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HOMOTOPY GERSTENHABER ALGEBRAS: EXAMPLES AND APPLICATIONS

T. Kadeishvili UDC 515.14

ABSTRACT. In this paper, we indicate several aspects of the notion of homotopy Gerstenhaber algebras and give some important examples and applications.

CONTENTS

1. Homotopy Gerstenhaber Algebras

The notion of a *homotopy G-algebra* (breifly, hGa) was introduced by Gerstenhaber and Voronov in [6, 16] as an additional structure on a dg algebra $(A, d, \cdot)$ that induces a Gerstenhaber algebra (see below) structure on homology.

An hGa is defined as a differential graded algebra (dga) $(A, d, \cdot)$ together with a sequence of operations

$$E_{1,k} : A \otimes A^{\otimes k} \to A, \quad k = 1, 2, 3, \dots, \quad \deg E_{1,k} = -k, \quad E_{1,0} = E_{0,1} = \mathrm{id},$$

and the notation $E_{1,k}(a \otimes b_1 \otimes \dots \otimes b_k) = E_{1,k}(a; b_1, \dots, b_k)$, subject to the following conditions:

$$dE_{1,k}(a; b_1, \dots, b_k) + E_{1,k}(da; b_1, \dots, b_k) + \sum_i E_{1k}(a; b_1, \dots, db_i, \dots, b_k)$$
$$= b_1 E_{1k}(a; b_2, \dots, b_k) + \sum_i E_{1k}(a; b_1, \dots, b_i b_{i+1}, \dots, b_k) + E_{1k}(a; b_1, \dots, b_{k-1}) b_k; \quad (1)$$

$$a_1 E_{1,k}(a_2; b_1, \dots, b_k) + E_{1,k}(a_1 \cdot a_2; b_1, \dots, b_k) + E_{1,k}(a_1; b_1, \dots, b_k) a_2$$
$$= \sum_{p=1,\dots,k-1} E_{1,p}(a_1; b_1, \dots, b_p) \cdot E_{1,m-p}(a_2; b_{p+1}, \dots, b_k); \quad (2)$$

$$E_{1,n}(E_{1,m}(a; b_1, \dots, b_m); c_1, \dots, c_n) =$$
$$= \sum E_{1,n-\sum l_i+m}(a; c_1, \dots, c_{k_1}, E_{1,l_1}(b_1; c_{k_1+1}, \dots, c_{k_1+l_1}),$$
$$c_{k_1+l_1+1}, \dots, c_{k_m}, E_{1,l_m}(b_m; c_{k_m+1}, \dots, c_{k_m+l_m}), c_{k_m+l_m+1}, \dots, c_n) \quad (3)$$

(note that for simplicity we work mod 2).

Let us present the above conditions in low dimensions.

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For $k = 1$, the condition (1) gives

$$dE_{1,1}(a;b) + E_{1,1}(da;b) + E_{1,1}(a;db) = a \cdot b + b \cdot a, \tag{4}$$

i.e., the operation $E_{1,1}$ is a $\smile_1$-type product, which measures the noncommutativity of A. Below we use the notation $E_{1,1} = \smile_1$.

For $k = 2$, the condition (2) gives

$$(a \cdot b) \smile_1 c + a \cdot (b \smile_1 c) + (a \smile_1 c) \cdot b = 0, \tag{5}$$

i.e., $E_{1,1} = \smile_1$ satisfies the so-called *Hirsch formula*.

For $k = 2$, the condition (1) gives

$$\begin{aligned} &a \smile_1 (b \cdot c) + b \cdot (a \smile_1 c) + (a \smile_1 b) \cdot c \\ &\qquad = dE_{1,2}(a;b,c) + E_{1,2}(da;b,c) + E_{1,2}(a;db,c) + E_{1,2}(a;b,dc), \end{aligned} \tag{6}$$

so the "left Hirsch formula" is satisfied just up to chain homotopy and the suitable homotopy is the operation $E_{1,2}$.

Moreover, the condition (3) yields

$$(a \smile_1 b) \smile_1 c - a \smile_1 (b \smile_1 c) = E_{1,2}(a;b,c) + E_{1,2}(a;c,b), \tag{7}$$

so this $\smile_1$ is not strictly associative, but the operation $E_{1,2}$ somehow measures the lack of its associativity.

The operations $E_{1,k}$ forming hGa have a good description in terms of the *surjection operad* χ (for the definition, see [3, 14]). Namely, to the dot product corresponds the element $(1,2) \in \chi_0(2)$; to the $E_{1,1} = \smile_1$ product corresponds $(1,2,1) \in \chi_1(2)$, and generally to the operation $E_{1,k}$ corresponds the element $E_{1,k} = (1,2,1,3,\ldots,1,k,1,k+1,1) \in \chi_k(k+1)$.

2. Three Aspects of Homotopy Gerstenhaber Algebras

2.1. The structure of Gerstenhaber algebras in the homology of an hGa. A Gerstenhaber algebra is defined as a graded commutative algebra $(H, \cdot)$ equipped additionally with a Lie bracket of degree -1 $[\ ,\ ] : H^p \otimes H^q \to H^{p+q-1}$ satisfying the Leibniz rule

$$[a, b \cdot c] = b \cdot [a,c] + [a,b] \cdot c.$$

For an hGa $(A, d, \cdot, \{E_{1,k}\})$, the operation $E_{1,1} = \smile_1$ is not associative, but the condition (7) implies the so-called "pre-Lie condition"

$$(a \smile_1 b) \smile_1 c - a \smile_1 (b \smile_1 c) = (a \smile_1 c) \smile_1 b - a \smile_1 (c \smile_1 b),$$

which guarantees that the commutator $[a,b] = a \smile_1 b - b \smile_1 a$ satisfies the Jacobi identity. Thus, it forms on the desuspension $s^{-1}A$ a structure of a dg-Lie algebra and thus induces the Lie bracket on homology $H(A)$. Moreover, the conditions (5) and (6) guarantee that the Lie bracket induced on $H(A)$ satisfies the Leibniz rule. Thus, $H(A)$ is a Gerstenhaber algebra.

2.2. The hGa structure and multiplication in the bar construction. For an hGa $(A, d, \cdot, \{E_{1,k}\})$, the sequence $\{E_{1,k}\}$ defines on the bar construction BA of a dga $(A, d, \cdot)$ a multiplication turning BA into a dg bialgebra. In fact, this means that an hGa is a B_∞-algebra in the sense of Getzler [7].

Indeed, the sequence of operations $\{E_{1,k}\}$ defines a homomorphism

$$E : BA \otimes BA \to A$$

by

$$E([\,] \otimes [a]) = E([a] \otimes [\,]) = a, \quad E([a] \otimes [b_1|\cdots|b_n]) = E_{1,n}(a; b_1, \ldots, b_n)$$

and

$$E([a_1|\cdots|a_m]\otimes[b_1|\cdots|b_n])=0$$

if $m>1$. Since the bar construction BA is a cofree coalgebra, E induces a graded coalgebra map $\mu_E: BA\otimes BA\to BA$. Then the conditions (1) and (2) are equivalent to the condition

$$dE+E(d_{BA}\otimes id+id\otimes d_{BA})+E\smile E=0,$$

that is, E is a twisting cochain, and this is equivalent to μ_E being a chain map. Moreover, the condition (3) is equivalent to μ_E being associative. Finally, we have that $(BA,d_{BA},\Delta_{BA},\mu_E)$ is a dg bialgebra.

2.3. An hGa as a "strong homotopy commutative" dga. This is the third aspect of an hGa [11]: it measures the noncommutativity of A.

Steenrod's $\smile_1$ product is the classical tool that measures the noncommutativity of a dg algebra:

$$d(a\smile_1 b)=da\smile_1 b+a\smile_1 db-a\cdot b+b\cdot a.$$

The existence of $\smile_1$ in a dga $(A,d,\cdot)$ guarantees the commutativity of $H(A)$, but a $\smile_1$ product satisfying just this condition is too poor for most applications. But a $\smile_1$ product that is a starting operation of some hGa structure is much powerful: it satisfies Hirsch formula (5), up to homotopy left Hirsch formula (6), pre-Lie condition (7), etc.

Note that if $(A,d,\cdot)$ is a commutative dga, its bar construction BA is a dg bialgebra with respect to the shuffle product, and if $(A,d,\cdot,\{E_{1,k}\})$ is an hGa, then BA is again a dg bialgebra but now with respect to μ_E.

3. Three Examples of Homotopy Gerstenhaber Algebras

Let us present three main examples of homotopy Gerstenhaber algebras.

3.1. Cochain complex of 1-reduced simplicial set $C^*(X)$ (see [2]). The hGa structure on $C^*(X)$ is a consequence of diagonal constructed on the cobar construction $\Omega C_*(X)$ by Baues in [2]. The starting operation $E_{1,1}$ is the classical Steenrod $\smile_1$ product.

3.2. Hochschild cochain complex $C^*(U,U)$ of an associative algebra U (see [7, 10]). The operations $E_{1,k}$ (now called *brace operations*) here were defined in [10] with the purpose of describing $A(\infty)$-algebras in terms of Hochschild cochains. Again the starting operation $E_{1,1}$ is the classical Gerstenhaber circle product, which is a sort of $\smile_1$-product.

3.3. The cobar construction of a dg *bialgebra* (see [12]). By definition, the cobar construction ΩC of a dg coalgebra $(C,d:C\to C,\Delta:C\to C\otimes C)$ is a dga. If C is additionally equipped with a multiplication $\mu: C\otimes C\to C$ turning it into a dg bialgebra, how is this structure reflected on the cobar construction ΩC? In [12], it is shown that μ gives rise to an hGa structure on ΩC. And again the starting operation $E_{1,1}$ is classical: it is the Adams $\smile_1$-product defined for ΩC in [1] using the *multiplication* of C.

4. Applications

4.1. An hGa on Hochschild complex. The hGa structure is an important tool for classical Gerstenhaber's deformation of associative algebras [5]. Here we indicate two more applications of this structure.

4.1a. Deligne's conjecture. Deligne's conjecture states that the little square operad acts on the Hochschild cochain complex $C^*(U,U)$ of an associative algebra.

As was already mentioned, $C^*(U,U)$ is an hGa with respect to brace operations constructed in [7, 10]. This fact was used for proving Deligne's conjecture in [14]: The elements of surjection operad $E_{1,k} = (1,2,1,\dots,1,k+1,1)$ together with the element $(1,2)$ generate the suboperad $F_2\chi$ which is equivalent to the little square operad (see [3, 14]).

4.1b. A_∞-algebras. An hGa structure on the Hochschild cochain complex allows one to interpret and classify minimal A_∞-algebras $(M,\{m_i\})$ (the minimality here means $m_1 = 0$, in this case (M,m_2) is an associative algebra). Each operation $m_i : M^{\otimes i} \to M$ can be interpreted as a Hochschild cochain $m_i \in C^*(M,M)$, and the Stasheff defining condition of A_∞-algebras (see [15]) looks like $dm = m \smile_1 m$, where $m = m_3 + m_4 + \cdots$, that is, m is a twisting cochain with respect to the $\smile_1$ product. Two minimal A_∞ algebras $(M,\{m_i\})$ and $(M,\{\bar{m}_i\})$ are isomorphic if and only if there exists $g = g_1 + g_2 + \cdots$, $g_i \in C^i(M,M)$ such that

$$\overline{m} = m + dg + g\cdot g + E_{1,1}(g;m) + \sum_{k=1}^{\infty} E_{1,k}(\overline{m};g,\dots,g),$$

(the gauge transformation). Using this equivalence in [10, 13], we developed the obstructions for equivalence (isomorphism in the category of A_∞-algebras) and degeneracy ($m_i = 0$ for $i > 2$) of minimal A_∞ structures.

The interpretation of minimal A_∞ structures as a twisting cochain in the Hochschild complex is useful for various problems in mathematics and physics. In [8], it is shown that to a dga $(A,d,\cdot)$ in homology $H(A)$ there exists the structure of a minimal A_∞-algebra $(H(A),\{m_i\})$ such that A_∞-algebras $(A,\ \{m_1 = d,\ m_2 = \cdot,\ m_3 = 0,\ m_4 = 0,\dots\})$ and $(H(A),\{m_i\})$ are weakly equivalent in the category of A_∞-algebras. The A_∞-algebra $(H(A),\{m_i\})$ is called a *minimal model* of a dga $(A,d,\cdot)$. The main property of minimal model is that two dg algebras A and A' are weakly equivalent if and only if their minimal models $(H(A),\{m_i\})$ and $(H(A'),\{m_i'\})$ are isomorphic. In [9], the existence of minimal model for an A_∞-algebra $(M,\{m_i\})$ was shown; the minimal structure here appears on $H(M,m_1)$. In [4], this fact was generalized for A_∞ categories and used to establish the equivalence of certain A_∞-categories that arise in the theory of open strings.

4.2. An hGa on the cochain complex $C^*(X)$. Homology of the bar construction $BC^*(X)$ gives cohomology modules of the loop space $H^*(\Omega X)$. The Baues hGa structure $(C^*(X),\delta,\smile,\{E_{1,k}\})$ (see [2]) defines on $BC^*(X)$ a multiplication μ_E and this describes the multiplicative structure on $H^*(\Omega X)$. Moreover, it allows one to obtain the second bar construction $BBC^*(X)$, which determines the cohomology modules of the second loop space $H^*(\Omega^2 X)$.

4.3. An hGa on the cobar construction of a bialgebra. The hGa structure on the cobar construction immediately implies the existence of the well-known Gerstenhaber algebra structure on the homology algebra of the double loop space.

We start with dg bialgebra $C = (C_*(\Omega X),d,\Delta_{AW},\mu_\Omega)$, where $C_*(\Omega X)$ is the chain complex of the loop space ΩX with the Alexander–Whitney diagonal Δ_{AW} and the product μ_Ω induced by loop multiplication. The cobar construction $\Omega C_*(\Omega X)$ determines a homology algebra of the double loop space $H_*(\Omega^2 X)$. Since in this case $\Omega C_*(\Omega X)$ is an hGa (see [12]), its homology $H(\Omega C_*(\Omega X)) = H_*(\Omega^2 X)$ is a Gerstenhaber algebra.

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T. Kadeishvili

A. Razmadze Mathematical Institute, Tbilisi, Georgia

E-mail: `kade@rmi.ge`