Homology, Homotopy and Applications, vol.7(2), 2005, pp.109–122

ON THE COBAR CONSTRUCTION OF A BIALGEBRA

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(*communicated by Tom Lada*)

Abstract

We show that the cobar construction of a DG-bialgebra is a homotopy G-algebra. This implies that the bar construction of this cobar is a DG-bialgebra as well.

1. Introduction

The cobar construction ΩC of a DG-coalgebra $(C, d : C \to C, \Delta : C \to C \otimes C)$ is, by definition, a DG-algebra. Suppose now that C is additionally equipped with a multiplication $\mu : C \otimes C \to C$ turning (C, d, Δ, μ) into a DG-bialgebra. How does this multiplication reflect on the cobar construction ΩC? It was shown by Adams [**1**] that in the mod 2 situation in this case, the multiplication of ΩC is homotopy commutative: there exists a $\smile_1$ product

$$\smile_1 : \Omega C \otimes \Omega C \to \Omega C$$

which satisfies the standard condition

$$d(a \smile_1 b) = da \smile_1 b + a \smile_1 db + a \cdot b + b \cdot a, \tag{1}$$

(since we work mod 2 the signs are ignored in the whole paper). In this note we show that this $\smile_1$ gives rise to a sequence of operations

$$E_{1,k} : \Omega C \otimes (\Omega C)^{\otimes k} \to \Omega C,\ \ k = 1, 2, 3, ...$$

which form on the cobar construction ΩC of a DG-bialgebra, a structure of *homotopy G-algebra* (hGa) in the sense of Gerstenhaber and Voronov [**8**].

There are two remarkable examples of homotopy G-algebras. The first one is the cochain complex of a 1-reduced simplicial set $C^*(X)$. The operations $E_{1,k}$ here are dual to cooperations defined by Baues in [**2**], and the starting operation $E_{1,1}$ is the classical Steenrod's $\smile_1$ product.

The second example is the Hochschild cochain complex $C^*(U, U)$ of an associative algebra U. The operations $E_{1,k}$ here were defined in [**11**] with the purpose of describing $A(\infty)$-algebras in terms of Hochschild cochains although the properties of those operations which were used as defining ones for the notion of homotopy G-algebra in [**8**] did not appear there. These operations were defined also in [**9**]. Again the starting operation $E_{1,1}$ is the classical Gerstenhaber's circle product which is sort of a $\smile_1$-product in the Hochschild complex.

Received November 13, 2003, revised December 1, 2003; published on April 22, 2005.
2000 Mathematics Subject Classification: 55P48.
Key words and phrases: Cobar construction, DG-bialgebra, Homotopy G-algebra.

In this paper we present a third example of a homotopy G-algebra: we construct the operations $E_{1,k}$ on the cobar construction ΩC of a DG-bialgebra C, and the starting operation $E_{1,1}$ is again classical, it is Adams's $\smile_1$-product.

The notion of hGa was introduced in [**8**] as an additional structure on a DG-algebra $(A, d, \cdot)$ that induces a Gerstenhaber algebra structure on homology. The source of the defining identities and the main example was the Hochschild cochain complex $C^*(U, U)$. Another point of view is that hGa is a particular case of $B(\infty)$-algebra. This is an additional structure on a DG-algebra $(A, d, \cdot)$ that induces a DG-bialgebra structure on the bar construction BA.

We emphasize the third aspect of hGa: this is a structure which measures the noncommutativity of A. There exists the classical tool which measures the noncommutativity of a DG-algebra $(A, d, \cdot)$, namely the Steenrod's $\smile_1$ product, satisfying the condition (1). The existence of such $\smile_1$ guarantees the commutativity of $H(A)$, but the $\smile_1$ product satisfying just the condition (1) is too poor for most applications. In many constructions some deeper properties of $\smile_1$ are needed, for example the compatibility with the dot product of A (the Hirsch formula)

$$(a \cdot b) \smile_1 c + a \cdot (b \smile_1 c) + (a \smile_1 c) \cdot b = 0. \tag{2}$$

For a hGa $(A, d, \cdot, \{E_{1,k}\})$ the starting operation $E_{1,1}$ is a kind of $\smile_1$ product: it satisfies the conditions (1) and (2). As for the symmetric expression

$$a \smile_1 (b \cdot c) + b \cdot (a \smile_1 c) + (a \smile_1 b) \cdot c,$$

it is just *homotopical to zero* and the appropriate homotopy is the operation $E_{1,2}$. The defining conditions of a hGa which satisfy higher operations $E_{1,k}$ can be regarded as generalized Hirsch formulas. So we can say that a hGa is a DG-algebra with a "good" $\smile_1$ product.

2. Notation and preliminaries

We work over Z_2. For a graded Z_2-module M we denote by sM the suspension of M, i.e. $(sM)^i = M^{i-1}$. Respectively $s^{-1}M$ denotes the desuspension of M, i.e. $(s^{-1}M)^i = M^{i+1}$. A *differential graded algebra* (DG-algebra) is a graded R-module $C = \{C^i\},\ i \in Z$, with an associative multiplication $\mu : C^i \otimes C^j \to C^{i+j}$ and a homomorphism (a differential) $d : C^i \to C^{i+1}$ with $d^2 = 0$ and satisfying the Leibniz rule $d(x \cdot y) = dx \cdot y + x \cdot dy$, where $x \cdot y = \mu(x \otimes y)$. We assume that a DG-algebra contains a unit $1 \in C^0$. A non-negatively graded DG-algebra C is *connected* if $C^0 = Z_2$. A connected DG-algebra C is *n-reduced* if $C^i = 0, 1 \leqslant i \leqslant n$. A DG-algebra is *commutative* if $\mu = \mu T$, where $T(x \otimes y) = y \otimes x$.

A *differential graded coalgebra* (DG-coalgebra) is a graded Z_2-module $C = \{C_i\},\ i \in Z$, with a coassociative comultiplication $\Delta : C \to C \otimes C$ and a homomorphism (a differential) $d : C_i \to C_{i+1}$ with $d^2 = 0$ and satisfying $\Delta d = (d \otimes id + id \otimes d)\Delta$. A DG-coalgebra C is assumed to have a counit $\epsilon : C \to Z_2,\quad (\epsilon \otimes id)\Delta = (id \otimes \epsilon)\Delta = id$. A non-negatively graded dgc C is *connected* if $C_0 = Z_2$. A connected DG-coalgebra C is *n-reduced* if $C_i = 0, 1 \leqslant i \leqslant n$. A *differential graded bialgebra* (DG-bialgebra) (C, d, μ, Δ) is a DG-coalgebra (C, d, Δ) with a morphism of DG-coalgebras $\mu : C \otimes C \to C$ turning (C, d, μ) into a DG-algebra.

2.1. Cobar and Bar constructions

Let M be a graded Z_2-vector space with $M^{i\leqslant 0}=0$ and let $T(M)$ be the tensor algebra of M, i.e. $T(M)=\oplus_{i=0}^{\infty}M^{\otimes i}$.

$T(M)$ is a free graded algebra: for a graded algebra A and a homomorphism $\alpha : M\to A$ of degree zero there exists its *multiplicative extension*, a unique morphism of graded algebras $f_\alpha : T(M)\to A$ such that $f_\alpha(a)=\alpha(a)$. The map f_α is given by $f_\alpha(a_1\otimes ...\otimes a_n)=\alpha(a_1)\cdot ...\cdot\alpha(a_n)$. Dually, let $T^c(M)$ be the tensor coalgebra of M, i.e. $T^c(M)=\oplus_{i=0}^{\infty}M^{\otimes i}$, and the comultiplication $\nabla : T^c(M)\to T^c(M)\otimes T^c(M)$ is given by

$$\nabla(a_1\otimes ...\otimes a_n)=\sum_{k=0}^{n}(a_1\otimes ...\otimes a_k)\otimes(a_{k+1}\otimes ...\otimes a_n).$$

$(T^c(M),\nabla)$ is a cofree graded coalgebra: for a graded coalgebra C and a homomorphism $\beta : C\to M$ of degree zero there exists its *comultiplicative extension*, a unique morphism of graded coalgebras $g_\beta : C\to T^c(M)$ such that $p_1g_\beta=\beta$, here $p_1 : T^c(M)\to M$ is the clear projection. The map g_β is given by

$$g_\beta(c)=\sum_n \beta(c^{(1)})\otimes ...\otimes\beta(c^{(n)}),$$

where $\Delta^n(c)=c^{(1)}\otimes ...\otimes c^{(n)}$ and $\Delta^n : C\to C^{\otimes n}$ is n-th iteration of the diagonal $\Delta : C\to C\otimes C$, i.e. $\Delta^1=id,\ \Delta^2=\Delta,\ \Delta^n=(\Delta^{n-1}\otimes id)\Delta$.

Let (C,d_C,Δ) be a connected DG-coalgebra and $\Delta=id\otimes 1+1\otimes id+\Delta'$. The (reduced) *cobar construction* ΩC on C is a DG-algebra whose underlying graded algebra is $T(sC^{>0})$. An element $(sc_1\otimes ...\otimes sc_n)\in(sC)^{\otimes n}\subset T(sC^{>0})$ is denoted by $[c_1,...,c_n]\in\Omega C$. The differential on ΩC is the sum $d=d_1+d_2$ which for a generator $[c]\in\Omega C$ is defined by $d_1[c]=[d_C(c)]$ and $d_2[c]=\sum[c',c'']$ for $\Delta'(c)=\sum c'\otimes c''$, and extended as a derivation. Let (A,d_A,μ) be a 1-reduced DG-algebra. The (reduced) *bar construction* BA on A is a DG-coalgebra whose underlying graded coalgebra is $T^c(s^{-1}A^{>0})$. Again an element $(s^{-1}a_1\otimes ...\otimes s^{-1}a_n)\in(s^{-1}A)^{\otimes n}\subset T^c(s^{-1}A^{>0})$ we denote as $[a_1,...,a_n]\in BA$. The differential of BA is the sum $d=d_1+d_2$ which for an element $[a_1,...a_n]\in BA$ is defined by

$$d_1[a_1,...,a_n]=\sum_{i=1}^{n}[a_1,...,d_Aa_i,...,a_n],\ d_2[a_1,...,a_n]=\sum_{i=1}^{n-1}[a_1,...,a_i\cdot a_{i+1},...,a_n].$$

2.2. Twisting cochains

Let (C,d,Δ) be a dgc, (A,d,μ) a dga. A twisting cochain [**5**] is a homomorphism $\tau : C\to A$ of degree $+1$ satisfying the Browns' condition

$$d\tau+\tau d=\tau\smile\tau, \tag{3}$$

where $\tau\smile\tau'=\mu_A(\tau\otimes\tau')\Delta$. We denote by $T(C,A)$ the set of all twisting cochains $\tau : C\to A$.

There are universal twisting cochains $C\to\Omega C$ and $BA\to A$ being clear inclusion and projection respectively. Here are essential consequences of the condition (3):
(i) The multiplicative extension $f_\tau : \Omega C\to A$ is a map of DG-algebras, so there is a bijection $T(C,A)\leftrightarrow Hom_{DG-Alg}(\Omega C,A)$;

(ii) The comultiplicative extension $g_\tau : C \to BA$ is a map of DG-coalgebras, so there is a bijection $T(C,A) \leftrightarrow Hom_{DG-Coalg}(C, BA)$.

3. Homotopy G-algebras

3.1. Products in the bar construction

Let $(A, d, \cdot)$ be a 1-reduced DG-algebra and BA its bar construction. We are interested in the structure of a multiplication

$$\mu : BA \otimes BA \to BA,$$

turning BA into a DG-bialgebra, i.e. we require that

(i) μ is a DG-coalgebra map;

(ii) is associative;

(iii) has the unit element $1_\Lambda \in \Lambda \subset BA$.

Because of the cofreeness of the tensor coalgebra $BA = T^c(s^{-1}A)$, a map of graded coalgebras

$$\mu : BA \otimes BA \to BA$$

is uniquely determined by the projection of degree $+1$

$$E = pr \cdot \mu : BA \otimes BA \to BA \to A.$$

Conversely, a homomorphism $E : BA \otimes BA \to A$ of degree $+1$ determines its coextension, a graded coalgebra map $\mu_E : BA \otimes BA \to BA$ given by

$$\mu_E = \sum_{k=0}^{\infty} (E \otimes ... \otimes E) \nabla^k_{BA\otimes BA},$$

where $\nabla^k_{BA\otimes BA} : BA \otimes BA \to (BA \otimes BA)^{\otimes k}$ is the k-fold iteration of the standard coproduct of tensor product of coalgebras

$$\nabla_{BA\otimes BA} = (id \otimes T \otimes id)(\nabla \otimes \nabla) : BA \otimes BA \to (BA \otimes BA)^{\otimes 2}.$$

The map μ_E is a *chain map* (i.e. it is a map of DG-coalgebras) if and only if E is a twisting cochain in the sense of E. Brown, i.e. satisfies the condition

$$dE + Ed_{BA\otimes BA} = E \smile E. \tag{4}$$

Indeed, again because of the cofreeness of the tensor coalgebra $BA = T^c(s^{-1}A)$ the condition $d_{BA}\mu_E = \mu_E d_{BA\otimes BA}$ is satisfied if and only if it is satisfied after the projection on A, i.e. if $pr \cdot d_{BA}\mu_E = pr \cdot \mu_E d_{BA\otimes BA}$ but this condition is nothing else than the Brown's condition (4).

The same argument shows that the product μ_E *is associative* if and only if $pr \cdot \mu_E(\mu_E \otimes id) = pr \cdot \mu_E(id \otimes \mu_E)$, or, having in mind $E = pr \cdot \mu_E$

$$E(\mu_E \otimes id) = E(id \otimes \mu_E). \tag{5}$$

A homomorphism $E : BA \otimes BA \to A$ consists of *components*

$$\{\bar{E}_{p,q} : (s^{-1}A)^{\otimes p} \otimes (s^{-1}A)^{\otimes q} \to A,\ p, q = 0, 1, 2, ...\},$$

where $\bar{E}_{pq}$ is the restriction of E on $(s^{-1}A)^{\otimes p} \otimes (s^{-1}A)^{\otimes q}$. Each component $\bar{E}_{p,q}$ can be regarded as an operation

$$E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A,\ p, q = 0, 1, 2, ...$$

The value of $E_{p,q}$ on the element $(a_1 \otimes ... \otimes a_p) \otimes (b_1 \otimes ... \otimes b_q)$ we denote by $E_{p,q}(a_1, ..., a_p; b_1 ..., b_q)$.

It is not hard to check that the multiplication μ_E induced by E (or equivalently by a collection of multioperations $\{E_{p,q}\}$) *has the unit* $1_\Lambda \in \Lambda \subset BA$ if and only if

$$E_{0,1} = E_{1,0} = id; \quad E_{0,k} = E_{k,0} = 0,\ k > 1. \tag{6}$$

So we can summarize:

Proposition 1. *The multiplication μ_E induced by a collection of multioperations $\{E_{p,q}\}$ turns BA into a DG-bialgebra, i.e. satisfies (i-iii), if and only if the conditions (4), (5), and (6) are satisfied.*

Let us interpret the condition (4) in terms of the components E_{pq}. The restriction of (4) on $A \otimes A$ gives

$$dE_{1,1}(a; b) + E_{1,1}(da; b) + E_{1,1}(a; db) = a \cdot b + b \cdot a. \tag{7}$$

This condition coincides with the condition (1), i.e. the operation $E_{1,1}$ is sort of a $\smile_1$ product, which measures the noncommutativity of A. Below we denote $E_{1,1}(a; b) = a \smile_1 b$.

The restriction on $A^{\otimes 2} \otimes A$ gives

$$\begin{gathered} dE_{2,1}(a, b; c) + E_{2,1}(da, b; c) + E_{2,1}(a, db; c) + E_{2,1}(a, b; dc) = \\ (a \cdot b) \smile_1 c + a \cdot (b \smile_1 c) + (a \smile_1 c) \cdot b, \end{gathered} \tag{8}$$

this means, that this $\smile_1$ satisfies the *left Hirsch formula* (2) up to homotopy and the appropriate homotopy is the operation $E_{2,1}$.

The restriction on $A \otimes A^{\otimes 2}$ gives:

$$\begin{gathered} dE_{1,2}(a; b, c) + E_{1,2}(da; b, c) + E_{1,2}(a; db, c) + E_{1,2}(a; b, dc) = \\ a \smile_1 (b \cdot c) + (a \smile_1 b) \cdot c + b \cdot (a \smile_1 c), \end{gathered} \tag{9}$$

this means, that this $\smile_1$ satisfies the *right Hirsch formula* (2) up to homotopy and the appropriate homotopy is the operation $E_{1,2}$.

Generally the restriction of (4) on $A^{\otimes m} \otimes A^{\otimes n}$ gives:

$$\begin{gathered} dE_{m,n}(a_1, ..., a_m; b_1, ..., b_n) + \sum_i E_{m,n}(a_1, ..., da_i, ..., a_m; b_1, .., b_n) \\ + \sum_i E_{m,n}(a_1, ..., a_m; b_1, ..., db_i, .., b_n) = \\ a_1 \cdot E_{m-1,n}(a_2, ..., a_m; b_1, ..., b_n) + E_{m-1,n}(a_1, ..., a_{m-1}; b_1, ..., b_n) \cdot a_m \\ +b_1 \cdot E_{m,n-1}(a_1, ..., a_m; b_2, ..., b_n) + E_{m,n-1}(a_1, ..., a_m; b_1, ..., b_{n-1}) \cdot b_m + \\ \sum_i E_{m-1,n}(a_1, ..., a_i \cdot a_{i+1}, ..., a_m; b_1, ..., b_n) + \\ \sum_i E_{m,n-1}(a_1, ..., a_m; b_1, ..., b_i \cdot b_{i+1}, ..., b_n) + \\ \sum_{p=1}^{m-1} \sum_{q=1}^{n-1} E_{p,q}(a_1, ..., a_p; b_1, .., b_q) \cdot E_{m-p,n-q}(a_{p+1}, ..., a_m; b_{q+1}, ..., b_n). \end{gathered} \tag{10}$$

Now let us interpret the associativity condition (5) in terms of the components

$E_{p,q}$. The restriction of (5) on $A \otimes A \otimes A$ gives

$$\begin{array}{c}(a \smile_1 b) \smile_1 c + a \smile_1 (b \smile_1 c) = E_{1,2}(a; b, c) + E_{1,2}(a; c, b)+ \\ E_{2,1}(a, b; c) + E_{2,1}(b, a; c).\end{array} \tag{11}$$

Generally the restriction of (5) on $A^{\otimes k} \otimes A^{\otimes l} \otimes A^{\otimes m}$ gives

$$\begin{array}{c}\sum_{r=1}^{l+m} \sum_{l_1+...+l_r=l, m_1+...+m_r=m} \\ E_{k,r}(a_1, ..., a_k; E_{l_1,m_1}(b_1, ..., b_{l_1}; c_1, ..., c_{m_1}), ..., \\ E_{l_r,m_r}(b_{l_1+...+l_{r-1}+1}, ..., b_l; c_{m_1+...+m_{r-1}+1}, ..., c_m) = \\ \sum_{s+1}^{k+l} \sum_{k_1+...+k_s=k, l_1+...+l_s=l} \\ E_{s,m}(E_{k_1 l_1}(a_1, ..., a_{k_1}; b_1, ..., b_{l_1}), ..., \\ E_{k_s,l_s}(a_{k_1+...+k_{s-1}+1}, ..., a_k; b_{l_1+...+l_{s-1}+1}, ..., b_l); c_1, ..., c_m)\end{array} \tag{12}$$

We define a *Hirsch algebra* as a DG-algebra $(A, d, \cdot)$ endowed with a sequence of multioperations $\{E_{p,q}\}$ satisfying (6), (10). This name is inspired by the fact that the defining condition (10) can be regarded as generalizations of classical Hirsch formula (2). This notion was used in [**12**], [**13**].

A Hirsch algebra we call *associative* if in addition the condition (12) is satisfied.

This structure is a particular case of a B_∞-algebra, see below. Moreover, the notion of *homotopy G-algebra*, described below, is a particular case of an associative Hirsch algebra.

3.2. Some particular cases

For a Hirsch algebra $(A, d, \cdot, \{E_{p,q}\})$ the operation $E_{1,1} = \smile_1$ satisfies (1), so this structure can be considered as a tool which measures the noncommutativity of the product $a \cdot b$ of A. We distinguish various levels of "noncommutativity" of A according to the form of $\{E_{p,q}\}$.

Level 1. Suppose for the collection $\{E_{p,q}\}$ all the operations except $E_{0,1} = id$ and $E_{1,0} = id$ are trivial. Then it follows from (7) that in this case A is a *strictly* commutative DG-algebra.

Level 2. Suppose all operations except $E_{0,1} = id$, $E_{1,0} = id$ and $E_{1,1}$ are trivial. In this case A is endowed with a "strict" $\smile_1$ product $a \smile_1 b = E_{1,1}(a; b)$: the condition (10) here degenerate to the following 4 conditions

$$d(a \smile_1 b) = da \smile_1 b + a \smile_1 db + a \cdot b + b \cdot a,$$

$$(a \cdot b) \smile_1 c + a \cdot (b \smile_1 c) + (a \smile_1 c) \cdot b = 0,$$

$$a \smile_1 (b \cdot c) + b \cdot (a \smile_1 c) + (a \smile_1 b) \cdot c = 0,$$

$$(a \smile_1 c) \cdot (b \smile_1 d) = 0.$$

The condition (12) degenerates to the associativity $\smile_1$

$$a \smile_1 (b \smile_1 c) = (a \smile_1 b) \smile_1 c.$$

As we see in this case we have very strong restrictions on the $\smile_1$-product. An example of a DG-algebra with such strict $\smile_1$ product is $(H^*(SX, Z_2), d = 0)$ with

$a \smile_1 b = 0$ if $a \neq b$ and $a \smile_1 a = Sq^{|a|-1}a$; another example is $C^*(SX, CX)$, where SX is the suspension and CX is the cone of a space X (see [**18**]).

Level 3. Suppose all operations except $E_{0,1} = id$, $E_{1,0} = id$ and $E_{1,k}$, $k = 1, 2, 3, ...$ are trivial. In this case the condition (10) degenerates into two conditions: at $A \otimes A^{\otimes k}$

$$\begin{gathered} dE_{1,k}(a; b_1, ..., b_k) + E_{1,k}(da; b_1, ..., b_k) + \textstyle\sum_i E_{1,k}(a; b_1, ..., db_i, ..., b_k) = \\ b_1 \cdot E_{1,k-1}(a; b_2, ..., b_k) + E_{1,k-1}(a; b_1, ..., b_{k-1}) \cdot b_k + \\ \textstyle\sum_i E_{1,k-1}(a; b_1, ..., b_i \cdot b_{i+1}, ..., b_k), \end{gathered} \tag{13}$$

and at $A^{\otimes 2} \otimes A^{\otimes k}$

$$\begin{gathered} E_{1,k}(a_1 \cdot a_2; b_1, .., b_k) = a_1 \cdot E_{1,k}(a_2; b_1, ..., b_k) + E_{1,k}(a_1; b_1, ..., b_k) \cdot a_2 + \\ \textstyle\sum_{p=1}^{k-1} E_{1,p}(a_1; b_1, ..., b_p) \cdot E_{1,m-p}(a_2; b_{p+1}, ..., b_k); \end{gathered} \tag{14}$$

moreover at $A^{\otimes n>2} \otimes A^{\otimes k}$ the condition is trivial. In particular the condition (8) here degenerates to Hirsch formula (2).

The associativity condition (12) in this case looks like

$$\begin{gathered} E_{1,n}(E_{1,m}(a; b_1, ..., b_m); c_1, ..., c_n) = \textstyle\sum_{0 \leqslant i_1 \leqslant ... \leqslant i_m \leqslant n} \sum_{0 \leqslant n_1 + ... + n_r \leqslant n} \\ E_{1,n-(n_1+...+n_j)+j}(a; c_1, ..., c_{i_1}, E_{1,n_1}(b_1; c_{i_1+1}, ..., c_{i_1+n_1}), c_{i_1+n_1+1}, ..., \\ c_{i_2}, E_{1,n_2}(b_2; c_{i_2+1}, ..., c_{i_2+n_2}), c_{i_2+n_2+1}, ..., \\ c_{i_m}, E_{1,n_m}(b_m; c_{i_m+1}, ..., c_{i_m+n_m}), c_{i_m+n_m+1}, ..., c_n), \end{gathered} \tag{15}$$

In particular the condition (11) here degenerates to

$$(a \smile_1 b) \smile_1 c + a \smile_1 (b \smile_1 c) = E_{1,2}(a; b, c) + E_{1,2}(a; c, b). \tag{16}$$

The structure of this level coincides with the notion of *Homotopy G-algebra*, see below.

Level 4. As the last level we consider a Hirsch algebra structure with no restrictions. An example of such structure is the cochain complex of a 1-reduced cubical set. Note that it is a *nonassociative* Hirsch algebra.

3.3. B_∞-algebra

The notion of a B_∞−algebra was introduced in [**2**], [**10**] as an additional structure on a DG-algebra $(A, \cdot, d)$ which turns the tensor coalgebra $T^c(s^{-1}A) = BA$ into a DG-bialgebra. So it requires a new differential

$$\widetilde{d} : BA \to BA$$

(which should be a coderivation with respect to standard coproduct of BA) and a new associative multiplication

$$\widetilde{\mu} : (BA, \widetilde{d}) \otimes (BA, \widetilde{d}) \to (BA, \widetilde{d})$$

which should be a map of DG-coalgebras, with $1_\Lambda \in \Lambda \subset BA$ as the unit element.

It is known that such $\widetilde{d}$ specifies on A a structure of A_∞-algebra in the sense of Stasheff [**19**], namely a sequence of operations $\{m_i : \otimes^i A \to A, i = 1, 2, 3, ...\}$ subject of appropriate conditions.

As for the new multiplication $\widetilde{\mu}$, it follows from the above considerations, that it is induced by a sequence of operations $\{E_{pq}\}$ satisfying (6), (12) and the modified condition (10) with involved A_∞-algebra structure $\{m_i\}$.

Thus the structure of associative Hirsch algebra is a particular B_∞-algebra structure on A when the standard differential of the bar construction $d_B : BA \to BA$ *does not change*, i.e. $\widetilde{d} = d_B$ (in this case the corresponding A_∞-algebra structure is degenerate: $m_1 = d_A, m_2 = \cdot, m_3 = 0, m_4 = 0, ...$).

Let us mention that a twisting cochain E satisfying (6) and (4), (but not (5) i.e. the induced product in the bar construction is not strictly associative), was constructed in [**14**] for the singular cochain complex of a topological space $C^*(X)$ using acyclic models. The condition (6) determines this twisting cochain E uniquely up to standard equivalence (homotopy) of twisting cochains in the sense of N. Berikashvili [**4**].

3.4. Strong homotopy commutative algebras

The notion of strong homotopy commutative algebra (shc-algebra), as a tool for measuring of noncommutativity of DG-algebras, was used in many papers: [**17**], [**20**], etc.

A shc-algebra is a DG-algebra $(A, d, \cdot)$ with a given twisting cochain $\Phi : B(A \otimes A) \to A$ which satisfies apropriate up to homotopy conditions of associativity and commutativity. Compare with the Hirsch algebra structure which is represented by a twisting cochain $E : BA \otimes BA \to A$. Standard contraction of $B(A \otimes A)$ to $BA \otimes BA$ allows one to establish a connection between these two notions.

3.5. DG-Lie algebra structure in a Hirsch algebra

A structure of an associative Hirsch algebra on A induces on the homology $H(A)$ a structure of Gerstenhaber algebra (G-algebra) (see [**6**], [**8**], [**21**]) which is defined as a commutative graded algebra $(H, \cdot)$ together with a Lie bracket of degree -1

$$[\ ,\ ] : H^p \otimes H^q \to H^{p+q-1}$$

(i.e. a graded Lie algebra structure on the desuspension $s^{-1}H$) that is a biderivation: $[a, b \cdot c] = [a, b] \cdot c + b \cdot [a, c]$.

The existence of this structure in the homology $H(A)$ is seen by the following argument.

Let $(A, d, \cdot, \{E_{p,q}\})$ be an associative Hirsch algebra, then in the desuspension $s^{-1}A$ there appears a structure of DG-Lie algebra: although the $\smile_1 = E_{1,1}$ is not associative, the condition (11) implies the pre-Jacobi identity

$$a \smile_1 (b \smile_1 c) + (a \smile_1 b) \smile_1 c = a \smile_1 (c \smile_1 b) + (a \smile_1 c) \smile_1 b$$

This condition guarantees that the commutator $[a, b] = a \smile_1 b + b \smile_1 a$ satisfies the Jacobi identity. Besides, condition (7) implies that $[\ ,\ ] : A^p \otimes A^q \to A^{p+q-1}$ is a chain map. Thus on $s^{-1}H(A)$ there appears the structure of graded Lie algebra. The up to homotopy Hirsh formulae (8) and (9) imply that the induced Lie bracket is a biderivation.

3.6. Homotopy G-algebra

An associative Hirsch algebra of level 3 in the literature is known as *Homotopy G-algebra* .

A *Homotopy G-algebra* in [**8**] and [**21**] is defined as a DG-algebra $(A, d, \cdot)$ with a given sequence of multibraces $a\{a_1, ..., a_k\}$ which, in our notation, we regard as a sequence of operations

$$E_{1,k} : A \otimes (\otimes^k A) \to A, \quad k = 0, 1, 2, 3, ...$$

which, together with $E_{01} = id$ satisfies the conditions (6), (13), (14) and (15).

The name *Homotopy G-algebra* is motivated by the fact that this structure induces on the homology $H(A)$ the structure of G-algebra (as we have seen in the previous section such a structure appears even on the homology of an associative Hirsch algebra).

The conditions (13), (14), and (15) in [**8**] are called *higher homotopies*, *distributivity* and *higher pre-Jacobi identities* respectively. As we have seen the first two conditions mean that $E : BA \otimes BA \to A$ is a twisting cochain, or equivalently $\mu_E : BA \otimes BA \to BA$ is a chain map, and the third one means that this multiplication is associative.

3.7. Operadic description

Appropriate language to describe such huge sets of operations is the operadic language. Here we use the *surjection operad* χ and the *Barratt-Eccles operad* $\mathcal{E}$ which are the most convenient E_∞ operads. For definitions we refer to [**3**].

The operations $E_{1,k}$ forming hGa have nice description in the *surjection operad*, see [**15**], [**16**], [**3**]. Namely, to the dot product corresponds the element $(1, 2) \in \chi_0(2)$, to $E_{1,1} = \smile_1$ product corresponds $(1, 2, 1) \in \chi_1(2)$, to the operation $E_{1,2}$ the element $(1, 2, 1, 3) \in \chi_2(3)$, etc. Generally to the operation $E_{1,k}$ corresponds the element

$$E_{1,k} = (1, 2, 1, 3, ..., 1, k, 1, k + 1, 1) \in \chi_k(k + 1). \tag{17}$$

We remark here that the defining conditions of a hGa (13), (14), (15) can be expressed in terms of operadic structure (differential, symmetric group action and composition product) and the elements (17) satisfy these conditions *already in the operad* χ. This in particular implies that *any χ-algebra is automatically a hGa*. Note that the elements (17) together with $(1, 2)$ generate the suboperad $F_2\chi$ which is equivalent to the little square operad. This fact and a hGa structure on the Hochschild cochain complex $C^*(U, U)$ of an algebra U are used by many authors to prove so called Deligne conjecture about the action of the little square operad on $C^*(U, U)$.

Now look at the operations $E_{p,q}$ which define a structure of Hirsch algebra. They *can not live* in χ: it is enough to mention that the Hirsch formula (2), as a part of defining conditions of hGa, is satisfied in χ, but for a Hirsch algebra this condition is satisfied up to homotopy $E_{2,1}$, see (8). We belive that $E_{p,q}$-s live in the Barratt-Eccles operad $\mathcal{E}$. In particular direct calculation shows that

$$\begin{aligned}
&E_{1,1} = ((1, 2), (2, 1)) \in \mathcal{E}_1(2);\\
&E_{1,2} = ((\mathbf{1}, 2, 3), (2, \mathbf{1}, 3), (2, 3, \mathbf{1})) \in \mathcal{E}_2(3);\\
&E_{2,1} = ((1, 2, \mathbf{3}), (1, \mathbf{3}, 2), (\mathbf{3}, 1, 2)) \in \mathcal{E}_2(3);\\
&E_{1,3} = ((\mathbf{1}, 2, 3, 4), (2, \mathbf{1}, 3, 4), (2, 3, \mathbf{1}, 4), (2, 3, 4, \mathbf{1})) \in \mathcal{E}_3(4);\\
&E_{3,1} = ((1, 2, 3, \mathbf{4}), (1, 2, \mathbf{4}, 3), (1, \mathbf{4}, 2, 3), (\mathbf{4}, 1, 2, 3)) \in \mathcal{E}_3(4);
\end{aligned}$$

and in general

$$E_{1,k} = ((\mathbf{1}, 2, ..., k+1), ..., (2, 3, ..., i, \mathbf{1}, i+1, ..., k+1), ..., (2, 3, ..., k+1, \mathbf{1}));$$
$$E_{k,1} = ((1, 2, ..., \mathbf{k+1}), ..., (1, 2, ..., i, \mathbf{k+1}, i+1, ..., k), ..., (\mathbf{k+1}, 1, 2, ..., k)).$$

As for other $E_{p,q}$-s we can indicate just

$$\begin{aligned} E_{2,2} = \quad & ((1,2,3,4),(1,3,4,2),(3,1,4,2),(3,4,1,2))+ \\ & ((1,2,3,4),(3,1,2,4),(3,1,4,2),(3,4,1,2))+ \\ & ((1,2,3,4),(1,3,2,4),(1,3,4,2),(3,1,4,2))+ \\ & ((1,2,3,4),(1,3,2,4),(3,1,2,4),(3,1,4,2)). \end{aligned}$$

We remark that the operadic *table reduction* map $TR : \mathcal{E} \to \chi$, see [**3**], maps $E_{k>1,1}$ and $E_{2,2}$ to zero, and $E_{1,k} \in \mathcal{E}_k(k+1)$ to $E_{1,k} \in \chi_k(k+1)$.

4. Adams $\smile_1$-product in the cobar construction of a bialgebra

Here we present the Adams $\smile_1$-product $\smile_1$: $\Omega A \otimes \Omega A \to \Omega A$ on the cobar construction ΩA of a DG-bialgebra $(A, d, \Delta : A \to A \otimes A, \mu : A \otimes A \to A)$ (see [**1**]). This will be the first step in the construction of an hGa structure on ΩA.

This $\smile_1$ product satisfies the Steenrod condition (1) and the Hirsch formula (2).

First we define the $\smile_1$-product of two elements $x = [a], y = [b] \in \Omega A$ of length 1 as $[a] \smile_1 [b] = [a \cdot b]$. Extending this definition by (2) we obtain

$$[a_1, a_2] \smile_1 [b] = ([a_1] \cdot [a_2]) \smile_1 [b] = [a_1] \cdot ([a_2] \smile_1 [b]) + ([a_1] \smile_1 [b]) \cdot [a_2] = \\ [a_1] \cdot [a_2 \cdot b] + [a_1 \cdot b] \cdot [a_2] = [a_1, a_2 \cdot b] + [a_1 \cdot b, a_2].$$

Further iteration of this process gives

$$[a_1, ..., a_n] \smile_1 [b] = \sum_i [a_1, ..., a_{i-1}, a_i \cdot b, a_{i+1}, ..., a_n].$$

Now let's define $[a] \smile_1 [b_1, b_2] = [a^{(1)} \cdot b, a^{(2)} \cdot b]$ where $\Delta a = a^{(1)} \otimes a^{(2)}$ is the value of the diagonal $\Delta : A \to A \otimes A$ on $[a]$. Inspection shows that the condition (1) for short elements

$$d([a] \smile_1 [b]) = d[a] \smile_1 [b] + [a] \smile_1 d[b] + [a] \cdot [b] + [b] \cdot [a].$$

is satisfied.

Generally we define the $\smile_1$ product of an element $x = [a] \in \Omega A$ of length 1 and an element $y = [b_1, ..., b_n] \in \Omega A$ of arbitrary length by

$$[a] \smile_1 [b_1, ..., b_n] = [a^{(1)} \cdot b_1, ..., a^{(n)} \cdot b_n];$$

here $\Delta^n(a) = a^{(1)} \otimes ... \otimes a^{(n)}$ is the n-fold iteration of the diagonal $\Delta : A \to A \otimes A$ and $a \cdot b = \mu(a \otimes b)$ is the product in A.

Extending this definition for the elements of arbitrary lengths $[a_1, ..., a_m] \smile_1 [b_1, ..., b_n]$ by the Hirsch formula (2) we obtain the general formula

$$[a_1, ..., a_m] \smile_1 [b_1, ..., b_n] = \sum_k [a_1, ..., a_{k-1}, a_k^{(1)} \cdot b_1, ..., a_k^{(n)} \cdot b_n, a_{k+1}, ..., a_m]. \quad (18)$$

Of course, so defined, the $\smile_1$ satisfies the Hirsch formula (2) automatically. It remains to prove the

Proposition 2. *This $\smile_1$ satisfies Steenrod condition (1)*

$$d_\Omega([a_1,...,a_m]\smile_1[b_1,...,b_n]) =$$
$$d_\Omega[a_1,...,a_m]\smile_1[b_1,...,b_n]+[a_1,...,a_m]\smile_1 d_\Omega[b_1,...,b_n]+$$
$$[a_1,...,a_m,b_1,...,b_n]+[b_1,...,b_n,a_1,...,a_m].$$

Proof. Let us denote this condition by $Steen_{m,n}$. The first step consists in direct checking of the conditions $Steen_{1,m}$ by induction on m. Furthermore, assume that $Steen_{m,n}$ is satisfied. Let us check the condition $Steen_{m+1,n}$ for $[a,a_1,...,a_m]\smile_1[b_1,...,b_n]$. We denote $[a_1,...,a_m]=x,\ [b_1,...,b_m]=y$. Using the Hirsch formula (2), $Steen_{m,n}$, and $Steen_{1,n}$ we obtain:

$$d([a,a_1,...,a_m]\smile_1[b_1,...,b_n]) =$$
$$d(([a]\cdot x)\smile_1 y)=d([a]\cdot(x\smile_1 y)+([a]\smile_1 y)\cdot x)=$$
$$=d[a]\cdot(x\smile_1 y)+[a]\cdot(dx\smile_1 y+x\smile_1 dy+x\cdot y+y\cdot x)+$$
$$(d[a]\smile_1 y+[a]\smile_1 dy+[a]\cdot y+y\cdot[a])\cdot x+([a]\smile_1 y)dx=$$
$$d[a]\cdot(x\smile_1 y)+[a]\cdot(dx\smile_1 y)+[a]\cdot(x\smile_1 dy)+[a]\cdot x\cdot y+[a]\cdot y\cdot x+$$
$$(d[a]\smile_1 y\cdot)x+([a]\smile_1 dy\cdot)x+[a]\cdot y\cdot x+y\cdot[a]\cdot x+([a]\smile_1 y)dx.$$

Besides, using Hirsch (2) formula we obtain

$$d[a,a_1,...,a_m]\smile_1[b_1,...,b_n]=$$
$$d([a]\cdot x)\smile_1 y=(d[a]\cdot x)\smile_1 y+([a]\cdot dx)\smile_1 y=$$
$$d[a]\cdot(x\smile_1 y)+(d[a]\smile_1 y)\cdot x+[a]\cdot(dx\smile_1 y)+([a]\smile_1 y)\cdot dx$$

and

$$[a,a_1,...,a_m]\smile_1 d[b_1,...,b_n]=$$
$$([a]\cdot x)\smile_1 dy=[a]\cdot(x\smile_1 dy)+([a]\smile_1 dy)\cdot x,$$

now it is evident that $Steen_{m+1,n}$ is satisfied. This completes the proof. □

5. Homotopy G-algebra structure on the cobar construction of a bialgebra

Below we present a sequence of operations

$$E_{1,k}:\Omega A\otimes(\Omega A)^{\otimes k}\to\Omega A,$$

which extends the above described $E_{1,1}=\smile_1$ to a structure of a homotopy G-algebra on the cobar construction of a DG-bialgebra . This means that $E_{1,k}$-s satisfy the conditions (13), (14) and (15).

For $x = [a] \in \Omega A$ of length 1, $y_i \in \Omega A$ and $k > 1$ we define $E_{1,k}([a]; y_1, ...y_k) = 0$ and extend for an arbitrary $x = [a_1, ...a_n]$ by (14). This gives

$$E_{1,k}([a_1, ..., a_n]; y_1, ..., y_k) = 0$$

for $n < k$ and

$$E_{1,k}([a_1, ..., a_k]; y_1, ..., y_k) = [a_1 \diamond y_1, ..., a_k \diamond y_k],$$

here we use the notation $a \diamond (b_1, ..., b_s) = (a^{(1)} \cdot b_1, ..., a^{(s)} \cdot b_s)$, so using this notation $[a] \smile_1 [b_1, ..., b_s] = [a \diamond (b_1, ..., b_s)]$. Further iteration by (14) gives the general formula

$$\begin{gathered} E_{1,k}([a_1, ..., a_n]; y_1, ..., y_k) = \\ \sum [a_1, ..., a_{i_1 - 1}, a_{i_1} \diamond y_1, a_{i_1+1}, ..., a_{i_k - 1}, a_{i_k} \diamond y_k, a_{i_k+1}, ..., a_n], \end{gathered} \tag{19}$$

where the summation is taken over all $1 \leqslant i_1 < ... < i_k \leqslant n$.

Of course, so defined, the operations $E_{1,k}$ automatically satisfy the condition (14). It remains to prove the

Proposition 3. *The operations $E_{1,k}$ satisfy the conditions (13) and (15).*

Proof. The condition (13) is trivial for $x = [a]$ of length 1 and $k > 2$. For $x = [a]$ and $k = 2$ this condition degenerates to

$$E_{1,1}([a]; y_1 \cdot y_2) + y_1 \cdot E_{1,1}([a]; y_2) - E_{1,1}([a]; y_1) \cdot y_2 = 0$$

and this equality easily follows from the definition of $E_{1,1} = \smile_1$. For a long $x = [a_1, ..., a_m]$ the condition (13) can be checked by induction on the length m of x using the condition (14).

Similarly, the condition (15) is trivial for $x = [a]$ of length 1 unless the case $m = n = 1$ and in this case this condition degenerates to

$$E_{1,1}(E_{1,1}(x; y); z) = E_{1,1}(x; E_{1,1}(y); z)) + E_{1,2}(x; y, z) + E_{1,2}(x; z, y).$$

This equality easily follows from the definition of $E_{1,1} = \smile_1$. For a long $x = [a_1, ..., a_m]$ the condition (15) can be checked by induction on the length m of x using the condition (14). □

Remark 1. *For a DG-coalgebra $(A, d, \Delta : A \to A \otimes A)$ there is a standard DG-coalgebra map $g_A : A \to B\Omega A$ from A to the bar of cobar of A. This map is the coextension of the universal twisting cochain $\phi_A : A \to \Omega A$ defined by $\phi(a) = [a]$ and is a weak equivalence, i.e. it induces an isomorphism of homology. Suppose A is a DG-bialgebra. Then the constructed sequence of operations $E_{1,k}$ define a multiplication $\mu_E : B\Omega A \otimes B\Omega A \to B\Omega A$ on the bar construction $B\Omega A$ so that it becomes a DG-bialgebra. Direct inspection shows that $g_A : A \to B\Omega A$ is multiplicative, so it is a weak equivalence of DG-bialgebras. Dualizing this statement we obtain a weak equivalence of DG-bialgebras $\Omega BA \to A$ which can be considered as a free (as an algebra) resolution of a DG-bialgebra A.*

References

[1] J. Adams, On the non-existence of elements of Hopf invariant one, Ann. Math. 72 (1960), 20-104.

[2] H. Baues, The double bar and cobar construction, Compositio Math.43 (1981), 331-341.

[3] C. Berger, B. Fresse, Combinatorial operad action on cochains, preprint (2001), math.AT/0109158.

[4] N. Berikashvili, On the differentials of spectral sequence, Proc. Tbil. Math. Inst., 51 (1976), 1-105.

[5] E. Brown, Twisted tensor products. Ann. of Math., 69 (1959), 223-246.

[6] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. of Math., 78 (1963), 267-288.

[7] M. Gerstenhaber, On the deformations of rings and algebras, Ann. of Math., 79, 1 (1964), 59-103.

[8] M. Gerstenhaber and A. Voronov, Higher operations on Hochschild complex, Functional Annal. Appl. 29 (1995), 1-6.

[9] E. Getzler, Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology, Israel Math. Conf. proc. 7 (1993), 65-78.

[10] E. Getzler and J.D. Jones, Operads, homotopy algebra, and iterated integrals for double loop spaces, preprint (1994), hep-th/9403055.

[11] T. Kadeishvili, The $A(\infty)$-algebra Structure and Cohomology of Hochschild and Harrison, Proc. of Tbil. Math. Inst., 91, (1988), 19-27.

[12] T. Kadeishvili and S. Saneblidze, On a multiplicative model of a fibration, Bull. of Georg. Acad. Sci., 153 (1996), 345-346.

[13] T. Kadeishvili and S. Saneblidze, The twisted Cartesian model for the double path space fibration, preprint (2002), math. AT/021022.

[14] L. Khelaia, On some chain operations, Bull. Georg. Acad. Sci. 96, 3 (1979), 529-531.

[15] J. McLure, J. Smith, A solution of Deligne's conjecture, preprint (1999), math.QA/9910126.

[16] J. McLure, J. Smith, Multivariable cochain operations and little n-cubes, preprint (2001), math.QA/0106024.

[17] H.J. Munkholm, The Eilenberg-Moore spectral sequence and strongly homotopy multiplicative maps, J.P.A.A., 5 (1974), 1-50.

[18] S. Saneblidze, The Hohchschild complex of a space is the complex of the Hochschild set, preprint (2000).

[19] J.D. Stasheff, Homotopy associativity of H-spaces. I, II, Trans. Amer. Math. Soc., 108 (1963), 275-312.

[20] B. Ndombol and J.-C. Thomas, Products in the Hochschild homology of spaces, Preprint (1998).

[21] A. Voronov, Homotopy Gerstenhaber algebras, preprint (1999), math.QA/9908040.

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