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FREE RESOLUTIONS FOR DIFFERENTIAL MODULES OVER DIFFERENTIAL ALGEBRAS

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ABSTRACT. A free resolution $(R, d+h) \to (M, d)$ for a DG-module (M, d) over a DG-algebra (A, d) is constructed in the sense of a perturbation of the differential in a free bigraded resolution $(R, d) \to M$ of the underlying graded module M over an underlying graded algebra A.

CONTENTS

Introduction

To obtain a differential homological algebra, i.e., to construct a free resolution for a differential graded (DG) module (M, d_M) over a DG-algebra (A, d_A), the bar resolution

$$\alpha : (A \otimes \overline{B}(A) \otimes M, d_H) \longrightarrow M$$

of the underlying graded module M over the underlying graded algebra A is usually taken, first with the *horizontal* differential

$$d_H\big(a[a_1|\cdots|a_n]m\big) = a \cdot a_1[a_2|\cdots|a_n] + \sum_k \pm a\big[a_1|\cdots|a_k \cdot a_{k+1}|\cdots|a_n\big]m \pm a[a_1|\cdots|a_{n-1}]a_n \cdot m.$$

Then the differentials d_A and d_M *automatically* induce a suitable perturbation of d_H – the *vertical* differential

$$d_V\big(a[a_1|\cdots|a_n]m\big) = d_A a_1[a_1|\cdots|a_n] + \sum_k \pm a\big[a_1|\cdots|d_A a_k|\cdots|a_n\big]m \pm a[a_1|\cdots|a_n]d_M m,$$

so that

$$\alpha : (A \otimes \overline{B}(A) \otimes M, d_H + d_V) \longrightarrow (M, d_M)$$

is a resolution of (M, d_M) over (A, d_A). This happens since the bar resolution is too *large* and *functorial* in a certain sense, which implies that d_A and d_M *induce* a perturbation $h = d_V$, which, in general, is not the case for smaller resolutions.

In this paper, we present the method of constructing the resolutions of differential graded (DG)-modules over DG-algebras.

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We are doing it *in the spirit of Gugenheim et al. [3, 4] that a differential homological algebra is obtained from a homological algebra by perturbing bigraded objects to graded filtered objects* (citation from [5]).

Namely, to construct a free resolution of a DG-module (M, d_M) over a DG-algebra (A, d_A), we, forgetting for a moment about the differentials d_A and d_M, begin with a *free bigraded resolution* of M over A:

$$\begin{array}{ccccccc} \multicolumn{7}{c}{\dotfill} \\ M_2 & \xleftarrow{\alpha} & R_{2,0} & \xleftarrow{d} & R_{2,1} & \longleftarrow & \cdots \\ M_1 & \xleftarrow{\alpha} & R_{1,0} & \xleftarrow{d} & R_{1,1} & \longleftarrow & \cdots \\ M_0 & \xleftarrow{\alpha} & R_{0,0} & \xleftarrow{d} & R_{0,1} & \longleftarrow & \cdots \end{array}, \tag{0.1}$$

where each column $R_{*,q}$ is a free A-module, i.e., $R_{*,q} = A \otimes V_{*,q}$ for a certain free graded Λ-module $V_{*,q}$, $dd = 0$, $\alpha d = 0$, and each row is acyclic. In this case, the total complex (R_n, d), $R_n = \sum\limits_{p+q=n} R_{p,q}$, of the bigraded complex $\{R_{p,q}, d\}$, together with α, is a free A-resolution of M. There is a large source of such resolutions, starting from *minimal* (which exists under some conditions on A) till, say, *maximal* in some sense, bar resolution $A \otimes \overline{B}(A) \otimes M$.

Now we assume that A and M are equipped with differentials d_A and d_M, respectively. The main result of our paper is the following theorem.

Theorem 1. *There exists a perturbed differential* $(d+h)$ *on* $R_{*,*}$ *such that the perturbation* $h : R_* \to R_{*-1}$ *consists of components*

$$h^k_{p,q} : R_{p,q} \longrightarrow R_{p-k,q+k-1}, \quad p, q = 0, 1, 2, \ldots; \quad k = 1, 2, \ldots, p,$$

so that the same α *forms a free resolution of* (M, d_M) *over* (A, d_A), *i.e.,*

$$\alpha : (R_*, d + h) \longrightarrow (M, d_M)$$

is a weak equivalence of $DG - (A, d_A)$*-modules.*

Here we note that the perturbation h is not uniquely defined. We describe the freedom in the construction of h and prove the suitable comparison theorem.

In the case where the ground ring Λ is a field, the bigraded resolution (0.1) is not only acyclic, but also contractible: there exist Λ-homomorphisms

$$\beta_* : M_* \longrightarrow R_{*,0}, \quad s_{p,*} : R_{p,*} \longrightarrow R_{p,*+1}, \quad p = 0, 1, \ldots,$$

such that

$$\alpha\beta = \mathrm{id}_M, \quad \beta\alpha + ds = \mathrm{id}_{R_{*,0}}, \quad sd + ds = \mathrm{id}\,.$$

This actually means that we have a *contraction*

$$(M, \beta, \alpha, (R_*, d), s),$$

where d and α preserve the action of A, but β and s are just Λ-mappings. In this case, it is possible to present *explicit* formulas for h in terms of the above contraction.

As is seen from the theorem, the perturbation h consists of *vertical* $h^1_{p,q}$ *down and right* $h^{k>1}_{p,q}$ components. In the bar resolution, the perturbation $h = d_V$ has only a vertical component h^1. The following simple example shows the necessity of these *down and right* components in the general case.

Example 1. Let $\Lambda = Z$, $A = Z$ (nongraded and nondifferential) and

$$(M, d_M) = \big(Z_2 \xleftarrow{d'_M} Z_4 \xleftarrow{d''_M} Z_2 \xleftarrow{0} 0 \longleftarrow \cdots\big)$$

with nontrivial d'_M and d''_M. Take the following bigraded resolution of M over $A = Z$:

$$\begin{array}{ccccccccc} \multicolumn{9}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ M_2 = Z_2 & \xleftarrow{\alpha_2} & R_{2,0} = Z & \xleftarrow{d_2} & R_{2,1} = Z & \longleftarrow & 0 & \longleftarrow & \cdots \\ M_1 = Z_4 & \xleftarrow{\alpha_1} & R_{1,0} = Z & \xleftarrow{d_1} & R_{1,1} = Z & \longleftarrow & 0 & \longleftarrow & \cdots \\ M_0 = Z_2 & \xleftarrow{\alpha_0} & R_{0,0} = Z & \xleftarrow{d_0} & R_{0,1} = Z & \longleftarrow & 0 & \longleftarrow & \cdots, \end{array} \tag{0.2}$$

where $d_0(x) = 2x$, $d_1(x) = 4x$, and $d_2(x) = 2x$. Taking into account the differential d_M, we *can* construct (not canonically) vertical components of the perturbation h

$$h^1_{1,0} : R_{1,0} = Z \longrightarrow R_{0,0} = Z; \quad h^1_{2,0} : R_{2,0} = Z \longrightarrow R_{1,0} = Z;$$
$$h^1_{1,1} : R_{1,1} = Z \longrightarrow R_{0,1} = Z; \quad h^1_{2,1} : R_{2,1} = Z \longrightarrow R_{1,1} = Z,$$

satisfying

$$d'_M \alpha_1 = \alpha_0 h^1_{1,0}, \quad d''_M \alpha_2 = \alpha_1 h^1_{2,0}, \quad h^1_{1,0} d_1 = d_0 h^1_{1,1}, \quad h^1_{2,0} d_2 = d_1 h^1_{2,1}.$$

For example, we can take

$$h^1_{1,0}(x) = x; \quad h^1_{2,0}(x) = 2x; \quad h^1_{1,1}(x) = 2x; \quad h^1_{2,1}(x) = x.$$

But the compositions $h^1_{1,0} h^1_{2,0}$ and $h^1_{1,1} h^1_{2,1}$ *cannot be trivial* and, therefore, $(d + h) : R_* \to R_*$ is not a differential. To correct this, we have to take one more component $h^2_{2,0} : R_{2,0} = Z \to R_{0,1} = Z$, $h^2_{2,0}(x) = x$. Then $(d+h)(d+h) = 0$ is guaranteed and $(R, d+h) \to (M, d_M)$ is a chain mapping inducing an isomorphism in the homology.

The method of constructing of resolutions for differential modules described above differs from another method *passing through the homology* used for the construction of free resolutions for $DG - \Lambda$-modules by Berikashvili in [1] and for constructing free models for commutative DG-algebras by Halperin and Stasheff in [5]. To obtain a free resolution (model) for a differential object (M, d), they first take a free bigraded resolution $\alpha : (R, d) \to H(M)$ of a nondifferential object $H(M)$ and, perturbing the differential d, obtain a free resolution $\alpha' : (R, d + h) \to (M, d)$.

Unfortunately, this method – passing through the homology – is not effective in the case of interest to us (to construct a free resolution for the DG-module (M, d_M) over a DG-algebra (A, d_A)): in general, the homology $H(M)$ is not an A-module.

Our approach is inspired by the approach of Huebschmann from [7], where a small resolution of a finite metacyclic group is constructed, and by the *blowing-up perturbation lemma* from [8], where a small model for a DG-algebra is constructed. This lemma allows one to transport perturbations from a smaller object to a larger one (from M to (R, d) in our case), in contrast to the basic perturbation lemma.

In Sec. 1, bigraded resolutions in the nondifferential situation, for a graded module over a graded algebra, are presented. In Sec. 2, the perturbation h is constructed, which gives a resolution for a differential graded module over a differential graded algebra. The suitable comparison theorem is proved, and the freedom in the construction of h is studied. In the final section, we consider Koszul resolutions.

1. Free Resolution for a Graded Module over a Graded Algebra

In this section, we prepare the grounds for the next: construct a bigraded resolution for a graded module over a graded algebra. This material is quite standard.

Let $A = \{A_{n \geq 0}\}$ be a graded algebra with unit and $M = \{M_{n \geq 0}\}$ be a graded A-module, i.e., the following structure mappings are given:

$$\mu : A_p \otimes A_q \longrightarrow A_{p+q}, \quad a \otimes a' \longrightarrow a \cdot a';$$
$$\nu : A_p \otimes M_q \longrightarrow M_{p+q}, \quad a \otimes m \longrightarrow a \cdot m,$$

satisfying the standard conditions

$$a \cdot (a' \cdot a'') = (a \cdot a') \cdot a'', \quad a \cdot (a' \cdot m) = (a \cdot a') \cdot m, \quad 1_A \cdot m = m.$$

A *free* graded A-module over a free graded Λ-module $V = \{V_n\}$ is just the tensor product $A \otimes V$; it has the standard universal property (see, e.g., [10, VI, 8.2]): for a graded A-module M and a morphism of graded Λ-modules $\psi : V \to M$ there exists a unique morphism of graded A-modules $f_\psi : A \otimes V \to M$ such that $f_\psi(1 \otimes v) = \psi(v)$. In fact, $f_\psi = \nu(\mathrm{id} \otimes \psi)$.

A *differential* graded A-module ($DG-A$-module) is a graded A-module M equipped with a differential $d_M : M_* \to M_{*-1}$, satisfying $d_M d_M = 0$, which is, in addition, a *derivation*, i.e., $d_M(a \cdot m) = (-1)^{dima} a \cdot d_M(m)$.

A *free resolution* (or *resolution*) of a graded A-module M (a free A-resolution of M) is defined as a $DG-A$-module (R, d), whose underlying graded A-module R is free, together with a weak equivalence of $DG-A$-modules $(R, d) \to (M, d_M = 0)$.

Below, we present such a free resolution of a special type, the so-called *bigraded* resolution.

By the *bigraded* free resolution of a graded A-module M, we mean a bigraded complex

$$\begin{array}{ccccccc}
\multicolumn{7}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
M_2 & \xleftarrow{\alpha} & R_{2,0} & \xleftarrow{d} & R_{2,1} & \longleftarrow & \cdots \\
M_1 & \xleftarrow{\alpha} & R_{1,0} & \xleftarrow{d} & R_{1,1} & \longleftarrow & \cdots \\
M_0 & \xleftarrow{\alpha} & R_{0,0} & \xleftarrow{d} & R_{0,1} & \longleftarrow & \cdots,
\end{array}$$

where each column $R_{*,q}$ is a free A-module, $dd = 0$, $\alpha d = 0$, and each row is acyclic. In this case, we have a weak equivalence of graded A-modules $(r, d) \to (M, d_M) = 0$, where (R, d), $R_n = \sum\limits_{p+q=n} R_{p,q}$ is the total complex of the bigraded complex $\{R_{p,q}, d\}$.

The freeness of each column $R_{*,q}$ means that it is the tensor product of A_* and a certain free graded vector space $V_{*,q}$, and, therefore, having a bigraded free resolution, we actually have a generating bigraded free Λ-module $\{V_{p,q}\}$ and $R_{p,q} = \sum\limits_{i=0}^{p} A_i \otimes V_{p-i,q}$ (actually, we have trigraded $R_{i,j,k} = A_i \otimes V_{j,k}$ and $R_{p,q} = \sum\limits_{i+j=p} R_{i,j,q}$).

Remark 1. We emphasize here that, in general, if R is the total complex of a certain bigraded A-module $R_{p,q}$, then an arbitrary differential on R can have many components

$$d^k_{p,q} : R_{p,q} \longrightarrow R_{p-k-1,q+k}, \quad k = -q, -q+1, \dots, -1, 0, 1, \dots, p-1,$$

and also the mapping $\alpha : R \to M$ can have components $\alpha_{p,q} : R_{p,q} \to M_{p+q}$, but here, in a bigraded resolution, we have only a horizontal differential, having just components $d^{-1}_{p,q}$, and α is also horizontal, i.e., it has only the components $\alpha_{p,0}$.

The way to construct such a resolution is standard, based on the following lemma.

Lemma 1. *For a graded A-module M there exist a free graded Λ-module V and a homomorphism of graded Λ-modules $\psi : V \to M$ such that the mapping of graded A-modules $\alpha : A \otimes V \to M$ given by $\alpha(a \otimes v) = a\psi(V)$ is surjective.*

Using this lemma, it is easy to construct a free bigraded A-resolution $R_{*,*} = A \otimes V_{*,*} \to M$ of M: let $V_{*,0}$ be V from the lemma, and let $R_{p,0} = \sum\limits_{i+j=p} A_i \otimes V_{j,0}$; then $\mathrm{Ker}(\alpha : R_{*,0} \to M)$ is an A-module as well and, using the lemma for $\mathrm{Ker}\,\alpha$, we obtain the surjective $d : R_{*,1} = A \otimes V_{*,1} \to \mathrm{Ker}\,\alpha$, etc.

Proof. We present two constructions for V and ψ, one very simple but large and another more complex but smaller.

Large construction. Let V be a free graded Λ-module which *covers* M, i.e., there exists an "onto" mapping of graded Λ-modules $\psi : V \to M$. Then $A \otimes V$ is a free graded A-module, and the graded A-module mapping $\alpha : A \otimes V \to M$ is onto.

Small construction. Now we assume that A is connected, i.e., $A_0 = \Lambda$.

Let $QM = M/(A_{>0} \cdot M)$ be the graded module of *indecomposables* of M, so that $(QM)_n = M_n/\big(\sum_{k=0}^{n-1} A_{n-k} \cdot M_k\big)$.

Let us take any free graded Λ-module V which covers QM, i.e., there exists a surjective mapping $\phi : V \to QM$. Because of freeness of V, we can construct $\psi : V \to M$ such that $p\psi = \phi$, where $p : M \to QM$ is the standard projection.

It remains to show that $\alpha : A \otimes V \to M$ given by $\alpha(a \otimes v) = a \cdot \psi(v)$ is onto.

Let us denote $R = A \otimes V$, i.e., $R_n = \sum_{k=0}^{n} A_{n-k} \otimes V_k$. We are going to prove the surjectivity of $\alpha_n : R_n \to M_n$ by induction on n.

For $n = 0$, we have: $R_0 = V_0$, $(QM)_0 = M_0$ and $\alpha_0 = \psi_0 = \phi_0$ is surjective by definition.

Now we suppose that

$$\alpha_k : R_k = \sum_{s=0}^{k} A_{k-s} \otimes V_s \longrightarrow M_k$$

is surjective for $k < n$. Multiplying by A_{n-k}, we obtain that

$$\sum_{s=0}^{k} A_{n-k} \cdot A_{k-s} \otimes V_s \longrightarrow A_{n-k} \cdot M_k$$

is also surjective and, therefore,

$$\sum_{s=0}^{k} A_{n-s} \otimes V_s \longrightarrow A_{n-k} \cdot M_k \tag{1.1}$$

is also surjective.

Now we take any $m \in M_n$. Since

$$\phi_n : V_n \to (QM)_n = M_n/\Big(\sum_{k=0}^{n-1} A_{n-k} \cdot M_k\Big)$$

is surjective, there exists $v \in V_n$ such that $p\psi(v) = p(m)$ and, therefore, $m - \psi(v) \in \sum_{k=0}^{n-1} A_{n-k} \cdot M_k$. But each $A_{n-k} \cdot M_k$ is covered by the image of (1.1); therefore, there exists $x \in \sum_{k=0}^{n-1} A_{n-k} \cdot M_k$ such that $m = \psi(v) + \alpha(x) = \alpha(v + x)$.

Both constructions allow one to produce free bigraded A-resolutions for M, a *large* one, using the first construction on each step and a *small* one, using the second. $\square$

Among various free A-resolutions of M, there are two remarkable ones — a *minimal resolution* and a *bar resolution* — which we describe now.

1.1. Minimal resolution. Here we assume that Λ is a field and A is connected. If, for a graded A-module M, we construct the above-mentioned small resolution, taking as V in Lemma 1 the graded vector space of indecomposables QM itself (and the same at all subsequent steps), we obtain even smaller, so-called *minimal* resolution.

Definition 1. A $DG-A$-module (R, d) is *minimal* if for any $r \in R$ the value of the differential $d(r)$ is decomposable, i.e., $d(r) \in A_{>0} \cdot R$.

This is the A-module version of Sullivan's notion of minimal commutative DG-algebra.

Let us consider what the minimality means for free R, i.e., for $R = A \otimes V$.

First, we examine the structure of a differential $d : A \otimes V \to A \otimes V$. Because of the freeness, d is defined by the restriction $\gamma = di : V \to A \otimes V \to A \otimes V$. Moreover, any Λ-homomorphism $\gamma : V \to A \otimes V$ of degree -1 defines a derivation $d_\gamma = (\mu_A \otimes \mathrm{id})(\mathrm{id} \otimes \gamma)$ which is a differential, i.e., $d_\gamma d_\gamma = 0$ if and only if $d_\gamma d_\gamma i = 0$, or

$$(\mu_A \otimes \mathrm{id})(\mathrm{id} \otimes \gamma)\gamma = 0. \tag{1.2}$$

Because of the connectedness of A, we have

$$A \otimes V = V \oplus A_{>0} \otimes V,$$

and, therefore, γ is the sum of two components, $\gamma = \gamma_1 + \gamma_2$,

$$\gamma_1 : V \longrightarrow V, \quad \gamma_2 : V \longrightarrow A_{>0} \otimes V,$$

which are called *linear* and *quadratic* parts, respectively.

It follows from (1.2) that $\gamma_1\gamma_1 = 0$ and, therefore, (V, γ_1) is a DG vector space over Λ.

Now we turn back to minimality. It is easy to observe that in terms of components, the minimality of $(A \otimes V, d_\gamma)$ means nothing other than $\gamma_1 = 0$.

Now we similarly examine the structure of a mapping of free graded A-modules $f : A \otimes V \to A \otimes V'$. Because of the freeness, f is determined by $\beta = fi : V \to A \otimes V'$. Moreover, any Λ-homomorphism $\beta : V \to A \otimes V'$ of degree 0 defines a mapping of graded A-modules $f_\beta = (\mu_A \otimes \mathrm{id})(\mathrm{id} \otimes \beta)$ which is a chain mapping, i.e., $d_{\gamma'} f_\beta = f_\beta d_\gamma$ if and only if $d_{\gamma'} f_\beta i = f_\beta d_\gamma i$, or

$$(\mu_A \otimes \mathrm{id})(\mathrm{id} \otimes \gamma')\beta = (\mu_A \otimes \mathrm{id})(\mathrm{id} \otimes \beta)\gamma. \tag{1.3}$$

Since $A \otimes V' = V' \oplus A_{>0} \otimes V'$, β is the sum of two components, $\beta = \beta_1 + \beta_2$,

$$\beta_1 : V \longrightarrow V', \quad \beta_2 : V \longrightarrow A_{>0} \otimes V',$$

the *linear* and *quadratic* parts, respectively.

It follows from (1.3) that $\gamma_1'\beta_1 = \beta_1\gamma_1$, so $\beta_1 : (V, \gamma_1) \to (V', \gamma_1')$ is a chain mapping.

We omit the proofs of the following two standard statements.

Proposition 1. *A mapping of $DG - A$-modules*

$$f_\beta : A \otimes V \longrightarrow A \otimes V'$$

is a weak equivalence if and only if the linear component

$$\beta_1 : (V, \gamma_1) \longrightarrow (V', \gamma_1')$$

is a weak equivalence.

Proposition 2. *A mapping of graded A-modules*

$$f_\beta : A \otimes V \longrightarrow A \otimes V'$$

is an isomorphism if and only if the linear component

$$\beta_1 : V \longrightarrow: V'$$

is an isomorphism.

Corollary 1. *Any weak equivalence of minimal free $DG - A$-modules is an isomorphism.*

Proof. If $f_\beta : A \otimes V \to A \otimes V'$ is a weak equivalence, then, by Proposition 1, $\beta_1 : (V, \gamma_1 = 0) \to: (V', \gamma_1' = 0)$ is a weak equivalence, but since $\gamma_1 = \gamma_1' = 0$, β_1 is an isomorphism and, therefore, by Proposition 2, f_β is also an isomorphism.

Now we can construct a minimal resolution for a graded A-module M using the small construction from the proof of Lemma 1. If Λ is a field, there is no need to pass to free graded Λ-modules $V_{*,*}$. It is possible to take $V_{n,0} = (QM)_n$ and $\psi_n : (QM)_n \to M_n$ to be a section of p_n in the exact sequence

$$0 \longrightarrow \sum_{k=0}^{n-1} A_{n-k} \cdot M_k \longrightarrow M_n \xrightarrow{p_k} (M)_n \longrightarrow 0,$$

so that in this case $R_{0,0} = (QM)_0 = M_0$ and

$$R_{n,0} = \sum_{k=0}^{n} A_{n-k} \otimes (QM)_k,$$

the mapping $\alpha_n : R_{n,0} \to M_n$ being given by

$$\alpha_n(a_{n-k} \otimes v_{k,0}) = a_{n-k}\psi_k(v_{k,0}),$$

where $v_{k,0} \in V_{k,0} = (QM)_k$. $\square$

Proposition 3. $\operatorname{Ker}\alpha_n \subset \sum_{k=0}^{n-1} A_{n-k} \otimes (QM)_k$, *i.e.,* $\operatorname{Ker}\alpha_n$ *has no (indecomposable) components in* $(QM)_n \subset R_{n,0}$.

Proof. Take any $r_{n,0} = \sum_{k=0}^{n} x_k \in R_{n,0}$ with $x_k \in A_{n-k} \otimes (QM)_k$ and suppose that $r_{n,0} \in \operatorname{Ker}\alpha_n$, i.e.,

$$\sum_{k=0}^{n} \alpha_n(x_k) = 0. \tag{1.4}$$

By the definition of α_n, we have $\alpha_n(x_k) \in A_{n-k} \cdot M_k$; therefore, acting on (1.4) by p_n, we obtain (since $p_n\alpha_n(x_k) = 0$ for $k < n$)

$$0 = p_n\alpha_n(x_n) = p_n\psi_n(x_n) = x_n.$$

$\square$

This proposition allows us to show that the obtained free A-resolution is minimal. Indeed, $\operatorname{Im}(d : R_{n,1} \to R_{n,0}) = \operatorname{Ker}(\alpha_n : R_{n,0} \to M_n)$, but according to the above proposition, $\operatorname{Ker}\alpha_n$ consists only of decomposable elements. Clearly, the same argument proves the decomposability of further differentials $d : R_{n,q} \to R_{n,q-1}$.

It follows from the standard comparison theorem and Theorem 4 that the minimal resolution for M is unique up to an isomorphism of $DG-A$-modules.

Remark 2. If A is a free commutative graded algebra, then the minimal resolution is the well-known Koszul resolution.

1.2. Bar resolution. In addition to the minimal resolution, there is one more remarkable resolution. This is the so-called two-sided bar construction

$$A \otimes \overline{B}(A) \otimes M \longrightarrow M,$$

where $\overline{B}(A)$ is the reduced bar construction of A. The bigraduation here is given by $a[a_1|...|a_q]m \in R_{p,q}$, where $p = |a| + \sum_k |a_k| + |m|$, $a, a_k \in A$, $m \in M$.

This resolution appears when we use the large construction from the proof of Lemma 1, taking M itself as V (and the same on subsequent steps).

The remarkable property of this resolution is that it is *functorial* in M and A, which gives some advantages in the case where A and M are equipped with differentials (see the Introduction).

2. Free Resolution for a DG-Module over a DG-Algebra

Now we assume that $(A, d_A : A_* \to A_{*-1})$ is a *connected* differential graded algebra (DGA-algebra), i.e., $A_n = 0$ for $n < 0$ and $A_0 = \Lambda$.

Let (M, d_M) be a DG-module over (A, d_A) (we say $DG-(A, d_A)$-module), i.e., the action $A \otimes M \to M$ is a chain mapping: $d_M(a \cdot m) = d_A(a) \cdot m + (-1)^{|a|} a \cdot d_M(m)$.

Let us forget for a moment about the differentials d_A and d_M and consider a *free bigraded resolution* of the underlying graded module M over the underlying graded algebra A

$$
\begin{array}{ccccccc}
\multicolumn{7}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
M_2 & \xleftarrow{\alpha} & R_{2,0} & \xleftarrow{d} & R_{2,1} & \longleftarrow & \cdots \\
M_1 & \xleftarrow{\alpha} & R_{1,0} & \xleftarrow{d} & R_{1,1} & \longleftarrow & \cdots ; \\
M_0 & \xleftarrow{\alpha} & R_{0,0} & \xleftarrow{d} & R_{0,1} & \longleftarrow & \cdots
\end{array}
$$

here each column $R_{*,q}$ is a free A-module, i.e., $R_{p,q} = \sum_{i=0}^{p} A_i \otimes V_{p-i,q}$ for a certain free bigraded Λ-module $\{V_{p,q}\}$. Since $A_0 = \Lambda$, we have

$$R_{p,q} = V_{p,q} \oplus (A_1 \otimes V_{p-1,q}) \oplus \cdots \oplus (A_p \otimes V_{0,q}) \tag{2.1}$$

and, therefore, $R_{p,q}$ is a direct sum of the *indecomposable* part $V_{p,q}$ and *decomposable* part $U_{p,q} = \sum_{i=1}^{p} A_i \otimes V_{p-i,q}$.

Since the resolution differential $d : R_{p,*} \to R_{p,*-1}$ is an A-mapping, i.e., $d(a \cdot v) = (-1)^{|a|} a \cdot d(v)$, we have

$$d(A_i \otimes V_{p-i,q}) \subset \sum_{j=i}^{p} A_j \otimes V_{p-j,q-1},$$

and, therefore, the decomposable part $(U_{p,*}, d)$ is a subcomplex in $(R_{p,*}, d)$.

Now we assume that both A and M are equipped with differentials d_A and d_M, respectively. Our aim is to construct on $R_{*,*}$ a perturbed differential $(d + h) : R_* \to R_{*-1}$ such that $(R_*, d + h)$ becomes a $DG-(A, d_A)$-module and the same α remains a weak equivalence

$$\alpha : (R_*, d + h) \longrightarrow (M, d_M).$$

Our perturbation $h : R_* \to R_{*-1}$ will consist of components $\{h^k_{p,q}\}$, where

$$h^k_{p,q} : R_{p,q} \longrightarrow R_{p-k,q+k-1}, \quad p, q = 0, 1, 2, \ldots; \quad k = 1, 2, \ldots, p, \tag{2.2}$$

i.e., $d + h$ in $R_{*,*}$ will have only *horizontal* (the differential d), *vertical* (the components $h^1_{p,q}$), and *down and right* (the components $h^{k>1}_{p,q}$) components.

Theorem 2. *On $R_{*,*}$, there exists a perturbed differential $(d + h)$ such that the perturbation h consists of components $h^k_{p,q}$, see* (2.2), *so that the same α forms a free resolution of (M, d_M) over (A, d_A), i.e.,*

$$\alpha : (R_*, d + h) \longrightarrow (M, d_M)$$

is a weak equivalence of (A, d_a)-modules.

The construction of the perturbation h will be based on the relative form of the standard comparison theorem of the homological algebra.

Relative comparison theorem. *Suppose that*

$$(\overline{U}, d) = U_{-1} \xleftarrow{d_{-1}} V_0 \oplus U_0 \xleftarrow{d_0} V_1 \oplus U_1 \longleftarrow \cdots$$

is a $DG-\Lambda$-module, where $(U_*,d)\subset(\overline{U},d)$ is a sub-$DG-\Lambda$-module with $\overline{U}_{-1}=U_{-1}$, and the direct complement V_* is a free graded Λ-module. Also, assume that an acyclic $DG-\Lambda$-module

$$(R,d)=R_{-1}\xleftarrow{d_{-1}}R_0\xleftarrow{d_0}R_1\longleftarrow\cdots$$

is given and that $F:U_{-1}\to R_{-1}$ is a Λ-homomorphism, already lifted to a mapping of $DG-\Lambda$-modules $f_*:(U_*,d)\to(R_*,d)$, i.e., $f_{-1}=F$.

(1) *Then there exists a lifting of F on the whole $\overline{U}$*

$$F_*:(\overline{U},d)\longrightarrow(R,d)$$

extending f_*.

(2) *Suppose that*

$$F'_*:(\overline{U},d)\longrightarrow(R,d)$$

is another lifting of F and that $H:U_*:\to R_{*+1}$ is a homotopy between the restrictions $F_*|U_*$ and $F'_*|U_*$; then there exists an extension of H on the whole $\overline{U_*}$, which realizes the homotopy between F and F'.

Proof. In order that $d+h$ be a correct differential satisfying the needed conditions, a perturbation h should have certain properties, which we now consider.

1. To agree with the action of (A,d_A) on R_*, the differential $d+h$ should satisfy

$$(d+h)(a\cdot r)=d_A(a)\cdot r+(-1)^{|a|}a\cdot(d+h)(r). \tag{2.3}$$

Having in mind (2.1), we construct $h^*_{*,*}$ on the indecomposable part $V_{*,*}$ of $R_{*,*}$ and extend it on the decomposable part $U_{*,*}$ by the rules

$$h^1_{p,q}(a_i\otimes v_{p-i,q})=d_A(a_i)\otimes v_{p-i,q}+(-1)^{|a_i|}a_i\cdot h^1_{p-i,q}(v_{p-i,q}), \tag{2.4}$$

$$h^{k>1}_{p,q}(a_i\otimes v_{p-i,q})=(-1)^{|a_i|}a_i\cdot h^k_{p-i,q}(v_{p-i,q}). \tag{2.5}$$

Then condition (2.3) is automatically satisfied.

2. For $(d+h)$ to be a differential, i.e., for $(d+h)(d+h)=0$, a perturbation $\{h^k_{p,q}\}$ should satisfy Brown's condition [2] $dh+hd+hh=0$ (i.e., h must be a twisting element). This condition, in terms of components, looks as follows:

$$dh^k_{p,q}+h^k_{p,q-1}d=-\sum_{i=1}^{k-1}h^{k-i}_{p-i,q+i-1}h^i_{p,q}. \tag{2.6}$$

Let us denote by $\Phi^k_{p,q}$ the right-hand side of Eq. (2.6):

$$\Phi^k_{p,q}=-\sum_{i=1}^{k-1}h^{k-i}_{p-i,q+i-1}h^i_{p,q}:R_{p,q}\longrightarrow R_{p-m,q+m-2}.$$

Then the condition $dh+hd=hh$ can be rewritten as

$$dh^k_{p,q}+h^k_{p,q-1}d=\Phi^k_{p,q}. \tag{2.7}$$

3. Finally, we note that in order that α be a chain mapping, a perturbation should satisfy the condition

$$\alpha h^1_{p,0}=d_M\alpha. \tag{2.8}$$

We will construct the collection $\{h^k_{p,q}\}$ by induction on k satisfying the conditions (2.2), (2.4), (2.5), (2.8), and (2.7).

For $k=1$, let us first consider the 1st and 0th rows of the bigraded resolution

$$\begin{array}{ccccccc} M_1 & \xleftarrow{\alpha_1} & R_{1,0}=V_{1,0}\oplus U_{1,0} & \xleftarrow{d} & R_{1,1}=V_{1,1}\oplus U_{1,1} & \longleftarrow & \cdots \\ M_0 & \xleftarrow{\alpha_0} & R_{0,0}=V_{0,0} & \xleftarrow{d} & R_{0,1}=V0,0 & \longleftarrow & \cdots \end{array},$$

where $U_{1,q}=A_1\otimes V_{1,q}$.

We write

$$d_M\alpha_1(a_1\otimes v_{0,0}) = d_M(a_1\cdot\alpha_0(v_{0,0})) = d_M(a_1)\cdot\alpha_0(v_{0,0}) \pm a_1\cdot d_M(v_{0,0}) = 0,$$

i.e., the zero mapping $0 : U_{1,*} \to R_{0,*}$ lifts $d_M : M_1 \to M_0$; therefore, we are in the situation of part one of the relative comparison theorem. Thus, there exists an extension of $0 : U_{1,*} \to R_{0,*}$, a chain mapping $h^1_{1,*} : R_{1,*} \to R_{0,*}$, i.e., $h^1_{1,*}$ satisfies the conditions

$$h^1_{1,*}(a_1\otimes v_{0,*}) = 0, \quad \alpha_0 h^1_{1,0} = d_M\alpha_1, \quad dh^1_{1,q>0} = h^1_{1,q-1}d;$$

these are exactly conditions (2.4), (2.8), and (2.7) for $k = 1$ and $p = 1$, respectively.

Now we suppose that $h^1_{k,*}$ are constructed for $k < p$.

We consider the pth and $(p-1)$th rows of the bigraded resolution

$$\begin{array}{ccccccc} M_p & \xleftarrow{\alpha_p} & R_{p,0} = V_{p,0}\oplus U_{p,0} & \xleftarrow{d} & R_{p,1} = V_{p,1}\oplus U_{p,1} & \longleftarrow & \cdots \\ M_{p-1} & \xleftarrow{\alpha_{p-1}} & R_{p-1,0} & \xleftarrow{d} & R_{p-1,1} & \longleftarrow & \cdots \end{array},$$

where $U_{p,q} = \sum\limits_{i=1}^{p} A_i\otimes V_{p-i,q}$.

The already defined $h^1_{k<p,*}$ determine $h^1_{p,*}|U_{p,*} : U_{p,*} \to R_{p,*}$ by condition (2.4):

$$h^1_{p,q}(a_i\otimes v_{p-i,q}) = d_A(a_i)\otimes v_{p-i,q} + (-1)^{|a_i|}h^1_{p-i,q}(v_{p-i,q}).$$

A routine verification shows that, actually, $h^1_{p,*} : U_{p,*} \to R_{p,*}$ is a chain mapping lifting $d_M : M_p \to M_{p-1}$.

Thus, we are in the situation of part one of the relative comparison theorem. Thus, there exists a chain mapping

$$h^1_{p,*} : R_{p,*} \longrightarrow R_{p,*}$$

lifting d_M and extending $h^1_{p,*}|U_{p,*}$, and, therefore, it satisfies

$$h^1_{p,q}(a_i\otimes v_{p-i,q}) = d_A(a_i)\otimes v_{p-1,q} + a_i\cdot h^1_{p-i,q)},$$
$$\alpha_{p-1}h^1_{p,0} = d_M\alpha_p, \quad dh^1_{p,q>0} = h^1_{p,q-1}d;$$

these are exactly conditions (2.4), (2.8), and (2.7) for $k = 1$, respectively. This completes the construction of components $h^1_{p,q}$.

Until we go to the next step of the induction, we note that the constructed *vertical* components $h^1_{*,*}$ are well connected with the *horizontal* differential d, but $h^1_{*,*}h^1_{*,*} = 0$ is not guaranteed and, therefore, $d + h^1_{*,*}$ is not a differential. The meaning of the next component $h^2_{*,*}$ is that $h^1_{*,*}h^1_{*,*}$ is *homotopic* to zero and $h^2_{*,*}$ is the suitable homotopy.

Now we suppose that $h^k_{p,q}$ are constructed for $k < n$ satisfying (2.4), (2.5), (2.8), and (2.7) for $k = 1$. Note that, in this case, $\Phi^n_{p,q}$ are also defined.

A standard routine calculation, using just (2.7), shows that

$$d\Phi^n_{p,0} = 0, \quad d\Phi^n_{p,q} = \Phi^n_{p,q-1}d.$$

This means that

$$\Phi^n_{p,*} : R_{p,*} \longrightarrow R_{p-n,*+n-2}$$

is a chain mapping (of degree $n-2$) lifting the zero mapping $0 : M_p \to R_{p-n,n-3}$.

As above, we are going to construct $h^n_{p,*}$ by induction on p starting, of course, from $p = n$.

Take the nth row and the part of the 0th row of the bigraded resolution

$$\begin{array}{ccccccc} M_n & \xleftarrow{\alpha_n} & R_{n,0} = V_{n,0}\oplus U_{n,0} & \xleftarrow{d} & R_{n,1} = V_{n,1}\oplus U_{n,1} & \longleftarrow & \cdots \\ R_{0,n-3} & \xleftarrow{d} & R_{0,n-2} & \xleftarrow{d} & R_{0,n-1} & \longleftarrow & \cdots \end{array},$$

where $U_{n,q}=\sum\limits_{i=1}^{n}A_i\otimes V_{n-i,q}$.

As mentioned above,

$$\Phi^n_{n,*}:R_{n,*}\longrightarrow R_{0,*+n-2}$$

is a chain mapping (of degree $n-2$) lifting the zero mapping $0:M_n\to R_{0,n-3}$. Moreover, it is not difficult to calculate just by dimensional reasoning that the restriction of $\Phi^n_{n,*}$ on decomposable $U_{n,*}$ is zero. Thus, by the second part of the relative comparison theorem, $\Phi^n_{n,*}$ is homotopic to the zero, and the suitable homotopy $h^n_{n,*}:R_{n,*}\to R_{0,*+n-1}$ can be chosen so that the restriction of $h^n_{n,*}$ on decomposable $U_{n,*}$ is zero. Thus, we have $h^n_{n,*}$ satisfying

$$h^n_{n,q}(a_i\otimes v_{n-i,q})=0,\quad dh^n_{n,q>0}+h^n_{n,q-1}d=\Phi^n_{n,q},$$

which are exactly conditions (2.5) and (2.7) for $k=n$ and $p=n$, respectively.

Now we suppose that $h^n_{k,*}$ are constructed for $k<p$. Note that these components determine $h^n_{p,q}$ on the decomposable $U_{p,q}$ by (2.5):

$$h^n_{p,q}(a_i\otimes v_{p-i,q})=(-1)^{|a_i|}a_i\cdot h^n_{p-i,q}(v_{p-i,q});$$

therefore, what remains is to define $h^n_{p,q}$ on $V_{p,q}$.

Take the pth row and the part of the $(p-n)$th row of the bigraded resolution

$$\begin{array}{ccccccc}
M_p & \xleftarrow{\alpha_p} & R_{p,0}=V_{p,0}\oplus U_{p,0} & \xleftarrow{d} & R_{p,1}=V_{p,1}\oplus U_{p,1} & \longleftarrow & \cdots\\
R_{p-n,n-3} & \xleftarrow{d} & R_{p-n,n-2} & \xleftarrow{d} & R_{p-n,n-1} & \longleftarrow & \cdots
\end{array},$$

where $U_{p,q}=\sum\limits_{i=1}^{n}A_i\otimes V_{p-i,q}$.

As is mentioned above,

$$\Phi^n_{p,*}:R_{p,*}\longrightarrow R_{p-n,*+n-2}$$

is a chain mapping (of degree $n-2$) lifting the zero mapping $0:M_p\to R_{p-n,n-3}$. Of course, the restriction on decomposable

$$\Phi^n_{p,*}:U_{p,*}\longrightarrow R_{p-n,*+n-2}$$

is also a chain mapping lifting the zero mapping. Moreover, conditions (2.5) and (2.7) which are satisfied by the components $h^{k<n}_{p,q}$ allow one to verify that

$$(dh^n_{p,q}+h^n_{p,q-1}d)(a_i\otimes v_{p-i,q})=\Phi^n_{p,q}(a_i\otimes v_{p-i,q}),\ \ i=1,2,\dots,p,$$

i.e., already existing $h^n_{p,*}|U_{p,*}$ realizes the homotopy of $\Phi^n_{p,*}|U_{p,*}$ to zero. Then by part two of the relative comparison theorem, this homotopy can be extended to the whole $R_{p,*}$, and we obtain $h^n_{p,*}:R_{p,*}\to R_{p-n,*+n-1}$ satisfying the conditions

$$h^n_{p,q}(a_i\otimes v_{p-i,q})=(-1)^{|a_i|}a_i\cdot h^n_{p-i,q}(v_{p-i,q}),$$

and

$$dh^n_{p,q}+h^n_{p,q-1}d=\Phi^n_{p,q},$$

which are exactly conditions (2.5) and (2.7) for $k=n$. This completes the construction of the perturbation h.

Thus, we have obtained the collection $h^k_{p,q}$ satisfying Brown's condition $dh=hh$. Thus, $(d+h)(d+h)=0$.

The differential $(d+h)$ preserves the action of (A,d) on R, i.e., satisfies $(d+h)(a\cdot v) = d_A(a)\cdot v + (-1)^{|a|}a\cdot(d+h)(v)$. Indeed,

$$
\begin{gathered}
(d+h)(a\cdot v) = d(a\cdot v) + (h^1 + h^{>1})(a\cdot v) \\
= (-1)^{|a|}a\cdot d(v) + d_A(a)\cdot v + (-1)^{|a|}a\cdot h^1(v) + (-1)^{|a|}a\cdot h^{>1}(v) \\
= d_A(a)\cdot v + a\cdot(d+h)(v).
\end{gathered}
$$

Moreover, $\alpha : (R_*, d+h) \longrightarrow M_*$ is a chain mapping. Indeed,

$$\alpha(d+h) = \alpha(d + h^1 + h^{>1}) = \alpha h^1 = d_M\alpha.$$

Finally, we mention that since d is horizontal and h goes *down and right*, the perturbed differential $(d+h)$ preserves the filtration $F_p(R_{*,*}) = \{R_{\leq p,*}\}$, and the standard spectral sequence argument shows that α is a weak equivalence. This completes the proof of the theorem. □

Remark 3. For the bar resolution

$$\alpha : (A\otimes \overline{B}(A)\otimes M, d_H \longrightarrow M$$

of the underlying graded module M over the underlying graded algebra A, the resolution (*horizontal*) differential is given by

$$
\begin{gathered}
d_H\big(a[a_1|\cdots|a_n]m\big) = a\cdot a_1[a_2|\cdots|a_n] \\
+\sum_k \pm a\big[a_1|\cdots|a_k\cdot a_{k+1}|\cdots|a_n\big]m \pm a[a_1|\cdots|a_{n-1}]a_n\cdot m
\end{gathered}
$$

and as the perturbation h, we can take the *vertical* differential

$$
\begin{gathered}
d_V\big(a[a_1|\cdots|a_n]m\big) \\
= d_A a_1[a_1|\cdots|a_n] + \sum_k a\big[a_1|\cdots|d_A a_k|\cdots|a_n\big]m + a[a_1|\cdots|a_n]d_M m;
\end{gathered}
$$

actually, $h = h^1$ and all higher components $h^{k>1}$ are trivial.

2.1. Comparison theorem. Suppose that $\alpha : (R,d) \to M$ and $\alpha' : (R',d') \to M$ are two bigraded A-resolutions of M.

Using the standard arguments of comparison of free resolutions, based on the above-mentioned relative comparison theorem, it is possible to construct a morphism of bigraded modules

$$\big\{f^0_{p,q} : R_{p,q} \longrightarrow R'_{p,q}\big\},$$

which defines a mapping of $DG-A$-modules $f^0 : (R,d) \to (R',d')$ and $\alpha' f^0 = \alpha$.

Having the differentials d_A and d_M in A and M, respectively, we can construct perturbations in each of these two bigraded resolutions and obtain filtered resolutions $\alpha : (R, d+h) \to (M, d_M)$ and $\alpha' : (R', d'+h') \to (M, d_M)$.

Theorem 3. *There exists a collection of homomorphisms*

$$\big\{f^k_{p,q} : R_{p,q} \longrightarrow R'_{p-k,q+k},\ p,q = 0,1,2,\ldots;\ k = 1,2,\ldots,p\big\} \tag{2.9}$$

such that, together with $\{f^0_{p,q}\}$, *it defines a mapping of* $DG-A$*-modules* $f = \sum\limits_{k=0}^{\infty} f^k_{p,q} : (R_*, d+h) \to (R'_*, d'+h')$.

Proof. In order that f be a mapping of $DG-A$-modules, a collection $f^*_{*,*}$ should satisfy certain conditions, which we now consider.

1. First of all, f should be a mapping of graded A-modules, i.e.,

$$f(a\cdot r) = a\cdot f(r). \tag{2.10}$$

Having in mind (2.1), we construct $f^*_{*,*}$ first on $V_{*,*}$, and then extend it on $R_{*,*}$ by the rule

$$f^k_{p,q}(a_i \otimes v_{p-i,q}) = a_i \cdot f^k_{p-i,q}(v_{p-i,q}). \tag{2.11}$$

Then condition (2.10) will be automatically satisfied.

2. To be a chain mapping, i.e., for

$$(d' + h')f = f(d + h), \tag{2.12}$$

a collection $f^*_{*,*}$ should satisfy

$$d' f^k_{p,q} + \sum_{i=0}^{k-1} h'^{k-i}_{p-i,q+i} f^i_{p,q} = f^k_{p,q-1} d + \sum_{i=1}^{k} f^{k-i}_{p-i,q+i} h'^i_{p,q} \tag{2.13}$$

or, denoting

$$\Psi^k_{p,q} = f^k_{p,q-1} d + \sum_{i=1}^{k} f^{k-i}_{p-i,q+i} h'^i_{p,q} - \sum_{i=0}^{k-1} h'^{k-i}_{p-i,q+i} f^i_{p,q},$$

this condition can be rewritten as

$$d' f^k_{p,q} = \Psi^k_{p,q}. \tag{2.14}$$

3. Finally, the condition $\alpha' f = \alpha$ in terms of components has the form

$$\alpha' f^0_{p,q} = \alpha, \tag{2.15}$$

and, therefore, it is actually a property of given $f^0_{p,q}$.

A collection $\{f^k_{p,q}\}$ satisfying the conditions (2.9), (2.11), and (2.14) can be constructed exactly by the same induction as in the proof of Theorem 2, which we omit here. ☐

2.2. Equivalence of perturbations. As is seen from the above inductive process of construction of h, there is some freedom in choosing the components $h^k_{p,q}$ on each step, so that the perturbation h is not uniquely defined. Here we describe this freedom, introducing, following [1], the set $D(M)$ — the set of equivalence classes of perturbations.

Thus, as above, let M be a graded A-module and $\alpha : (R, d) \to M$ be its free bigraded A-resolution, i.e., $\alpha : R_{k,0} \to M_k$ and $d : R_{p,q} \to R_{p,q-1}$.

Now we introduce the following class of A-endomorphisms:

$$G = \{(\mathrm{id} + f) : R_* \longrightarrow R_*\}$$

of the totalization R_* of the bigraded A-module $R_{*,*}$, where f consists (is the sum) only of the following components:

$$\{f^k_{p,q} : R_{p,q} \longrightarrow R_{p-k,q+k},\ p, q = 0, 1, 2, \dots;\ k = 1, 2, \dots, p\},$$

i.e., there are only the identity and *down and right* components in f.

It is not difficult to verify that

(i) each $(\mathrm{id} + f) \in G$ is an isomorphism;

(ii) for $(\mathrm{id} + f), (\mathrm{id} + g) \in G$, the composition $(\mathrm{id} + f)(\mathrm{id} + g) = (\mathrm{id} + f + g + fg)$ also belongs to G.

Therefore, G is a group with respect to the composition operation.

Now let P be the set of all perturbations $h = \{h^k_{p,q}\}$ on $(R_{*,*}, d)$, satisfying (2.2), (2.4), (2.5), and (2.6).

The group G acts on P as follows:

$$(\mathrm{id} + f) * h = (\mathrm{id} + f)h(\mathrm{id} + f)^{-1} + (df - fd)(\mathrm{id} + f)^{-1}. \tag{2.16}$$

We have to show that $h' = (\mathrm{id} + f) * h$ belongs to P. It is clear that h' satisfies (2.2), since f acts down and right. To show that h' satisfies (2.6), let us rewrite (2.16) as

$$h - h' = h'f + fh + df - fd \tag{2.17}$$

or

$$(d + h')(\mathrm{id} + f) = (\mathrm{id} + f)(d + h), \tag{2.18}$$

so that the isomorphism $(\mathrm{id}+f)$ is a chain mapping and, therefore, it is easy to conclude that $(d+h')(d+h')=0$, which is equivalent to (2.6). It is also clear that $d+h'$ is an A-derivation (i.e., it satisfies (2.4) and (2.5)), since $d+h$ has this property and $\mathrm{id}+f$ is an isomorphism of $DG-A$-modules. Thus, $h'\in P$.

Denote by $D_R(M)$ the set of orbits of P with respect to the action of G. We will call perturbations from the same orbit equivalent.

Now we are able to describe the freedom in the construction of h corresponding to a given differential d_M.

Proposition 4. *Let $h^k_{p,q}$ and $h'^k_{p,q}$ be two perturbations satisfying conditions* (2.2), (2.4), (2.5), (2.6), *and* (2.8). *Then these perturbations are equivalent.*

Remark 4. Actually, the equivalence of h and h' means that there exists an isomorphism of (A,d_A)-resolutions $(\mathrm{id}+f):(R_*,d+h)\to(R_*,d+h')$, for which $\alpha'(\mathrm{id}+f)=\alpha$. Therefore, different perturbations define isomorphic free resolutions.

Proof. This proposition is an immediate consequence of Proposition 3, taking $R=R'$ and $f^0=\mathrm{id}$. □

According to this proposition, we have a correct mapping from the set of all A-differentials on M:

$$Diff_A(M)=\left\{d_M:M_*\longrightarrow M_{*-1},\ d_Md_M=0,\ d_M(a\cdot m)=(-1)^{\dim a}a\cdot m\right\}$$

to the set of equivalence classes of perturbations $D_R(M)$.

Proposition 5. *There exists a bijection between* $\mathrm{Diff}_A(M)$ *and* $D_R(M)$.

Proof. Let us construct a converse mapping $D_R(M)\to\mathrm{Diff}_A(M)$. For a given perturbation $\{h^k_{p,q}\}$ satisfying (2.2), (2.4), (2.5), and (2.6), the first component $h^1_{p,0}:R_{p,0}\to R_{p-1,0}$ induces the correct homomorphism

$$d_M:M_p=R_{p,0}/\operatorname{Ker}\alpha_p\longrightarrow M_{p-1}=R_{p-1,0}/\operatorname{Ker}\alpha_{p-1};$$

the condition $d_Md_M=0$ follows from $h^1_{p-1,0}h^1_{p,0}=dh^2_{p,0}$ (see condition (2.6)) and $d_M(a\cdot m)=a\cdot m$ follows from (2.4). Moreover, if h is equivalent to h', then, in particular, $h^1_{p,0}-h'^1_{p,0}=df^1_{p,0}$, and, therefore, they define the same d_M. □

This proposition implies that, actually, $D_R(M)$ does not depend on the bigraded resolution $(R_{*,*},d)$ and, therefore, we can denote it as $D(M)$.

3. Application: Koszul Resolution

In this section, we apply our main theorem to the Koszul resolution of Λ over a free commutative graded algebra. We start from some notation and facts from [5].

Assume that Λ is a field of characteristic 0, X is a connected graded vector space over Λ, and ΛX is a free commutative graded Λ-algebra generated by X, i.e., it is the tensor product of the polynomial algebra $P(X_{\mathrm{even}})$ and exterior algebra $E(X_{\mathrm{odd}})$.

The Koszul resolution of Λ the over commutative graded algebra ΛX is given by

$$\Lambda\xleftarrow{\ \alpha\ }\Lambda X\xleftarrow{\ d_K\ }\Lambda X\otimes\Lambda^1\overline{X}\xleftarrow{\ d_K\ }\Lambda X\otimes\Lambda^2\overline{X}\longleftarrow\ \cdots,\tag{3.1}$$

where $\overline{X}$ is the suspension of X, i.e., $\overline{X}_p=X_{p-1}$, and, therefore, there exists an isomorphism $x\leftrightarrow\overline{x}$; $\Lambda^n\overline{X}$ denotes the subspace of $\Lambda\overline{X}$ spanned by $\overline{x_1}\cdots\overline{x_n}$ and $\overline{x_i}\in\overline{X}$, α is the clear projection, and the Koszul differential d_K is given by

$$d_K(a\otimes\overline{x}_1\cdots\overline{x}_n)=\sum_i\pm a\cdot x_i\otimes\overline{x}_1\cdots\overline{x}_{i-1}\cdot\overline{x}_{i+1}\cdots\overline{x}_n,$$

$a\in\Lambda X$, $\overline{x}\in\overline{X}$.

We consider the Koszul resolution as a *bigraded* resolution: $a\otimes\overline{x_1}\cdots\overline{x_q}\in R_{p,q}$, where $p=|a|+\sum_{k=1}^{q}|x_k|$ and the Koszul differential d_K is *horizontal*; it maps $R_{p,q}$ to $R_{p,q-1}$.

Of course, the Koszul resolution is contractible as a $DG-\Lambda$-module, there exist Λ-homomorphisms

$$\eta : \Lambda \longrightarrow \Lambda X, \quad s : \Lambda X \otimes \Lambda \overline{X}$$

such that

$$\alpha\eta = \mathrm{id}, \quad \eta\alpha + d_K s = \mathrm{id}, \quad s d_K + d_K s = \mathrm{id}\,.$$

Assuming that X has a well-ordered basis $\{x_i\}_{i\in I}$, it is possible to give explicit formulas for s: for an element $x_{k_1}^{p_1}\cdots x_{k_m}^{p_m}\otimes \overline{x}_{t_1}^{q_1}\cdots\overline{x}_{t_n}^{q_n}\in \Lambda X\otimes\Lambda\overline{X}$ with $x_{k_1}<\cdots<x_{k_m}$ and $x_{t_1}<\cdots<x_{t_n}$, we define

$$s\big(x_{k_1}^{p_1}\cdots x_{k_m}^{p_m}\otimes \overline{x}_{t_1}^{q_1}\cdots\overline{x}_{t_n}^{q_n}\big)=0$$

if $x_{k_m}<x_{t_n}$,

$$s\big(x_{k_1}^{p_1}\cdots x_{k_m}^{p_m}\otimes \overline{x}_{t_1}^{q_1}\cdots\overline{x}_{t_n}^{q_n}\big)=\frac{1}{q_n+1}x_{k_1}^{p_1}\cdots x_{k_m}^{p_m-1}\otimes \overline{x}_{t_1}^{q_1}\cdots\overline{x}_{t_n}^{q_n+1}$$

if $x_{k_m}=x_{t_n}$, and

$$s\big(x_{k_1}^{p_1}\cdots x_{k_m}^{p_m}\otimes \overline{x}_{t_1}^{q_1}\cdots\overline{x}_{t_n}^{q_n}\big)=x_{k_1}^{p_1}\cdots x_{k_m}^{p_m-1}\otimes \overline{x}_{t_1}^{q_1}\cdots\overline{x}_{t_n}^{q_n}\cdot\overline{x}_{k_m}$$

if $x_{k_m}>x_{t_n}$. Note that a similar contraction is written in [6] and [9].

Now we suppose that ΛX is equipped with a differential $D:\Lambda X\to\Lambda X$ turning $(\Lambda X, D)$ into a commutative DG-algebra. According to the main theorem, there exists a perturbation h of the Koszul differential d_K such that

$$\alpha : (\Lambda X\otimes\Lambda\overline{X}, d_K+h)\longrightarrow\Lambda$$

is a resolution of Λ over $(\Lambda X, D)$.

Using the contraction s it is possible to give an algorithm for computing the perturbation h.

First, let us mention that $\Lambda X\otimes\Lambda\overline{X}$ is a free commutative graded algebra with generators $x_i\otimes 1$ and $1\otimes\overline{x_i}$, $x_i\in\{x_i\}_{i\in I}$, and, therefore, it suffices to define h on $x_i\otimes 1$ and $1\otimes\overline{x_i}$ and then extend as a derivation.

First, we define $h^1(x_i\otimes 1)=D(x_i)\otimes 1$ and

$$h^1(1\otimes\overline{x}_i)=s(D\otimes\mathrm{id})d_K(1\otimes\overline{x_i})=s(Dx_i\otimes 1).$$

Extending it as a derivation, we obtain

$$h^1:\Lambda X\otimes\Lambda\overline{X}\longrightarrow\Lambda X\otimes\Lambda\overline{X}.$$

We define the next component h^2 on generators as $h^2(x_i\otimes 1)=0$ and

$$h^2(1\otimes\overline{x}_i)=s\big(h^1h^1(1\otimes\overline{x_i})\big),$$

and again extend it as a derivation.

By induction, assuming that $h^{k<m}$ are already constructed, we define $h^m(x_i\otimes 1)=0$ and

$$h^m(1\otimes\overline{x}_i)=s\Big(\sum_{k=1}^{m-1}h^{m-k}h^k(1\otimes\overline{x}_i)\Big),$$

and extend it as a derivation.

Example. Let us take $\Lambda X=\Lambda(a,b,u,z)$ with $|a|=|b|=1$, $|u|=3$, and $|z|=5$. Assuming the order $a<b<u<z$, we can construct the contraction s. In particular, $s(x\otimes 1)=1\otimes\overline{x}$ for $x=a,b,c,u,z$; $s(a\cdot b\otimes 1)=a\otimes\overline{b}$; $s(b\cdot u\otimes 1)=b\otimes\overline{u}$; $s(a\cdot b\otimes b)=\dfrac{1}{2}a\otimes\overline{b}^2$, etc.

Now we suppose that ΛX is equipped with a differential given by

$$D(a)=0,\quad D(b)=0,\quad D(u)=a\cdot b,\quad D(z)=b\cdot u.$$

Then, using the procedure described above, we obtain for h^1

$$h^1(1 \otimes \overline{a}) = 0, \quad h^1(1 \otimes \overline{b}) = 0,$$

$$\begin{aligned} h^1(1 \otimes \overline{u}) &= s(D \otimes \mathrm{id})d_k(1 \otimes \overline{u}) = s(D \otimes \mathrm{id})(u \otimes 1) \\ &= s(a \cdot b \otimes 1) = a \otimes \overline{b}, \\ h^1(1 \otimes \overline{z}) &= s(D \otimes \mathrm{id})d_k(1 \otimes \overline{z}) = s(D \otimes \mathrm{id})(z \otimes 1) \\ &= s(b \cdot u \otimes 1) = b \otimes \overline{u}. \end{aligned}$$

Extending it as a derivation, we obtain

$$\begin{gathered} h^1(1 \otimes \overline{a}^m \cdot \overline{b}^n \cdot \overline{u}^p \cdot \overline{z}^q) \\ = p \cdot a \otimes \overline{a}^m \cdot \overline{b}^{n+1} \cdot \overline{u}^{p-1} \cdot \overline{z}^q + q \cdot b \otimes \overline{a}^m \cdot \overline{b}^n \cdot \overline{u}^{p+1} \cdot \overline{z}^{q-1}. \end{gathered}$$

For h^2, we obtain

$$h^2(1 \otimes \overline{a}) = 0, \quad h^2(1 \otimes \overline{b}) = 0, \quad h^2(1 \otimes \overline{u}) = 0,$$

$$h^2(1 \otimes \overline{z}) = sh^1h^1(1 \otimes \overline{z}) = sh^1(b \otimes \overline{u}) = s(a \cdot b \otimes \overline{b}) = \frac{1}{2} a \otimes \overline{b}^2,$$

and, extending it as a derivation,

$$h^2(1 \otimes \overline{a}^m \cdot \overline{b}^n \cdot \overline{u}^p \cdot \overline{z}^q) = \frac{1}{2} q \cdot a \otimes \overline{a}^m \cdot \overline{b}^{n+2} \cdot \overline{u}^p \cdot \overline{z}^{q-1}.$$

A straightforward verification shows that $h^1h^2 + h^2h^1 = 0$ and $h^2h^2 = 0$, which yields $h^3 = h^4 = \ldots = 0$.

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