=\mathrm{id} & & & \\ & & & & & E_{11} & & & & & \\ & & & & E_{12} & & E_{21} & & & & \\ & & E_{13} & & & E_{22} & & & E_{31} & & \\ \dots & \dots & & \dots & & \dots & & \dots & & \dots & \dots, \end{array} \tag{6}$$

We investigate how condition (3) influences low-dimensional E_{pq}'s.

The restriction on $\left(\bigotimes^1 A\right) \otimes \left(\bigotimes^1 A\right)$ gives

$$d_A E_{11}(a|b) + E_{11}(d_A a|b + a|d_A b) = a \cdot b - b \cdot a,$$

i.e., the operation E_{11} is a $\cup_1$-product, which measures the noncommutativity of A.

The restriction to $\left(\bigotimes^1 A\right) \otimes \left(\bigotimes^2 A\right)$ gives

$$\begin{aligned} d_A E_{12}(a|b \otimes c) + E_{12}(d_A a|b \otimes c + a|d_A b \otimes c + a|b \otimes d_A c) \\ = E_{11}(a|bc) + bE_{11}(a|c) + E_{11}(a|b)c \end{aligned} \tag{7}$$

or, using the notation $E_{11}(a|b) = a \cup_1 b$,

$$\begin{aligned} d_A E_{12}(a|b \otimes c) + E_{12}(d_A a|b \otimes c + a|d_A b \otimes c + a|b \otimes d_A c) \\ = a \cup_1 (bc) + b(a \cup_1 c) + (a \cup_1 b)c; \end{aligned}$$

this means that $\cup_1$ satisfies the so-called *right Hirsch formula* up to homotopy and the appropriate homotopy is the operation E_{12}.

The restriction of (3) to $\left(\bigotimes^2 A\right) \otimes \left(\bigotimes^1 A\right)$ gives

$$\begin{aligned} d_A E_{21}(a \otimes b|c) + E_{21}(d_A a \otimes b|c + a \otimes d_A bc + a \otimes b|d_A c) \\ = E_{11}(ab|c) + aE_{11}(b|c) + E_{11}(a|c)b \end{aligned}$$

or, using the notation $E_{11}(a|b) = a \cup_1 b$,

$$\begin{aligned} d_A E_{21}(a \otimes b|c) + E_{21}(d_A a \otimes b|c + a \otimes d_A b|c + a \otimes b|d_A c) \\ (ab) \cup_1 c + a(b \cup_1 c) + (a \cup_1 c)b; \end{aligned} \tag{8}$$

this means that $\cup_1$ satisfies the so-called *left Hirsch formula* up to homotopy and the appropriate homotopy is the operation E_{21}.

Generally, the restriction of (3) to $(\bigotimes^m A) \otimes (\bigotimes^n A)$ gives

$$\begin{aligned} d_A E_{m,n}(a_1 \otimes \dots \otimes a_m|b_1 \otimes \dots \otimes b_n) + \sum_i E_{m,n}(a_1 \otimes \dots \otimes d_A a_i \otimes \dots \otimes a_m|b_1 \otimes \dots \otimes b_n) \\ + \sum_i E_{m,n}(a_1 \otimes \dots \otimes a_m|b_1 \otimes \dots \otimes d_A b_i \otimes \dots \otimes b_n) \\ = a_1 E_{m-1,n}(a_2 \otimes \dots \otimes a_m|b_1 \otimes \dots \otimes b_n) + E_{m-1,n}(a_1 \otimes \dots \otimes a_{m-1}|b_1 \otimes \dots \otimes b_n)a_m \\ + b_1 E_{m,n-1}(a_1 \otimes \dots \otimes a_m|b_2 \otimes \dots \otimes b_n) + E_{m,n-1}(a_1 \otimes \dots \otimes a_m|b_1 \otimes \dots \otimes b_{n-1})b_m \\ + \sum_i E_{m-1,n}(a_1 \otimes \dots \otimes a_i \cdot a_{i+1} \otimes \dots \otimes a_m|b_1 \otimes \dots \otimes b_n) \\ + \sum_i E_{m,n-1}(a_1 \otimes \dots \otimes a_m|b_1 \otimes \dots \otimes b_i \cdot b_{i+1} \otimes \dots \otimes b_n) \\ + \sum_{\substack{p=1,\dots,m-1,\\ q=1,\dots,n-1}} E_{p,q}(a_1 \otimes \dots \otimes a_p|b_1 \otimes \dots \otimes b_q) \cdot E_{m-p,n-q}(a_{p+1} \otimes \dots \otimes a_m|b_{q+1} \otimes \dots \otimes b_n). \end{aligned} \tag{9}$$

Now we investigate how the associativity condition (4) influences the components E_{pq}.

The restriction to $\left(\bigotimes^1 A\right) \otimes \left(\bigotimes^1 A\right) \otimes \left(\bigotimes^1 A\right)$ gives

$$\begin{aligned} E_{11}(E_{11}(a_1|a_2)|a_3) - E_{11}(a_1|E_{11}(a_2|a_3)) = E_{12}(a_1|a_2 \otimes a_3) + E_{12}(a_1|a_3 \otimes a_2) \\ -E_{21}(a_1 \otimes a_2|a_3) + E_{21}(a_2 \otimes a_1|a_1) \end{aligned}$$

or

$$\begin{aligned} (a_1 \cup_1 a_2) \cup_1 a_3 - a_1 \cup_1 (a_2 \cup_1 a_3) = E_{12}(a_1|a_2 \otimes a_3) + E_{12}(a_1|a_3 \otimes a_2) \\ -E_{21}(a_1 \otimes a_2|a_3) + E_{21}(a_2 \otimes a_1|a_1); \end{aligned} \tag{10}$$

note that this condition plays an important role in the definition in the desuspension of a Hirsch algebra of DG-Lie algebra structure (see below).

Remark. Thus, the operations E_{12} and E_{21}, which initially are tools for measuring the deviations from the Hirsch formulas (see (7) and (8)), simultaneously measure the deviation from the associativity of the $\cup_1$-product.

Generally, the restriction of (3) to $\left(\bigotimes^k A\right) \otimes \left(\bigotimes^l A\right) \otimes (\bigotimes^m A)$ gives

$$\begin{aligned} &\sum_{r=1}^{l+m} \sum_{\substack{l_1+\cdots+l_r=l \\ m_1+\cdots+m_r=m}} E_{kr}(a_1 \otimes \cdots \otimes a_k|E_{l_1m_1}(b_1 \otimes \cdots \otimes b_{l_1}|c_1 \otimes \cdots \otimes c_{m_1}) \otimes \dots \\ &\qquad \otimes E_{l_rm_r}(b_{l_1+\cdots+l_{r-1}+1} \otimes \cdots \otimes b_l|c_{m_1+\cdots+m_{r-1}+1} \otimes \cdots \otimes c_m) \\ &\qquad = \sum_{s=1}^{k+l} \sum_{\substack{k_1+\cdots+k_s=k \\ l_1+\cdots+l_s=l}} E_{sm}(E_{k_1l_1}(a_1 \otimes \cdots \otimes a_{k_1}|b_1 \otimes \cdots \otimes b_{l_1}) \otimes \dots \\ &\otimes E_{k_sl_s}(a_{k_1+\cdots+k_{s-1}+1} \otimes \cdots \otimes a_k|b_{l_1+\cdots+l_{s-1}+1} \otimes \cdots \otimes b_l)|c_1 \otimes \cdots \otimes c_m). \end{aligned} \tag{11}$$

We summarize the obtained results in the following theorem.

Theorem 1. *A multiplication $\mu : BA \otimes BA \to BA$, which turns the bar construction BA into a DG-Hopf algebra, specifies on A the set of multioperations* (6) $E_{m,n} : (\bigotimes^m A) \otimes (\bigotimes^n A) \to A$ *satisfying conditions* (5), (9), *and* (11).

In particular, the operation E_{11} is a $\cup_1$-product, which measures the noncommutativity of A and satisfies both (left and right) Hirsch formulas up to homotopy.

A DG-algebra endowed with such a structure is called a *Hirsch algebra*. This name is inspired by the fact that the defining conditions (9) and (11) can be considered as generalizations of the classical Hirsch formula

$$(a \cdot b) \cup_1 c = a \cdot (b \cup_1 c) + (a \cup_1 c) \cdot b.$$

This structure is a particular case of a B_∞-algebra (see below).

1.2. Levels of noncommutativity. We distinguish between various levels of "noncommutativity" of A according to the form of the appropriate twisting cochain E.

Level 1. If the twisting cochain E has the form

$$\begin{array}{ccccccccccc} & & & E_{01} = \mathrm{id} & & E_{10} = \mathrm{id} & & & \\ & & & & 0 & & & & \\ & & & 0 & & 0 & & & , \\ & & 0 & & 0 & & 0 & & \\ \dots & \dots & & \dots & \dots & \dots & & \dots & \dots \end{array}$$

i.e., E has just two nonzero components $E_{01} = \mathrm{id}$ and $E_{10} = \mathrm{id}$, then (1) implies that A is a *strictly* commutative DG-algebra.

Level 2. Suppose that E has the form

$$\begin{array}{ccccccccccc} & & & E_{01}=\mathrm{id} & & & & E_{10}=\mathrm{id} & & & \\ & & & & & E_{11} & & & & & \\ & & & 0 & & & & 0 & & & , \\ & & 0 & & & 0 & & & 0 & & \\ \cdots & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & \cdots \end{array}$$

i.e., E has just three nonzero components $E_{01} = \mathrm{id}$, $E_{10} = \mathrm{id}$, and E_{11}. In this case, A is endowed with the "strict" $\cup_1$-product $a \cup_1 b = E_{1,1}(a \otimes b)$, condition (9) yields

$$\begin{gathered} d_A(a \cup_1 b) = d_A a \cup_1 b + a \cup_1 d_A b + ab - ba, \\ a \cup_1 (bc) = b(a \cup_1 c) + (a \cup_1 b)c, \\ (ab) \cup_1 c = a(b \cup_1 c) + (a \cup_1 c)b, \\ (a \cup_1 c) \cdot (b \cup_1 d) = 0, \end{gathered}$$

and condition (11) means the associativity $\cup_1$:

$$a \cup_1 (b \cup_1 c) = (a \cup_1 b) \cup_1 c.$$

As we have seen, we have very strong restrictions on the $\cup_1$-product. The trivial example of a DG-algebra with such a strict $\cup_1$-product is $(H^*(SX, Z_2), d = 0)$ with $a \cup_1 b = 0$ if $a \neq b$ and $a \cup_1 a = Sq^{|a|-1}a$. Another example (S. Saneblidze) is $C^*(SX, CX)$, where SX is the suspension and CX is the cone of a space X.

Level 3. This is the "one-line" case where E has the form

$$\begin{array}{ccccccccccc} & & & E_{01}=\mathrm{id} & & & & E_{10}=\mathrm{id} & & & \\ & & & & & E_{11} & & & & & \\ & & & E_{12} & & & & 0 & & & , \\ & & E_{13} & & & 0 & & & 0 & & \\ \cdots & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & \cdots \end{array} \tag{12}$$

i.e., all components of E are zero except for E_{01}, E_{10}, and E_{1k}, $k = 1, 2, 3, \ldots$. We note that this case is especially interesting in this article.

In this case, condition (9) breaks into two conditions:

$$\begin{aligned} & d_A E_{1,k}(a|b_1 \otimes \cdots \otimes b_k) + E_{1,k}(d_A a|b_1 \otimes \cdots \otimes b_k) \\ & \quad + \sum_i E_{1k}(a|b_1 \otimes \cdots \otimes d_A b_i \otimes \cdots \otimes b_k) \\ & = b_1 E_{1k}(a|b_2 \otimes \cdots \otimes b_k) + \sum_i E_{1k}(a|b_1 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_k) + E_{1k}(a|b_1 \otimes \cdots \otimes b_{k-1})b_k \end{aligned} \tag{13}$$

at $\left(\bigotimes^1 A\right) \otimes \left(\bigotimes^k A\right)$ and

$$\begin{aligned} & a_1 E_{1,k}(a_2|b_1 \otimes \cdots \otimes b_k) + E_{1,k}(a_1 \cdot a_2|b_1 \otimes \cdots \otimes b_k) + E_{1,k}(a_1|b_1 \otimes \cdots \otimes b_k)a_2 \\ & \quad = \sum_{p=1,\ldots,k-1} E_{p,1}(a_1|b_1 \otimes \cdots \otimes b_p) \cdot E_{1,m-p}(a_2|b_{p+1} \otimes \cdots \otimes b_k) \end{aligned} \tag{14}$$

at $\left(\bigotimes^2 A\right) \otimes \left(\bigotimes^k A\right)$; at $\left(\bigotimes^{n>2} A\right) \otimes \left(\bigotimes^k A\right)$, the condition is trivial.

In this case, the associativity condition (11) has the form

$$\sum_{o\le i_1\le\cdots\le i_j\le m}\ \sum_{m_1+\cdots+m_r=m} E_{1r}(a|c_1\otimes\cdots\otimes c_{i_1}\otimes E_{1m_1}(b_1|c_{i_1+1}\otimes\cdots\otimes c_{i_1+m_1})\otimes\dots \\ \otimes c_{i_j}\otimes E_{1_r m_j}(b_l|c_{m_1+\cdots+m_{j-1}+1}\otimes\cdots\otimes c_m) = E_{1m}(E_{1l}(a|b_1\otimes\cdots\otimes b_l)|c_1\otimes\cdots\otimes c_m). \tag{15}$$

Actually, the structure of this level coincides with the notion of homotopy G-algebra (see below). Note that we can also consider the case where

$$\begin{array}{ccccccccccc} & & & E_{01}=\mathrm{id} & & & & E_{10}=\mathrm{id} & & & \\ & & & & & E_{11} & & & & & \\ & & & 0 & & & & E_{21} & & & \\ & & 0 & & & 0 & & & & E_{31} & \\ \dots & \dots & & \dots & & \dots & & \dots & & \dots & \dots \end{array}$$

(level $3'$) with suitable conditions.

Level 4. We consider the case of a twisting cochain $E=\{E_{pq}\}$ with no other restrictions but (5), (9), and (11). This is nothing other than the Hirsch algebra structure defined above.

1.3. B_∞**-algebra.** The notion of B_∞-algebra was introduced in [2,7] as an additional structure on a DG-algebra $(A,\cdot,d)$, which turns the tensor coalgebra $T^c(s^{-1}A)=BA$ into a DG-Hopf algebra. Therefore, it requires a new differential

$$\widetilde{d}: BA\to BA,$$

which should be a coderivation with respect to the standard coproduct of BA, and a new associative multiplication

$$\widetilde{\mu}:(BA,\widetilde{d})\otimes(BA,\widetilde{d})\to(BA,\widetilde{d}),$$

which should be a map of DG-coalgebras with $1_\Lambda\in\Lambda\subset BA$ as a unit element.

It is known (see, e.g., [11, 14, 19]) that such $\widetilde{d}$ specifies on A the structure of an A_∞-algebra in the sense of Stasheff [20], namely, a sequence of operations $\left\{m_i:\bigotimes^i A\to A,\ i=1,2,3,\dots\right\}$ subject to appropriate conditions.

As for the new multiplication $\widetilde{\mu}$, it follows from the above considerations that it is induced by a sequence of operations $\{E_{pq}\}$ satisfying (5), (11), and the modified condition (9) with involved A_∞-algebra structure $\{m_i\}$.

Thus, the Hirsch algebra structure (the above-mentioned Level 4 and, consequently, other levels), in fact, is a particular B_∞-algebra structure on A, where the standard differential of the bar construction $d_B: BA\to BA$ *does not change*, i.e., $\widetilde{d}=d_B$ (in this case, the corresponding A_∞-algebra structure is degenerate: $\{m_1=d_A,\ m_2=\mu_A,\ m_3=0,\ m_4=0,\dots\}$).

Let us mention that a sequence of cochain operations $\{E_{pq}\}$ satisfying (5) and (9) (but not (11), i.e., the induced product in the bar construction is not strictly associative) was constructed in [16] for the singular cochain complex of a topological space $C^*(X)$ by using acyclic models, the initial condition $E_{01}=E_{10}=\mathrm{id}$ and $E_{0k}=E_{k0}=0$ for $k>1$, determining a twisting cochain E uniquely up to equivalence of twisting cochains in this case.

1.4. DG**-Lie algebra structure in a Hirsch algebra.** Let $(A,d,\cdot,\{E_{pq}\})$ be a Hirsch algebra; then there appears a structure of a DG-Lie algebra in the desuspension $s^{-1}A$: although $\cup_1=E_{11}$ is not associative, condition (10), which is a particular case of condition (11), implies the pre-Jacobi identity

$$a\cup_1(b\cup_1 c)-(a\cup_1 b)\cup_1 c = a\cup_1(c\cup_1 b)-(a\cup_1 c)\cup_1 b,$$

which guarantees that the commutator

$$[a,b]=a\cup_1 b-b\cup_1 a$$

satisfies the Jacobi identity. In addition, condition (1) implies that $[\ ,\ ] : A^p \otimes A^q \to A^{p+q-1}$ is a chain map.

The structure of a Hirsch algebra on A induces the structure of the Gerstenhaber algebra (G-algebra) on homology $H(A)$ [6], which is defined as a commutative graded algebra $(H, \cdot)$ with a Lie bracket

$$[\ ,\ ] : H^p \otimes H^q \to H^{p+q-1}$$

of degree -1, i.e., a graded Lie-algebra structure on the desuspension $s^{-1}H$, which is a biderivation:

$$[ab, c] = a[b, c] + [a, c]b$$

(this is a graded version of a Poisson algebra).

The existence of this structure in the homology $H(A)$ of a Hirsch algebra $(A, d, \cdot, \{E_{1k}\})$ can be proved as follows. As was mentioned above, $s^{-1}A$ is a DG-Lie algebra if A is a Hirsch algebra. Thus, there appears the structure of graded Lie algebra on $s^{-1}H(A)$. The Hirsch formulas (7) and (8) (up to homotopy) imply that the induced Lie bracket is a biderivation.

1.5. Homotopy G-algebra. A Hirsch algebra of particular type of level 3 is known as a homotopy G-algebra.

In [6, 23], a *homotopy G-algebra* is defined as a DG-algebra $(A, d, \cdot)$ with a given sequence of multi-braces $a\{a_1, \dots, a_k\}$, which, in our notation, we regard as a sequence of operations

$$E_{1k} : A \otimes \left(\bigotimes^k A\right) \to A, \quad k = 1, 2, 3, \dots,$$

which, together with $E_{01} = \mathrm{id}$, satisfies conditions (5) and (13)–(15).

The name *homotopy G-algebra* is motivated by the fact that this structure induces the structure of Gerstenhaber algebra (G-algebra) on homology $H(A)$.

As was mentioned above, such a sequence defines a twisting cochain

$$E : BA \otimes BA \to A;$$

conditions (13) and (14) mean nothing other than that E satisfies condition (3) and, consequently, defines a product on the bar construction $\mu_E : BA \otimes BA \to BA$. We emphasize that this twisting cochain E is of special type; it is of level 3, i.e., it is a "one-line" twisting cochain like (12): all its components, except perhaps E_{1k}, are zero.

1.6. Strong homotopy commutative algebras. The notion of a strong homotopy commutative algebra (shc-algebra) as a tool for measuring the noncommutativity of DG-algebras was used in [9, 17, 22], etc.

An shc-algebra is a DG-algebra $(A, d, \cdot)$ with a twisting cochain

$$\Phi : B(A \otimes A) \to A,$$

which satisfies certain conditions (up to homotopy) of associativity and commutativity (actually, Φ induces a DG-algebra map $\Omega B(A \otimes A) \to A$).

We note that the fact that an shc-structure measures the noncommutativity of A is the result *only of the existence* of the twisting cochain Φ and not of the homotopy commutativity of it: in [17, Proposition 4.8], the $\cup_1$-product in A is defined in terms of Φ by the formula

$$a \cup_1 b = \Phi[(1 \otimes a) \otimes (b \otimes 1) + (a \otimes 1) \otimes (1 \otimes b)].$$

There is the shuffle map (a DG-coalgebra map)

$$\mathrm{Sh} : BA \otimes BA \to B(A \otimes A);$$

thus, each shc-algebra structure, i.e., twisting cochain Φ, induces a twisting cochain $E = \Phi \circ \mathrm{Sh} : BA \otimes BA \to A$ of level 4 in the above description, which, in fact, is an *almost* Hirsch-algebra structure on A: we cannot expect the strict associativity of the product in BA induced by E since Φ is associative only up to homotopy.

Conversely, the shuffle map Sh is a weak equivalence of DG-coalgebras; thus, it induces a bijection between *equivalence classes* of twisting cochains $E : BA \otimes BA \to A$ and $\Phi : B(A \otimes A) \to A$. This means that to each twisting cochain E (to a Hirsch algebra structure), there corresponds a class of twisting cochains Φ (class of shc-algebra structures) such that $E \sim \Phi \circ \mathrm{Sh}$.

We note that, as a rule, an shc-algebra structure (i.e., the twisting cochain Φ) is constructed by using acyclic models; hence, it is not uniquely determined and thus there is no guarantee that the induced $E = \Phi \circ \mathrm{Sh}$ will be of level 3 (i.e., of "one-line" form consisting just of components E_{1k}). Therefore, the induced structure is not generally a homotopy G-algebra. We emphasize that for the homotopy G-algebra structure (for the twisting cochain E), there are explicit formulas in the particular cases mentioned above.

2. Twisting Elements

2.1. Twisting elements in homotopy G-algebras. There is a very useful notion of *twisting element* introduced by Brown in [4]. Roughly speaking, for a cochain algebra C^*, a twisting element is defined as a 1-dimensional element $a \in C^1$ such that $da = aa$. Later, Berikashvili in [3] (see also [17, 19] in various contexts) introduced an equivalence relation in the set of all twisting elements $\mathrm{Tw}(C^*)$, namely, he introduced the action of the group G of invertible elements in C^0 on the set $\mathrm{Tw}(C^*)$ by the formula

$$g * a = g \cdot a \cdot g^{-1} + dg \cdot g^{-1}.$$

The set of orbits $D(C^*) = \mathrm{Tw}(C^*)/G$ is a functor on C^* with very interesting properties (for example, D sends weak equivalences to isomorphisms), which has important applications in the homology theory of fibrations.

In this section, we construct an analogue of the notion of twisting element for homotopy G-algebras (and the appropriate group action), replacing the product in the equation $da = aa$ by the $\cup_1$-product.

Let $(C^{*,*}, d, \cdot, \{E_{1,k}\})$ be a *bigraded* homotopy G-algebra. We mean the following:

$$d(C^{m,n}) \subset C^{m+1,n}, \quad C^{m,n} \cdot C^{p,q} \subset C^{m+p,n+q},$$
$$E_{1,k}(C^{m,n}|C^{p_1,q_1} \otimes \cdots \otimes C^{p_k,q_k}) \subset C^{m+p_1+\cdots+p_k-k,n+q_1+\cdots+q_k}.$$

Below, we introduce two versions of the notion of *twisting elements* in a homotopy G-algebra and the appropriate group actions. The first version controls the degeneracy of A_∞-algebra structures and the second version controls deformations of algebras.

Version 1. A *twisting element* in $C^{*,*}$ is the element

$$m = m^3 + m^4 + \cdots + m^p + \ldots; \quad m^p \in C^{p,2-p},$$

satisfying the condition $dm = E_{1,1}(m|m) = m \cup_1 m$. This condition can be rewritten in the form

$$dm^p = \sum_{i=3}^{p-1} m^i \cup_1 m^{p-i+2}. \tag{16}$$

In particular, $dm^3 = 0$, $dm^4 = m^3 \cup_1 m^3$, $dm^5 = m^3 \cup_1 m^4 + m^4 \cup_1 m^3$, $\ldots$. The set of all twisting elements is denoted by $\mathrm{Tw}(C^{*,*})$.

Consider the set $G = \{g = g^2 + g^3 + \cdots + g^p + \ldots,\ g^p \in C^{p,1-p}\}$. This set is a group with respect to the operation

$$\overline{g} * g = \overline{g} + g + \sum_{k=1}^{\infty} E_{1,k}(\overline{g}|g \otimes \cdots \otimes g); \tag{17}$$

in particular,

$$(\overline{g} * g)^2 = \overline{g} + g^2, \quad (\overline{g} * g)^3 = \overline{g}^3 + g^3 + \overline{g}^2 \cup_1 g^3,$$
$$(\overline{g} * g)^3 = \overline{g}^4 + g^3 + \overline{g}^2 \cup_1 g^3 + \overline{g}^3 \cup_1 g^2 + E_{1,2}(\overline{g}^2|g^2 \otimes g^2).$$

This operation is associative and has the unit $e = 0+0+\dots$, and the opposite g^{-1} can be found inductively from the equation $g * g^{-1} = e$.

The group G acts on the set $Tw(C^{*,*})$ by the rule $g * m = \overline{m}$, where

$$\overline{m} = m + dg + g \cdot g + E_{1,1}(g|m) + \sum_{k=1}^{\infty} E_{1,k}(\overline{m}|g \otimes \dots \otimes g); \tag{18}$$

in particular,

$$\overline{m}^3 = m^3 + dg^2, \quad \overline{m}^4 = m^4 + dg^3 + g^2 \cdot g^2 + g^2 \cup_1 m^3 + m^3 \cup_1 g^2,$$
$$\overline{m}^5 = m^5 + dg^4 + g^2 \cdot g^3 + g^3 \cdot g^2 + g^2 \cup_1 m^4 + g^3 \cup_1 m^3 + \overline{m}^3 \cup_1 g^3 + \overline{m}^4 \cup_1 g^2 + E_{1,2}(\overline{m}^3|g^2 \otimes g^2).$$

Note that, although on the right-hand side of this formula, $\overline{m}$ participates, but it has lower dimension than $\overline{m}$ on the left-hand side; thus, the components of $\overline{m}$ can be found from this equation inductively. The resulting $\overline{m}$ is also a twisting element. We denote by $D(C^{*,*})$ the set of orbits $\mathrm{Tw}(C^{*,*})/G$.

This group action allows us to perturb twisting elements. Let $g^n \in C^{n,1-n}$ be an arbitrary element. Then for

$$g = 0 + \dots + 0 + g^n + 0 + \dots,$$

the twisting element $\overline{m} = g * m$ has the form

$$\overline{m} = m_3 + \dots + m_n + (m_{n+1} + dg_n) + \overline{m}^{n+2} + \overline{m}^{n+3} + \dots.$$

For an arbitrary twisting element $m = m^3 + m^4 + \dots$, the first component $m^4 \in C^{3,-1}$ is a cocycle. If its class in the cohomology module $H^{3,-1}(C^{*,*})$ is zero, then $m^3 = dg^2$ for some $g^2 \in C^{2,-1}$. Perturbing m by $g = g^2 + 0 + 0 + \dots$, we can eliminate the first component m^3, i.e., we obtain the twisting element $\overline{m} \sim m$, which has the form

$$\overline{m} = 0 + \overline{m}^4 + \overline{m}^5 + \dots;$$

now the component $\overline{m}^4$ becomes a cocycle. If its class is zero, then we can eliminate it, etc. Finally, we obtain the following proposition.

Proposition 1. *If, for a bigraded homotopy G-algebra $C^{*,*}$, all homology modules $H^{n,2-n}(C^{*,*})$ are trivial for $n \geq 3$, then $D(C^{*,*}) = 0$, i.e., each twisting element is equivalent to the trivial twisting element.*

Version 2. In this case, we define a twisting element by the formula

$$b = b_1 + b_2 + \dots + b_n + \dots, \quad b_n \in C^{2,n},$$

where

$$db_n = \sum_{i=2}^{n-1} b_i \cup_1 b_{n-i}.$$

In particular,

$$db_1 = 0, \quad db_2 = b_1 \cup_1 b_1, \quad db_3 = b_1 \cup_1 b_2 + b_2 \cup_1 b_1, \quad \text{etc.}$$

The set of all twisting elements is denoted by $\mathrm{Tw}'(C)$.

We consider the group

$$G' = \{g = g_1 + g_2 + \dots + g_p + \dots, \ g_p \in C^{1,p}\}$$

with the operation

$$g' * g = g' + g + \sum_{k=1}^{\infty} E_{1,k}(g'|g \otimes \dots \otimes g).$$

In particular,

$$(g' * g)_1 = g'_1 + g_1, \quad (g' * g)_2 = g'_2 + g_2 + g'_1 \cup_1 g_1,$$
$$(g' * g)_3 = g'_3 + g_3 + g'_1 \cup_1 g_2 + g'_2 \cup_1 g_1 + E_{1,2}(g'_1|g_1 \otimes g_1).$$

As above, this operation is associative and has the unit $e = 0 + 0 + \dots$, and the opposite element g^{-1} can be found inductively from the equation $g * g^{-1} = e$.

The group G' acts on the set $\mathrm{Tw}'(C^{*,*})$ by the rule $g * b = b'$, where

$$b' = b + dg + g \cdot g + E_{1,1}(g|b) + \sum_{k=1}^{\infty} E_{1,k}(b'|g \otimes \dots \otimes g); \tag{19}$$

in particular,

$$b'_1 = b_1 + dg_1, \quad b'_2 = b_2 + dg_2 + g_1 \cdot g_1 + g_1 \cup_1 b_2 + b'_1 \cup_1 g_1,$$

$$b'_3 = b_3 + dg_3 + g_1 \cdot g_2 + g_2 \cdot g_1 + g_1 \cup_1 b_2 + g_2 \cup_1 b_1 + b'_1 \cup_1 g_2 + b'_2 \cup_1 g_1 + E_{1,2}(b'_1|g_1 \otimes g_1).$$

The components of b' can be found from this equation inductively. The resulting b' is also a twisting element. We denote by $D'(C^{*,*})$ the set of orbits $\mathrm{Tw}'(C^{*,*})/G'$.

As above, this group action allows us to perturb twisting elements, and we obtain the following proposition.

Proposition 2. *If for a bigraded homotopy G-algebra $C^{*,*}$, all homology modules $H^{2,n}(C^{*,*})$ are trivial for $n \geq 1$, then $D'(C^{*,*}) = 0$, i.e., each twisting element is equivalent to the trivial twisting element.*

2.2. Twisting elements in a DG-Lie algebra. There is a modified notion of twisting element in a DG-Lie algebra $(L, d, [\ ,\ ])$. This is an element $a \in L^1$ such that $da = \frac{1}{2}[a,a]$ (in the literature, this equation is called the Maurer–Cartan equation or master equation). A systematic investigation of this notion can be found in [10].

As was described above, for a homotopy G-algebra $(C, \cdot, d\{E_{1k}\})$ in the desuspension $s^{-1}A$, there appears the structure of a DG-Lie algebra with the bracket $[a,b] = a \cup_1 b - b \cup_1 a$. Note that if $C^{*,*}$ is a bigraded homotopy G-algebra, then $s^{-1}C^{*,*}$, where $(s^{-1}C^{*,*})^{p,q} = C^{p-1,q}$, is a bigraded DG-Lie algebra.

Suppose that $m = m^3 + m^4 + \dots + m^p + \dots$, $m^p \in C^{p,2-p}$, is a twisting element in A of Version 1. The defining equation $dm = m \cup_1 m$ can be rewritten in terms of brackets as $dm = \frac{1}{2}[m,m]$; therefore, the same m can be considered as a Lie twisting element.

The same is true for a twisting element $b = b_1 + b_2 + \dots + b_n + \dots$, $b_n \in C^{2,n}$, of Version 2: the condition $db = b \cup_1 b$ in terms of brackets becomes $db = \frac{1}{2}[b,b]$.

The following question of Huebschmann remains unsolved: How to rewrite formulas (18) and (19) of transformation of twisting elements in terms of brackets.

3. Examples of Hirsch Algebras

3.1. Cochain algebra of a simplicial set. An example of a Hirsch algebra is the cochain complex $C^*(S)$ of a 1-reduced simplicial set S. In [2], Baues constructed the strictly associative product in BA, where $A = C^*(S)$. Examining the appropriate twisting cochain, one can obtain that it is a "one-line" cochain of Level 3; thus, it forms the structure of a Hirsch algebra.

3.2. Hochschild cochain complex. Let A be an algebra and M be a two-sided module on A. The Hochschild cochain complex $C^*(A;M)$ of A with coefficients in M is defined as $C^n(A;M) = \mathrm{Hom}(\otimes^n A, M)$ with the differential $\delta : C^{n-1}(A;M) \to C^n(A;M)$ given by the formula

$$\delta f(a_1 \otimes \dots \otimes a_n) = a_1 f(a_2 \otimes \dots \otimes a_n)$$

$$+ \sum_{k=1}^{n-1} f(a_1 \otimes \dots \otimes a_{k-1} \otimes a_k a_{k+1} \otimes \dots \otimes a_n) + f(a_1 \otimes \dots \otimes a_{n-1})a_n.$$

If M is an algebra over A, then in the Hochschild complex, there appears the $\cup$-product

$$f \cup g(a_1 \otimes \dots \otimes a_{n+m}) = f(a_1 \otimes \dots \otimes a_n) \cdot g(a_{n+1} \otimes \dots \otimes a_{n+m}),$$

which turns $C^*(A;M)$ into a cochain algebra.

We focus our attention on the case $M = A$. In [13], explicit formulas for operations, which specify on the Hochschild cochain complex $C^*(A;A)$ the structure of a homotopy G-algebra, are given. We describe this structure below. Note that the same operations were described in [7] for constructing the $B(\infty)$-algebra structure on $C^*(A;A)$ (see also [6, 23].

In [5], Gerstenhaber defined the product $f \circ g$ in the Hochschild complex $C^*(A, A)$ by the formula

$$f \circ g(a_1 \otimes \cdots \otimes a_{n+m-1}) \\ = \sum_{k=0}^{n-1} f(a_1 \otimes \cdots \otimes a_k \otimes g(a_{k+1} \otimes \cdots \otimes a_{k+m}) \otimes a_{k+m+1} \otimes \cdots \otimes a_{n+m-1}).$$

The Gerstenhaber product has the following properties:

$$\delta(f \circ g) = \delta f \circ g + f \circ \delta g + f \cup g - g \cup f,$$
$$(f \cup g) \circ h = f \cup (g \circ h) + (f \circ h) \cup g.$$

This means that the product $f \circ g$ has the properties of a $\cup_1$-product: if we use the notation $f \circ g = f \cup_1 g$, then the first condition gives the standard condition on the $\cup_1$ product:

$$\delta(f \cup_1 g) = \delta f \cup_1 g + f \cup_1 \delta g + f \cup g - g \cup f,$$

and the second condition gives the left Hirsch formula

$$(f \cup g) \cup_1 h = f \cup (g \cup_1 h) + (f \cup_1 h) \cup g.$$

As for the right Hirsch formula, there is a different kind of $\cup_1$-product of a cochain and a *couple of cochains*: for $f \in C^p(A;A)$, $g \in C^q(A;A)$, and $h \in C^r(A;A)$, we define $f \cup_1 (g,h) \in C^{p+q+r-2}(A;A)$ by

$$(f \cup_1 (g,h))(a_1 \otimes \cdots \otimes a_{p+q+r-2}) \\ = \sum_{k,l} f(a_1 \otimes \cdots \otimes a_k \otimes g(a_{k+1} \otimes \cdots \otimes a_{k+q}) \otimes a_{k+m+1} \otimes \dots \\ \otimes a_l \otimes h(a_{l+1} \otimes \cdots \otimes a_{l+r}) \otimes a_{l+r+1} \otimes \cdots \otimes a_{p+q+r-2}).$$

A straightforward verification shows that the $\cup_1$-product in $C^p(A;A)$ satisfies the right Hirsch formula up to homotopy and the appropriate homotopy is $f \cup_1 (g,h)$, i.e., the condition

$$\delta(f \cup_1 (g,h)) + \delta f \cup_1 (g,h) + f \cup_1 (\delta g, h) + f \cup_1 (g, \delta h) \\ = f \cup_1 (g \cup h) + g \cup (f \cup_1 h) + (f \cup_1 g) \cup h$$

holds.

Let us mention also the following property of the introduced product: the product $f \cup_1 (g,h)$ measures the nonassociativity of the $\cup_1$-product:

$$f \cup_1 (g \cup_1 h) - (f \cup_1 g) \cup_1 h = f \cup_1 (g,h) + f \cup_1 (h,g). \tag{20}$$

Remark. In [5] (see also [21]), a DG-Lie algebra structure was introduced in the desuspension of the Hochschild complex $s^{-1}C^*(A;A)$. Actually, the Lie bracket $[f,g]$ is the commutator of the $\cup_1$-product: $[f,g] = f \cup_1 g - g \cup_1 f$. Although the $\cup_1$-product is not associative, condition (20) allows us to verify that the Jacobi identity holds.

In [13], the author defined the generalized $\cup_1$-products of a Hochschild cochain and a sequence of cochains:

$$(f \cup_1 (g_1, \dots, g_i))(a_1 \otimes \cdots \otimes a_n) \\ = \sum f(a_1 \otimes \cdots \otimes a_{k_1} \otimes g_1(a_{k_1+1} \otimes \dots a_{k_1+n_1}) \otimes \cdots \otimes a_{k_i} \otimes g_i(a_{k_i+1} \otimes \dots a_{k_i+n_i}) \otimes \cdots \otimes a_n).$$

A straightforward verification shows that the collection $\{E_{1k}\}$ given by

$$E_{1k}(f|g_1 \otimes \cdots \otimes g_k) = f \cup_1 (g_1, \dots, g_k)$$

satisfies conditions (5) and (13)–(15); thus it forms the structure of a homotopy G-algebra on the Hochschild complex $C^*(A; A)$.

3.3. Cobar construction of a Hopf algebra. As the third example of a Hirsch algebra, we present the cobar construction of a Hopf algebra.

The cobar construction ΩA of a coalgebra $(A, \nabla : A \to A \otimes A)$ is the DG-algebra

$$\Omega A = T(A) = \Lambda + A + A \otimes A + A \otimes A \otimes A + \dots$$

with the product

$$(a_1 \otimes \cdots \otimes a_p) \cdot (a_{p+1} \otimes \cdots \otimes a_{p+q}) = a_1 \otimes \cdots \otimes a_{p+q}$$

(i.e., it is a free graded algebra generated by A) and the differential

$$d_\Omega(a_1 \otimes \cdots \otimes a_n) = \sum_i a_1 \otimes \cdots \otimes \nabla a_i \otimes \cdots \otimes a_n.$$

What additional structure appears on ΩA if A is a *Hopf algebra*, i.e., if it is equipped additionally with a product $A \otimes A \to A$, which is a coalgebra map? It is shown in [1] that if the ground ring is Z_2, then there exists a $\cup_1$-product in ΩA given by the formula

$$(a_1 \otimes \cdots \otimes a_p) \cup_1 (b_1 \otimes \cdots \otimes b_q) = \sum_i a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i^{(1)} \cdot b_1 \otimes \cdots \otimes a_i^{(q)} \cdot b_q \otimes a_{i+1} \otimes \cdots \otimes a_p,$$

where $\nabla^q(a_i) = a_i^{(1)} \otimes \cdots \otimes a_i^{(q)}$ is the q-fold iteration of ∇ and $a \cdot b$ is the product in A. It is clear that this $\cup_1$-product is functorial on the category of Hopf algebras.

Let us introduce the following notation. For $a \in A$ and $b_1 \otimes \cdots \otimes b_q \in \bigotimes^q A$, we define $a \vee (b_1 \otimes \cdots \otimes b_q) \in \bigotimes^q A$ as follows:

$$a^{(1)} \cdot b_1 \otimes \cdots \otimes a^{(q)} \cdot b_q.$$

Thus, the definition of the Adams $\cup_1$-product has the form

$$(a_1 \otimes \cdots \otimes a_p) \cup_1 (b_1 \otimes \cdots \otimes b_q) = \sum_i a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i \vee (b_1 \otimes \cdots \otimes b_q) \otimes a_{i+1} \otimes \cdots \otimes a_p.$$

Below we show that there exist functorial operations

$$E_{1k} : (\Omega A) \otimes \left(\bigotimes^k \Omega A\right) \to \Omega A$$

with $E_{11} = \cup_1$, which satisfy conditions (13), (14), and (4), i.e., which form the structure of a homotopy G-algebra on ΩA.

There is a formula for the operation E_{1k}. Let $\alpha = a_1 \otimes \cdots \otimes a_n \in \Omega A$ and $\beta_1, \beta_2, \dots, \beta_k \in \Omega A$; then we define

$$\begin{aligned} E_{1k}(\alpha|\beta_1 \otimes \cdots \otimes \beta_k) = \sum a_1 \otimes \cdots \otimes a_{i_1-1} \otimes a_{i_1} \vee \beta_1 \otimes a_{i_1+1} \otimes \dots \\ \otimes a_{i_k-1} \otimes a_{i_k} \vee \beta_k \otimes a_{i_k+1} \otimes \cdots \otimes a_n, \end{aligned} \tag{21}$$

where the summation is taken over all $1 \le i_1 < \cdots < i_k \le n$. It is clear that $E_{1k}(\alpha|\beta_1 \otimes \cdots \otimes \beta_k) = 0$ if $n < k$.

Remark. The way of obtaining this formula is as follows. We take the initial condition $E_{1k}(a_1|\beta_1 \otimes \cdots \otimes \beta_k) = 0$ and extend the products $E_{1k}(a_1 \otimes a_2|\beta_1 \otimes \cdots \otimes \beta_k)$ by using condition (14).

Theorem. *The operations* $E_{1k} : (\Omega A) \otimes \left(\bigotimes^k \Omega A\right) \to \Omega A$ *given by* (21) *are functorial on the category of Hopf algebras and satisfy conditions* (13), (14), *and* (4); *thus they form the structure of a Hirsch algebra on* ΩA.

4. Applications

4.1. Multiplicative twisted tensor product. In this section, we present the result of [15]: the construction of a *multiplicative version* of Brown's [4] twisted tensor product.

4.1.1. Twisting cochains. Let $(C, d, \nabla : C \to C \otimes C)$ be a DG-coalgebra and $(A, d, \mu : A \otimes A \to A)$ be a DG-algebra (the differentials $d : C \to C$ and $d : A \to A$ are assumed to be of degree $+1$). A twisting cochain is a homomorphism

$$\varphi : C \to A$$

of degree 1 satisfying the condition

$$d\varphi + \varphi d = \varphi \cup \varphi. \tag{22}$$

The given twisting cochain $\varphi : C \to A$ determines the following three important maps:

(1) A *DG-coalgebra map* $f_\varphi : C \to BA$ from C to the bar construction BA given by the rule

$$f_\varphi = \sum_{n=0}^{\infty} (\varphi \otimes \cdots \otimes \varphi)\nabla^n,$$

where $\nabla^0 = \epsilon : C \to \Lambda$ is the coaugmentation, $\nabla^1 = \mathrm{id}$, and $\nabla^n = (\nabla^{n-1} \otimes \mathrm{id})\nabla$ is the iteration of the coproduct ∇.

(2) A *DG-algebra map* $g_\varphi : \Omega C \to A$ from the cobar construction ΩC to A given by the rule

$$g_\varphi|_{\otimes^n C} = \mu^n(\varphi \otimes \cdots \otimes \varphi),$$

where $\mu^0 : \Lambda \to A$ is the unit of A, $\mu^1 = \mathrm{id}$, and $\mu^n = \mu(\mu^{n-1} \otimes \mathrm{id})$ is the iteration of the product μ.

Remark. Let us denote by $T(C, A)$ the set of all twisting cochains $\varphi : C \to A$. Then the assignments $\varphi \longmapsto f_\varphi$ and $\varphi \longmapsto g_\varphi$ form the *bijections*

$$\mathrm{Hom}_{DG\text{-alg}}(\Omega C, A) \longleftrightarrow T(C, A) \longleftrightarrow \mathrm{Hom}_{DG\text{-Coalg}}(C, BA),$$

which realize the adjunction of the functors B and Ω.

(3) A *twisted differential* $d_\varphi : A \otimes C \to A \otimes C$ given by the rule

$$d_\varphi(a \otimes c) = da \otimes c + a \otimes dc + \varphi \cap (a \otimes c),$$

where

$$\varphi \cap (a \otimes c) = (\mu \otimes \mathrm{id})(\mathrm{id} \otimes \varphi \otimes \mathrm{id})(\mathrm{id} \otimes \nabla).$$

The tensor product $A \otimes C$ equipped with the differential d_φ is called a *twisted tensor product* and is denoted by $A \otimes_\varphi C$ (this notion belongs to Brown [4]). This construction has substantial applications in the homology theory of fibrations.

4.1.2. Multiplicative twisting cochains (commutative case). Now suppose that (C, d, ∇, μ) is a *DG-Hopf algebra* and (A, d, μ) is a *commutative DG-algebra*. Then the bar construction BA is a DG-Hopf algebra with respect to the shuffle product $\mu_{\mathrm{Sh}} : BA \otimes BA \to BA$.

A twisting cochain $\varphi : C \to A$ is called *multiplicative* if, in addition to the standard Brown condition (22), the following condition holds:

$$\varphi(ab) = \eta a \cdot \varphi(b) + \varphi(a) \cdot \eta(b)$$

(this notion was introduced by Prute in [18]). This condition is equivalent to the condition of $f_\varphi : C \to BA$ being a map of DG-Hopf algebras. Note that this condition can be reformulated in the following form: φ

factors through indecomposables $QC = C/C_+ \cdot C_+$, i.e., there exists ψ such that the diagram (see [14])

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & A \\ {\scriptstyle\psi}\downarrow & \nearrow & \\ QC & & \end{array}$$

is commutative.

On the other hand, the tensor product $A \otimes C$ is a graded algebra (since C and A are algebras). When is the twisted differential d_φ compatible with this product? As is shown in [18], this happens if φ is multiplicative.

4.1.3. Multiplicative twisting cochains (noncommutative case). The result of this section was announced in [15].

Assume that (C, d, ∇, μ) is a *DG-Hopf algebra* and $(A, d, \mu, \{E_{1,k}\})$ is a *homotopy G-algebra*. Then, as we know, there appears the product $\mu_E : BA \otimes BA \to BA$ in the bar construction BA.

A twisting cochain $\varphi : C \to A$ is called *multiplicative* if, in addition to the standard Brown condition (22), the following condition holds:

$$\begin{gathered} \varphi(ab) = \eta a \cdot \varphi(b) + \varphi(a) \cdot \eta(b) + E_{11}(\varphi(a)|\varphi(b)) \\ + E_{12}(\varphi(a)|(\varphi \otimes \varphi)\nabla^2(b)) + E_{13}(\varphi(a)|(\varphi \otimes \varphi \otimes \varphi)\nabla^3(b)) + \dots . \end{gathered} \tag{23}$$

This condition is equivalent to the condition of $f_\varphi : C \to BA$ being multiplicative, i.e., a map of DG-Hopf algebras.

Generally, even if φ is multiplicative in this sense, the twisted differential d_φ is not a derivation with respect to the standard multiplication of tensor product $A \otimes C$. There appears the need *to twist* the multiplication in $A \otimes C$. There is a formula for this twisted multiplication:

$$\mu_\varphi = (\mu_A \otimes \mu_C)(1 \otimes E_{1,*} \otimes 1 \otimes 1)(1 \otimes 1 \otimes f_\varphi \otimes 1 \otimes 1)(1 \otimes 1 \otimes \nabla \otimes 1)(1 \otimes T \otimes 1).$$

Direct inspections prove the following theorem.

Theorem 2. *Let (C, d, ∇, μ) be a DG-Hopf algebra, $(A, d, \mu, \{E_{1,k}\})$ be a homotopy G-algebra, and $\varphi : C \to A$ be a multiplicative twisting cochain (i.e., it satisfies (22) and (23)). Then the twisted differential $d_\varphi : A \otimes C \to A \otimes C$ is a derivation with respect to the twisted multiplication $\mu_\varphi : (A \otimes C) \otimes (A \otimes C) \to A \otimes C$, i.e., the twisted tensor product $(A \otimes C, d_\varphi, \mu_\varphi)$ is a DG-algebra.*

4.2. Deformation of algebras. This is just an illustrative application. Using the homotopy G-algebra structure, the notion of twisting element, and gauge transformation, we obtain the well-known result of Gerstenhaber [5].

Let $(A, \cdot)$ be an algebra over a field k, $k[[t]]$ be the algebra of formal power series in variable t, and $A[[t]] = A \otimes k[[t]]$ be the algebra of formal power series with coefficients in A.

A deformation of an algebra $(A, \cdot)$ is defined as a sequence of homomorphisms

$$B_i : A \otimes A \to A, \quad i = 0, 1, 2, \dots, \quad B_0(a \otimes b) = a \cdot b,$$

satisfying the *associativity* condition

$$\sum_{i+j=n} B_i(a \otimes B_j(b \otimes c)) = \sum_{i+j=n} B_i(B_j(a \otimes b) \otimes c) \tag{24}$$

for all $n \geq 1$.

Such a sequence determines the *star product*

$$a \star b = a \cdot b + B_1(a \otimes b)t + B_2(a \otimes b)t^2 + B_3(a \otimes b)t^3 + \dots \in A[[t]],$$

which can be naturally extended to a $k[[t]]$-bilinear product

$$\star : A[[t]] \otimes A[[t]] \to A[[t]],$$

and condition (24) guarantees that this product is associative.

Two deformations $\{B_i\}$ and $\{B'_i\}$ are called *equivalent* if there exists a sequence of homomorphisms

$$\{g_i : A \to A;\ i = 0, 1, 2, \dots;\ g_0 = \mathrm{id}\}$$

such that

$$\sum_{r+s=n} g_r(B_s(a \otimes B) = \sum_{i+j+k=n} B'_i(g_j(a) \otimes g_k(b)). \tag{25}$$

In this case, the sequence $\{g_i\}$ determines the power series

$$g = \mathrm{id} + g_1 t + g_2 t^2 + \cdots = \sum g_i t^i : A \to A[[t]];$$

therefore, the appropriate natural $k[[t]]$-linear map $(A[[t]], \star) \to (A[[t]], \star')$ is a multiplicative isomorphism.

A deformation $\{B_i\}$ is called *trivial* if $\{B_i\}$ is equivalent to $\{B_0, 0, 0, \dots\}$. In this case, the deformed algebra $(A[[t]], \star)$ is isomorphic to $A[[t]]$. An algebra A is called *rigid* if each deformation is trivial.

As was mentioned above, the Hochschild complex $C^*(A, A)$ for an algebra A is a homotopy G-algebra. Then the tensor product

$$C^*(A, A)[[t]] = C^*(A, A) \otimes k[[t]]$$

is a *bigraded* Hirsch algebra:

$$C^{p,q} = C^p(A, A) \cdot t^q, \quad d(f \cdot t^q) = \delta f \cdot t^q, \quad f \cdot t^p \cup g \cdot t^q = (f \cup g) \cdot t^{p+q},$$
$$E_{1k}(f \cdot t^p | g_1 \cdot t^{q_1} \otimes \cdots \otimes g_k \cdot t^{q_k}) = E_{1k}(f | g_1 \otimes \cdots \otimes g_k) \cdot t^{p+q_1+\cdots+q_k}.$$

Each deformation $\left\{B_i : \bigotimes^i A \to A,\ i = 1, 2, 3, \dots\right\}$ can be interpreted as a twisting element $B = B_1 \cdot t + B_2 \cdot t^2 + \cdots \in C^{2,*}$: the associativity condition (24) can be rewritten in the form

$$\delta B_n \cdot t^n = \sum_{i+j=n} B_i \cdot t^i \cup_1 B_j \cdot t^j.$$

If two deformations are equivalent, then the appropriate Hochschild twisting elements B and B' are also equivalent and condition (25) can be rewritten in the form

$$B' = B + \delta g + g \cup g + g \cup_1 B + \sum_{k=1}^{\infty} E_{1,k}(B' | g \otimes \cdots \otimes g).$$

Thus, the set of equivalence classes of deformations is bijective to $D'(C^{*,*})$. It is clear that

$$H^{p,q}(C^{*,*}) = \mathrm{Hoch}^p(A, A) \cdot t^q;$$

then Proposition 2 implies the result of Gerstenhaber: if $\mathrm{Hoch}^2(A, A) = 0$, then A is rigid.

4.3. Degeneracy of $A(\infty)$-algebras. In this section, we study the problem of degeneracy of an $A(\infty)$-algebra structure by using the homotopy G-algebra structure in the Hochschild complex. Actually, these results are given in [13, 14].

4.3.1. $A(\infty)$**-algebras.** The notion of $A(\infty)$-algebra was introduced by Stasheff in [20]. This notion generalizes the notion of DG-algebra.

An $A(\infty)$-algebra is a graded module M with a given sequence of operations

$$\left\{m_i : \left(\bigotimes^i M\right) \to M,\ i = 1, 2, \ldots,\ \deg m_i = 2 - i\right\},$$

which satisfies the condition

$$\sum_{i+j=n+1} \sum_{k=0}^{n-j} m_i(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_n) = 0. \tag{26}$$

In particular, for the operation $m_1 : M \to M$ we have $\deg m_1 = 1$ and $m_1 m_1 = 0$; thus, m_1 can be considered as a differential on M. The operation $m_2 : M \otimes M \to M$ has degree 0 and satisfies the condition

$$m_1 m_2(a_1 \otimes a_2) + m_2(m_1 a_1 \otimes a_2) + m_2(a_1 \otimes m_1 a_2) = 0,$$

i.e., m_2 can be considered as a multiplication on M and m_1 is a derivation with respect to it. Thus, (M, m_1, m_2) is a DG-algebra (maybe nonassociative). For the operation m_3, we have $\deg m_3 = -1$ and

$$m_1 m_3(a_1 \otimes a_2 \otimes a_3) + m_3(m_1 a_1 \otimes a_2 \otimes a_3) + m_3(a_1 \otimes m_1 a_2 \otimes a_3)$$
$$+ m_3(a_1 \otimes a_2 \otimes m_1 a_3) + m_2(m_2(a_1 \otimes a_2) \otimes a_3) + m_2(a_1 \otimes m_2(a_2 \otimes a_3)) = 0;$$

thus, the product m_2 is *homotopy associative* and the appropriate chain homotopy is m_3 (some authors call $A(\infty)$-algebras *strong homotopy associative DG-algebras*).

The main meaning of defining condition (26) of an $A(\infty)$-algebra $(M, \{m_i\})$ is as follows. The sequence of operations $\{m_i\}$ determines on the bar construction

$$BM = T^c(s^{-1}M) = \Lambda + s^{-1}M + s^{-1}M \otimes s^{-1}M + s^{-1}M \otimes s^{-1}M \otimes s^{-1}M + \ldots$$

a coderivation

$$d_m(a_1 \otimes \cdots \otimes a_n) = \sum_{k,j} a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_n),$$

and the Stasheff condition (26) is equivalent to $d_m d_m = 0$; thus, (BM, d_m) is a DG-coalgebra called a *bar construction* of the $A(\infty)$-algebra $(M, \{m_i\})$.

A morphism of $A(\infty)$-algebras $f : (M, \{m_i\}) \to (M', \{m'_i\})$ is defined as a DG-coalgebra map of the bar constructions

$$f : B(M, \{m_i\}) \to B(M', \{m'_i\}),$$

which (since the tensor coalgebra $T^c(s^{-1}M)$ is cofree) is uniquely determined by the projection

$$f : B(M, \{m_i\}) \to B(M', \{m'_i\}) \to M',$$

which, in fact, is a collection of homomorphisms

$$\left\{f_i : \left(\bigotimes^i M\right) \to M',\ i = 1, 2, \ldots,\ \deg f_i = 1 - i\right\},$$

subject to some conditions (see, e.g., [12,14]). In particular, $f_1 m_1 = m_1 f_1$, i.e., $f_1 : (M, m_1) \to (M', m'_1)$ is a chain map. We define a weak equivalence of $A(\infty)$-algebras as a morphism $\{f_i\}$, where f_1 is a homology isomorphism.

An $A(\infty)$-algebra $(M, \{m_i\})$ is called *minimal* if $m_1 = 0$; in this case, (M, m_2) is a *strictly* associative graded algebra. Assume that

$$f : (M, \{m_i\}) \to (M', \{m'_i\})$$

is a weak equivalence of minimal $A(\infty)$-algebras. Then $f_1 : (M, m_1 = 0) \to (M', m'_1 = 0)$, which by definition is a weak equivalence, is an isomprphism. It is easy to verify that in this case f is an isomorphism of $A(\infty)$-algebras; thus, a weak equivalence of minimal $A(\infty)$-algebras is an isomorphism. This fact motivates the word *minimal* in this notion.

Now assume that $(H, \{m_i\})$ is a *minimal* $(m_1 = 0)$ $A(\infty)$-algebra. Such an $A(\infty)$-algebra is called *degenerate* if it is isomorphic to the $A(\infty)$-algebra $(M, \{0, m_2, 0, 0, \dots\})$, i.e., to the ordinary associative graded algebra (M, m_2). We discuss below the problem of degeneracy of such $A(\infty)$-algebras.

4.3.2. Hochschild cohomology and $A(\infty)$-algebra structures. Assume that $(H, \mu : H \otimes H \to H)$ is a graded algebra. We consider the Hochschild cochain complex of H with coefficients in itself, which is bigraded in this case: $C^{m,n}(H,H) = \mathrm{Hom}^n\left(\bigotimes^m H, H\right)$. It is clear that the coboundary operator δ maps $C^{m,n}(H,H)$ to $C^{m+1,n}(H,H)$. Let us denote the nth homology module of the complex $(C^{*,k}(H,H), \delta)$ by $\mathrm{Hoch}^{n,k}(H,H)$.

In addition, for $f \in C^{m,n}(H,H)$ and $g \in C^{p,q}(H,H)$, one has $f \cup g \in C^{m+p,n+q}(H,H)$ and $f \cup_1 g \in C^{m+p-1,n+q}(H,H)$. Moreover, the above constructed operations $\{E_{1k}\}$, which form the structure of a homotopy G-algebra on the Hochschild complex, behave with bigrading as follows:

$$E_{1k}(f|g_1 \otimes \dots \otimes g_k) \in C^{m+p_1+\dots+p_k-k,n+q_1+\dots+q_k}(H,H);$$

thus, the Hochschild complex $C^{*,*}(H,H)$ is a *bigraded* homotopy G-algebra in this case.

Now assume that $(H, \{m_i\})$ is a *minimal* $(m_1 = 0)$ $A(\infty)$-algebra with $m_2 = \mu$. Each operation $m_i : \left(\bigotimes^i H\right) \to H$ can be considered as a Hochschild cochain in $C^{i,2-i}(H,H)$. Condition (26) can be rewritten as

$$\delta m_k = \sum_{i=3}^{k-1} m_i \cup_1 m_{k-i+2};$$

thus, $m = m_3 + m_4 + \dots$ is a *twisting element* in $C^{*,*}(H,H)$ and, therefore, each minimal $A(\infty)$-algebra structure on H can be considered as a Hochscild twisting element and vice versa.

Assume that $(H, \{m_i\})$ and $(H, \{m'_i\})$ are two minimal $A(\infty)$-algebras. Then the appropriate twisting elements m and m' lie in the same orbit if and only if $A(\infty)$-algebras $(H, \{m_i\})$ and $(H, \{m'_i\})$ are isomorphic: if $m' = p * m$, then $\{p_i\} : (H, \{m_i\}) \to (H, \{m'_i\})$ with $p_0 = \mathrm{id}$ is an *isomorphism* of $A(\infty)$-algebras. Thus, using Proposition 1, we obtain the following theorem.

Theorem 3. *If, for a graded algebra (H, μ), its Hochschild cohomology modules* $\mathrm{Hoch}^{n,2-n}(H,H)$ *are trivial for $n \geq 3$, then each minimal $A(\infty)$-algebra structure $\{m_i\}$ on H is* degenerate*, i.e., there exists an isomorphism of $A(\infty)$-algebras*

$$(H, \{m_i\}) \cong (H, \{m_2 = \mu, 0, 0, \dots\}).$$

4.3.3. $A(\infty)$-algebra structure in homologies of a DG-algebra. Let (A, d, μ) be a DG-algebra and $(H(A), \mu^*)$ be its homology algebra. Although the product in $H(A)$ is associative, there appears the structure of a (generally nondegenerate) minimal $A(\infty)$-algebra, which extends the usual structure of a graded algebra of $H(A)$. Namely, in [12] the following result was proved (see also [8, 19]).

Theorem 4. *If for a DG-algebra, all homology Λ-modules $H_i(A)$ are free, then there exist the structure of a minimal $A(\infty)$-algebra $(H(A), \{m_i\})$ on $H(A)$ and a weak equivalence of $A(\infty)$-algebras*

$$\{f_i\} : (H(A), \{m_i\}) \to (A, \{m_1 = d, m_2 = \mu, 0, 0, \dots\})$$

such that $m_1 = 0$, $m_2 = \mu^*$, and $f_1^* = \mathrm{id}_{H(A)}$. This structure is unique up to isomorphism in the category of $A(\infty)$-algebras.

In particular, such an $A(\infty)$-algebra structure appears in cohomologies of a space or in homologies of a topological group of an H-space. It is clear that the cohomology algebra (or Pontryagin algebra) equipped with such an $A(\infty)$-algebra structure carries more information about the space than the cohomology algebra itself. Some applications of this structure are given in [12, 14].

Therefore, the cases where this additional structure is not needed are especially interesting; in these cases, the $A(\infty)$-algebra $(H(A), \{m_i\})$ is degenerate and a DG-algebra A is called *formal.* Theorem 3 gives a sufficient condition of formality of A in terms of the Hochschild cohomology of $H(A)$.

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