DOI 10.1007/s10958-016-3064-y
Journal of Mathematical Sciences, Vol. 218, No. 6, Month, 2016

B_∞-ALGEBRA STRUCTURE IN HOMOLOGY OF A HOMOTOPY GERSTENHABER ALGEBRA

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UDC 512.7

Abstract. The minimality theorem states, in particular, that on cohomology $H(A)$ of a dg algebra there exists sequence of operations $m_i : H(A)^{\otimes i} \to H(A)$, $i = 2, 3, \dots$, which form a minimal A_∞-algebra $(H(A), \{m_i\})$. This structure defines on the bar construction $BH(A)$ a correct differential d_m so that the bar constructions $(BH(A), d_m)$ and BA have isomorphic homology modules. It is known that if A is equipped additionally with a structure of homotopy Gerstenhaber algebra, then on BA there is a multiplication which turns it into a dg bialgebra. In this paper, we construct algebraic operations $E_{p,q} : H(A)^{\otimes p} \otimes H(A)^{\otimes q} \to H(A)$, $p, q = 0, 1, 2, \dots$, which turn $(H(A), \{m_i\}, \{E_{p,q}\})$ into a B_∞-algebra. These operations determine on $BH(A)$ correct multiplication, so that $(BH(A), d_m)$ and BA have isomorphic homology algebras.

CONTENTS

1. Introduction

For a differential graded algebra (A, d, μ), the differential $d : A_* \to A_{*-1}$ and the multiplication $\mu : A_p \otimes A_q \to A_{p+q}$ define on the bar construction BA a differential $d_B : BA \to BA$ which turns (BA, d_B, ∇) into a dg coalgebra. For example, for $A = C_*(G)$ the bar construction $BC_*(G)$ gives cohomology *modules* of the classifying space $H_*(B_G)$, but not the homology *algebra*. For $A = C^*(X)$ the bar construction $BC^*(X)$ gives cohomology *modules* of the loop space $H^*(\Omega X)$, but not the cohomology *algebra*.

There exists the notion of homotopy Gerstenhaber algebra (see [19, 20]), briefly hGa, which allows one to construct a correct multiplication on the bar construction. This is an additional structure on a dg algebra (A, d, μ), which consists of a sequence of operations

$$E_{1,k} : A \otimes A^{\otimes k} \to A, \quad k = 1, 2, \dots,$$

which determine on BA a multiplication turning it into a dg bialgebra.

Our aim is to transfer these structures to homology level, i.e., from A to $H(A)$.

Note that the homology $H(A)$ is also a dga with trivial differential and induced multiplication $\mu^* : H(A) \otimes H(A) \to H(A)$, but generally the bar constructions BA and $BH(A)$ have different homologies.

Translated from Sovremennaya Matematika i Ee Prilozheniya (Contemporary Mathematics and Its Applications), Vol. 97, Proceedings of the International Conference "Lie Groups, Differential Equations, and Geometry," June 10–22, 2013, Batumi, Georgia, Part 2, 2015.

In [8], the so-called minimality theorem is proved, which states that for a dg algebra (A, d, μ), its homology $H(A)$ (all $H_i(X)$ are assumed free) can be equipped with a sequence of multi-operations

$$m_i : H(A)^{\otimes i} \to H(A), \quad i = 1, 2, \dots, \quad m_1 = 0,\ m_2 = \mu^*,$$

turning $(H(A), \{m_i\})$ into a minimal A_∞-algebra in sense of Stasheff [18], which is weakly equivalent to dga (A, d, μ). These A_∞ operations $\{m_i\}$ determine on $BH(A)$ new, perturbed differential $d_m : BH(A) \to BH(A)$ so that BA and $BH(A)$ have isomorphic homologies.

The aim of this paper is to construct for a hGa $(A, d, \mu, \{E_{1k}\})$ on its homology A_∞-algebra $(H(A), \{m_i\})$ certain additional structure, the so-called B_∞ algebra (see [7]), consisting of multi-operations

$$E_{p,q} : H(A)^{\otimes p} \otimes H(A)^{\otimes q} \to H(A), \quad p, q = 0, 1, 2, \dots,$$

which determines on $BH(A)$ the correct multiplication so that the bar constructions $BH(A)$ and BA will have isomorphic homology algebras.

Remark. This can be summarized as follows: If A is a dg algebra, hGa, or a commutative dg algebra, then $H(A)$ becomes respectively an A_∞ (see [8]), B_∞ (present paper), or C_∞ (see [11]) algebra (a commutative version of A_∞).

2. Preliminaries

In this section, we give some notions and construction needed in the sequel.

2.1. A_∞-algebras. The notion of an A_∞-algebra was introduced by Stasheff in [18]. This notion generalizes the notion of a dg algebra.

An A_∞-algebra is a graded module M with a given sequence of operations

$$\Big\{m_i : M^{\otimes i} \to M, \quad i = 1, 2, \dots, \quad \deg m_i = i - 2\Big\},$$

which satisfies the following conditions:

$$\sum_{i+j=n+1} \sum_{k=0}^{n-j} \pm m_i\Big(a_1 \otimes \dots \otimes a_k \otimes m_j(a_{k+1} \otimes \dots \otimes a_{k+j}) \otimes \dots \otimes a_n\Big) = 0, \tag{1}$$

(we ignore the signs).

In particular, for the operation $m_1 : M \to M$ we have $\deg m_1 = -1$ and $m_1 m_1 = 0$; this m_1 can be regarded as a differential on M. The operation $m_2 : M \otimes M \to M$ is of degree 0 and satisfies

$$m_1 m_2(a_1 \otimes a_2) + m_2(m_1 a_1 \otimes a_2) + m_2(a_1 \otimes m_1 a_2) = 0,$$

i.e., m_2 can be regarded as a multiplication on M and m_1 is a derivation with respect to it. Thus, (M, m_1, m_2) is a sort of (maybe nonassociative) dg algebra. For the operation m_3: $\deg m_3 = 1$ and

$$\begin{aligned} m_1 m_3(a_1 \otimes a_2 \otimes a_3) &+ m_3(m_1 a_1 \otimes a_2 \otimes a_3) + m_3(a_1 \otimes m_1 a_2 \otimes a_3) \\ &+ m_3(a_1 \otimes a_2 \otimes m_1 a_3) + m_2\big(m_2(a_1 \otimes a_2) \otimes a_3\big) + m_2\big(a_1 \otimes m_2(a_2 \otimes a_3)\big) = 0; \end{aligned}$$

thus, the product m_2 is *homotopy associative* and the appropriate chain homotopy is m_3 (some authors call A_∞-algebras *strong homotopy associative DG-algebras*).

The main meaning of defining condition (1) of an A_∞-algebra $(M, \{m_i\})$ is the following. The sequence of operations $\{m_i\}$ determines on the bar construction

$$BM = T^c(sM) = \Lambda + sM + sM \otimes sM + sM \otimes sM \otimes sM + \cdots$$

(here Λ is the ground ring and $(sM)_k = M_{k+1}$ is the standard suspension) a coderivation

$$d_m(a_1 \otimes \cdots \otimes a_n) = \sum_{k,j} a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_n,$$

and the Stasheffs condition (1) is equivalent to $d_m d_m = 0$; thus (BM, d_m) is a dg coalgebra, which is called the *bar construction* of A_∞-algebra $(M, \{m_i\})$.

A morphism of A_∞-algebras $f : (M, \{m_i\}) \to (M', \{m'_i\})$ is defined as a dg coalgebra map of the bar constructions

$$f : B\big(M, \{m_i\}\big) \to B\big(M', \{m'_i\}\big),$$

which, due to cofreeness of the tensor coalgebra $T^c(sM)$, is uniquely determined by the projection

$$f : B\big(M, \{m_i\}\big) \to B\big(M', \{m'_i\}\big) \to M',$$

which, in fact, is a collection of homomorphisms

$$\Big\{f_i : M^{\otimes i} \to M', \quad i = 1, 2, \ldots, \quad \deg f_i = i - 1\Big\},$$

subject of some conditions (see, e.g., [8, 11]). In particular, $f_1 m_1 = m_1 f_1$, i.e.

$$f_1 : (M, m_1) \to (M', m'_1)$$

is a chain map. We define a weak equivalence of A_∞-algebras as a morphism $\{f_i\}$, where f_1 is a homology isomorphism.

An A_∞-algebra $(M, \{m_i\})$ is called *minimal* if $m_1 = 0$; in this case (M, m_2) is a *strictly* associative graded algebra (see (1) for $n = 3$). Assume that

$$f : \big(M, \{m_i\}\big) \to \big(M', \{m'_i\}\big)$$

is a weak equivalence of minimal A_∞-algebras; then

$$f_1 : (M, m_1 = 0) \to (M', m'_1 = 0),$$

which by definition should induce isomorphism of homology, is automatically an isomorphism. It is easy to verify that in this case f is an isomorphism in the category of A_∞-algebras, thus a weak equivalence of minimal A_∞-algebras is an isomorphism. This fact motivates the word *minimal* in this notion: the Sullivan minimal model has a similar property.

An A_∞-algebra $(M, \{m_i\})$ with $m_{>2} = 0$ is just a dg algebra, and an A_∞-algebra morphism

$$\{f_i\} : (M, \{m_1, m_2, 0, 0, \ldots\}) \to (M', \{m'_1, m'_2, 0, 0, \ldots\})$$

with $f_{>1} = 0$ is just a multiplicative chain map; thus the category of dg algebras is a subcategory of A_∞-algebras.

2.2. Twisting cochains. Let $(K, d_K, \nabla_K : K \to K \otimes K)$ be a dg coalgebra, and (A, d_A, μ) be a dg algebra. $Hom(K, A)$ is a dg algebra with differential $d\phi = d_A\phi + \phi d_K$ and multiplication $\phi \smile \psi = \mu(\phi \otimes \psi)\nabla_K$.

A twisting cochain is defined as a homomorphism $\phi : K \to A$ of degree -1 (that is $\phi : K_* \to A_{*-1}$) satisfying the Brown condition[1] $d\phi = \phi \smile \phi$ (see [4]). Let $T(K, A)$ be the set of all twisting cochains.

Two twisting cochains are equivalent if there exists $c : K \to A$ such that

$$\psi = \phi + dc + \psi \smile c + c \smile \phi,$$

notation $\phi \sim_c \psi$.

[1]In other words the Maurer–Cartan equation or the master equation.

The Berikasvili functor $D(K, A)$ is defined as follows:

$$D(K, A) = \frac{M(K, A)}{\sim}.$$

A dg algebra map $f : A \to A'$ induces the map $T(K, A) \to T(K, A')$: if ϕ is a twisting cochain so is $f \circ \phi$. Moreover, if $\phi \sim_c \psi$, then $f \circ \phi \sim_{f \circ c} f \circ \psi$. Thus, we have a map $D(f) : D(K, A) \to D(K, A')$.

Theorem 1 (Berikashvili [3])**.** *If $f : A \to A'$ is a weak equivalence of dg algebras (homology isomorphism), then*

$$D(f) : D(K, A) \to D(K, A')$$

is a bijection.

2.2.1. Twisting cochains and the bar construction. Any twisting cochain $\phi : K \to A$ induces a dg coalgebra map $B(\phi) : K \to BA$ by

$$B(\phi) = \sum_i (\phi \otimes \cdots \otimes \phi)\nabla_K^i,$$

where $\nabla_K^i : K \to K^{\otimes i}$ is the iteration of comultiplication ∇_K:

$$\nabla_K^0 = id, \quad \nabla_K^2 = \nabla_K, \quad \nabla_K^n = (id \otimes \nabla_K)\nabla_K^{n-1}.$$

Conversely, any dg coalgebra map $f : K \to BA$ is $B(\phi)$ for $\phi = p \circ f : K \to BA \to A$. In fact, we have a bijection $Mor_{dgcoalg}(K, BA) \leftrightarrow T(K, A)$.

Moreover, if $\phi \sim_c \psi$, then $B(\phi)$ and $B(\psi)$ are homotopic in the category of dg coalgebras: the chain homotopy $D(c) : K \to BA$ is given by

$$D(c) = \sum_{k,j} \Big(\underbrace{\psi \otimes \cdots \otimes \psi}_{j \text{ times}} \otimes c \otimes \phi \otimes \cdots \otimes \phi \Big),$$

and, in addition, $D(c)$ is a $(B(\phi) - B(\psi))$-coderivation, i.e.,

$$\nabla_B D(c) = \big(B(\psi) \otimes D(c) + D(c) \otimes B(\phi)\big)\nabla_K.$$

Thus, we have a bijection $[K, BA] \leftrightarrow D(K, A)$, where $[K, BA]$ denotes the set of chain homotopy classes in the category of dg coalgebras.

2.2.2. A_∞-twisting cochains. Now we want to replace a dg algebra (A, d_A, μ) with an A_∞-algebra $(M, \{m_i\})$.

An A_∞-twisting cochain we define as a homomorphism $\phi : K \to M$ of degree -1 satisfying the condition

$$\sum_{k=1}^{\infty} m_k(\phi \otimes \cdots \otimes \phi)\nabla^k = 0.$$

Let $T_\infty(K, M)$ be the set of all A_∞-twisting cochains.

Two twisting A_∞-cochains are said to be equivalent if there exists $c : K \to M$ such that

$$\psi = \phi + \sum_{k,j} \Big(\underbrace{\psi \otimes \cdots \otimes \psi}_{j \text{ times}} \otimes c \otimes \phi \otimes \cdots \otimes \phi \Big)\nabla^k,$$

notation $\phi \sim_c \psi$.

By $D_\infty(K, M)$ we denote the factor set

$$D_\infty(K, M) = \frac{T_\infty(K, M)}{\sim}.$$

Assume that

$$f = \{f_i\} : (M, \{m_i\}) \to (M', \{m_i'\})$$

is a morphism of A_∞-algebras and $\phi : K \to M$ is an A_∞-twisting cochain. Then it is possible to show that $f(\phi) : K \to M'$ given by

$$f(\phi) = \sum_i f_i(\phi \otimes \cdots \otimes \phi)\nabla_K^i$$

is also an A_∞-twisting cochain. Moreover, if $\phi \sim_c \psi$, then $f(\phi) \sim_{c'} f(\psi)$ with $c' : K \to M'$ given by

$$c' = \sum_{i,j} f_i\Big(\underbrace{\psi \otimes \cdots \otimes \psi}_{j \text{ times}} \otimes c \otimes \phi \ \otimes \cdots \otimes \phi\Big)\nabla^i.$$

Thus, we have a map

$$D_\infty(f) : D_\infty(K, M) \to D_\infty(K, M').$$

The following theorem was proved in [9].

Theorem 2. *If*

$$f = \{f_i\} : (M, \{m_i\}) \to (M', \{m_i'\})$$

is a weak equivalence of A_∞-algebras, then

$$D_\infty(f) : D_\infty(K, M) \to D_\infty(K, M')$$

is a bijection.

2.2.3. A_∞-twisting cochains and the B construction. Any A_∞-twisting cochain $\phi : K \to (M, \{m_i\})$ induces a dg coalgebra map $B(\phi) : K \to B(M, \{m_i\})$ by

$$B(\phi) = \sum_i (\phi \otimes \cdots \otimes \phi)\nabla^i.$$

Conversely, any dg coalgebra map $K \to B(M, \{m_i\})$ is $B(\phi)$ for $\phi = p \circ f : K \to B(M, \{m_i\}) \to M$. In fact, we have a bijection

$$Mor_{dgcoalg}(K, BM) \leftrightarrow T_\infty(K, M).$$

Moreover, if $\phi \sim_c \psi$, then $B(\phi)$ and $B(\psi)$ are homotopic in the category of dg coalgebras: a chain homotopy $D(c) : K \to B(M, \{m_i\})$ given by

$$D(c) = \sum_{k,j} \Big(\underbrace{\psi \otimes \cdots \otimes \psi}_{j \text{ times}} \otimes c \otimes \phi \otimes \cdots \otimes \phi\Big)\nabla^k$$

which is a $B(\phi) - B(\psi)$-coderivation.

Thus, we have a bijection $[K, BM] \leftrightarrow D_\infty(K, M)$.

2.3. B_∞-algebras. The notion of B_∞-algebra was introduced in [2, 7] as an additional structure on a dg module (A, d) which turns the bar construction BA into a dg bialgebra. So it requires a new differential

$$\widetilde{d} : BA \to BA,$$

which should be a coderivation with respect to standard coproduct of BA, and a new associative multiplication

$$\widetilde{\mu} : (BA, \widetilde{d}) \otimes (BA, \widetilde{d}) \to (BA, \widetilde{d}),$$

which should be a map of dg coalgebras, with $1_\Lambda \in \Lambda \subset BA$ as a unit element.

It is mentioned above (see, e.g., [8, 11, 17]) that such $\widetilde{d}$ specifies on A a structure of A_∞-algebra, namely a sequence of operations $\{m_i : \otimes^i A \to A, i = 1, 2, \dots\}$ subject to appropriate conditions.

As for the new multiplication

$$\widetilde{\mu} : B\big(A, \{m_i\}\big) \otimes B\big(A, \{m_i\}\big) \to B\big(A, \{m_i\}\big)$$

by the definition of a dg bialgebra it must be a map of dg coalgebras. Consequently, it is uniquely determined by an A_∞-twisting element, say

$$E_{*,*} : B\big(A, \{m_i\}\big) \otimes B\big(A, \{m_i\}\big) \to \big(A, \{m_i\}\big).$$

In turn, such a twisting cochain is represented by a sequence of operations

$$\Big\{E_{pq} : A^{\otimes p} \otimes A^{\otimes q} \to A, \quad p, q = 0, 1, 2, \dots\Big\}$$

satisfying certain coherency condition together with A_∞ operations $\{m_i\}$.

Therefore, a B_∞-algebra is a graded module equipped with two sets of algebraic multi-operations $(A, \{m_i\}, \{E_{p,q}\})$.

A particular case of a B_∞-algebra of type $m_{\geq 3} = 0$ is called the Hirsch algebra, and a particular case of a Hirsch algebra with $E_{>1,q} = 0$ satisfying certain additional conditions is called the homotopy Gerstenhaber algebra (see below). We refer the reader to [12, 14] for more explanations of these structures. In fact, the present description is enough for this paper.

2.3.1. Homotopy G-algebras. A *homotopy G-algebra* (briefly, hGa) is a dg algebra with "good" $\smile_1$ product. The general notion was introduced in [19, 20].

Definition 1. A homotopy G-algebra is defined as a dg algebra $(A, d, \cdot)$ with a given sequence of operations

$$E_{1,k} : A \otimes (A^{\otimes k}) \to A, \quad k = 0, 1, 2, \dots$$

(the value of the operation $E_{1,k}$ on $a \otimes b_1 \otimes \cdots \otimes b_k \in A \otimes (A \otimes \cdots \otimes A)$ we write as $E_{1,k}(a; b_1, \dots, b_k)$), which satisfies the conditions

$$E_{1,0} = id, \tag{2}$$

$$dE_{1,k}(a; b_1, \dots, b_k) + E_{1,k}(da; b_1, \dots, b_k) + \sum_i E_{1,k}(a; b_1, \dots, db_i, \dots, b_k)$$

$$= b_1 \cdot E_{1,k-1}(a; b_2, \dots, b_k) + E_{1,k-1}(a; b_1, \dots, b_{k-1}) \cdot b_k + \sum_i E_{1,k-1}(a; b_1, \dots, b_i \cdot b_{i+1}, \dots, b_k), \tag{3}$$

$$E_{1,k}(a_1 \cdot a_2; b_1, \dots, b_k) = a_1 \cdot E_{1,k}(a_2; b_1, \dots, b_k) + E_{1,k}(a_1; b_1, \dots, b_k) \cdot a_2$$
$$+ \sum_{p=1}^{k-1} E_{1,p}(a_1; b_1, \dots, b_p) \cdot E_{1,m-p}(a_2; b_{p+1}, \dots, b_k), \tag{4}$$

$$E_{1,n}(E_{1,m}(a; b_1, \dots, b_m); c_1, \dots, c_n)$$
$$= \sum_{0 \leq i_1 \leq j_1 \leq \cdots \leq i_m \leq j_m \leq n} E_{1,n-(j_1+\cdots+j_m)+(i_1+\cdots+i_m)+m}\Big(a;\ c_1,\ \dots,\ c_{i_1},\ E_{1,j_1-i_1}(b_1; c_{i_1+1}, \dots, c_{j_1}),$$
$$c_{j_1+1},\ \dots,\ c_{i_2},\ E_{1,j_2-i_2}(b_2; c_{i_2+1}, \dots, c_{j_2}),\ c_{j_2+1},\ \dots, \tag{5}$$
$$c_{i_m},\ E_{1,j_m-i_m}(b_m; c_{i_m+1}, \dots, c_{j_m}), c_{j_m+1},\ \dots,\ c_n\Big). \tag{6}$$

Let us present these conditions in low dimensions.

The condition (3) for $k = 1$ has the form

$$dE_{1,1}(a; b) + E_{1,1}(da; b) + E_{1,1}(a; db) = a \cdot b + b \cdot a. \tag{7}$$

So the operation $E_{1,1}$ is a sort of $\smile_1$ product: it is the chain homotopy that measures the noncommutativity of A. Below we denote $a \smile_1 b = E_{1,1}(a; b)$.

Condition (4) for $k = 1$ has the form

$$(a \cdot b) \smile_1 c + a \cdot (b \smile_1 c) + (a \smile_1 c) \cdot b = 0; \tag{8}$$

this means that the operation $E_{1,1} = \smile_1$ satisfies the *left Hirsch formula*.

Condition (3) for $k = 2$ has the form

$$dE_{1,2}(a;b,c) + E_{1,2}(da;b,c) + E_{1,2}(a;db,c) + E_{1,2}(a;b,dc) \\ = a \smile_1 (b \cdot c) + (a \smile_1 b) \cdot c + b \cdot (a \smile_1 c). \tag{9}$$

This means that this $\smile_1$ satisfies the *right Hirsch formula* just up to homotopy and the appropriate homotopy is the operation $E_{1,2}$.

Condition (6) for $n = m = 2$ has the form

$$(a \smile_1 b) \smile_1 c + a \smile_1 (b \smile_1 c) = E_{1,2}(a;b,c) + E_{1,2}(a;c,b). \tag{10}$$

This means that the same operation $E_{1,2}$ measures also the deviation from the associativity of the operation $E_{1,1} = \smile_1$.

2.3.2. hGa as a $B(\infty)$-algebra. Here we show that a hGa structure on A is a particular $B(\infty)$-algebra structure: it induces on $B(A) = (T^c(sA), d_B)$ an *associative* multiplication but does not change the differential d_B (see [5, 7, 12, 14]).

Let us extend our sequence $\{E_{1,k},\ k = 0,1,2,\dots\}\}$ to the sequence

$$\left\{E_{p,q} : (A^{\otimes p}) \otimes (A^{\otimes q}) \to A, \quad p,q = 0,1,\dots\right\}$$

adding

$$E_{0,1} = id, \quad E_{0,q>1} = 0, \quad E_{1,0} = id, \quad E_{p>1,0} = 0, \tag{11}$$

and $E_{p>1,q} = 0$.

This sequence defines a map $E : B(A) \otimes B(A) \to A$ by

$$E\big([a_1,\dots,a_m] \otimes [b_1,\dots,b_n]\big) = E_{p,q}(a_1,\dots,a_m;b_1,\dots,b_n).$$

Conditions (3) and (4) mean exactly

$$dE + E(d_B \otimes id + id \otimes d_B) = E \smile E,$$

i.e., E is a twisting cochain. Thus its coextesion is a dg coalgebra map

$$\mu_E : B(A) \otimes B(A) \to B(A).$$

Condition (6) can be rewritten as

$$E(\mu_E \otimes id - id \otimes \mu_E) = 0,$$

so this condition means that the multiplication μ_E is associative. Condition (11) implies that $[\,] \in \Lambda \subset B(A)$ is the unit for this multiplication.

Finally, we obtained that $(B(A), d_B, \Delta, \mu_E)$ is a dg bialgebra; thus a hGa is a $B(\infty)$-algebra.

2.3.3. Three examples of hGas. There are three remarkable examples of homotopy G-algebras.

The first one is the cochain complex of 1-reduced simplicial set $C^*(X)$. The operations $E_{1,k}$ here are dual to cooperations defined by Baues in [2], and the starting operation $E_{1,1}$ is the classical Steenrod $\smile_1$ product.

The second example is the Hochschild cochain complex $C^*(U,U)$ of an associative algebra U. The operations $E_{1,k}$ here were defined in [10] with the purpose of describing $A(\infty)$-algebras in terms of Hochschild cochains, although the properties of those operations which where used as defining ones for the notion of homotopy G-algebra in [20] did not appear there. These operations where defined also in [6]. Again the starting operation $E_{1,1}$ is the classical Gerstenhaber circle product, which is sort of $\smile_1$-product in the Hochschild complex. These operations were used in [16] in the proof of the Deligne hypothesis.

The third example is the the cobar construction ΩC of a dg-bialgebra C. The cobar construction ΩC of a DG-coalgebra $(C, d : C \to C, \Delta : C \to C \otimes C)$ is, by definition, a DG-algebra. Now assume that C is additionally equipped with a multiplication $\mu : C \otimes C \to C$ turning (C, d, Δ, μ) into a DG-bialgebra. How does this multiplication μ reflect on the cobar construction ΩC? There arises a natural hGa structure, and the operations $E_{1,k}$ are constructed in [13]. Again, the starting operation $E_{1,1}$ is classical; it is the Adams $\smile_1$-product defined for ΩC in [1].

3. B_∞-Algebra Structure in Homology of a hGa

Here we turn to the main goal of this paper.

Now assume that $(A, d, \mu, \{E_{1,k}\})$ is a hGa. Note that the sequence of operations $\{E_{1,k}\}$ determines a twisting cochain $E : BA \otimes BA \to A$.

By the minimality theorem (see [8]), on $H(A)$ there exists a structure of minimal A_∞-algebra $(H(A), \{m_i\})$ and a weak equivalence of A_∞-algebras

$$f = \{f_i\} : \big(H(A), \{m_i\}\big) \to \Big(A, \{m_1 = d, m_2 = \mu, m_3 = 0, m_4 = 0, \dots\}\Big).$$

This weak equivalence induces a weak equivalence of dg coalgebras

$$\widetilde{B}(f) : \widetilde{B}\big(H(A), \{m_i\}\big) \to BA.$$

Composing the tensor product

$$\widetilde{B}(f) \otimes \widetilde{B}(f) : \widetilde{B}\big(H(A), \{m_i\}\big) \otimes \widetilde{B}\big(H(A), \{m_i\}\big) \to BA \otimes BA$$

with the twisting cochain $E : BA \otimes BA \to A$ determined by hGa structure operations $\{E_{1,k}\}$, we obtain a twisting cochain

$$E \circ (\widetilde{B}(f) \otimes \widetilde{B}(f)) : \widetilde{B}\big(H(A), \{m_i\}\big) \otimes \widetilde{B}\big(H(A), \{m_i\}\big) \to BA \otimes BA \to A.$$

Our aim is to lift this twisting cochain to a A_∞-twisting cochain

$$E_{*,*} : \widetilde{B}\big(H(A), \{m_i\}\big) \otimes \widetilde{B}\big(H(A), \{m_i\}\big) \to \big(H(A), \{m_i\}\big),$$

which, in turn, will define a needed B_∞ algebra structure on $(H(A), \{m_i\})$.

The existence of $E_{*,*}$ follows from the bijection

$$\begin{aligned} D_\infty(f) : D_\infty\Big(\widetilde{B}\big(H(A), \{m_i\}\big) \otimes \widetilde{B}\big(H(A), \{m_i\}\big), \big(H(A), \{m_i\}\big)\Big) & \\ \longrightarrow D\Big(\widetilde{B}\big(H(A), \{m_i\}\big) \otimes \widetilde{B}\big(H(A), \{m_i\}\big), A\Big), & \end{aligned}$$

which is guaranteed by Theorem 2. In particular, we can take $E_{*,*}$ from the preimage of the class of twisting cochain $E \circ (\widetilde{B}(f) \otimes \widetilde{B}(f))$.

These twisting cochains can be observed from the diagram

$$\begin{array}{ccc} \widetilde{B}\big(H(A), \{m_i\}\big) \otimes \widetilde{B}\big(H(A), \{m_i\}\big) & \xrightarrow{E_{*,*}} & \big(H(A), \{m_i\}\big) \\ \Big\downarrow{\scriptstyle \widetilde{B}(f) \otimes \widetilde{B}(f)} & & \Big\downarrow{\scriptstyle f} \\ BA \otimes BA & \xrightarrow[E]{} & A \end{array}$$

This diagram does not commute, but the twisting cochains $f \circ E_{*,*}$ and $E \circ (\widetilde{B}(f) \otimes \widetilde{B}(f))$ are equivalent. Consequently the diagram of induced dg coalgebra maps commutes up to homotopy.

To summarize, the obtained A_∞-twisting cochain $E_{*,*}$ determines on the A_∞-algebra $(H(A),\{m_i\})$ a structure B_∞-algebra, which in its turn determines a (nonassociative generally) multiplication $\widetilde{B}(E_{*,*})$ on the $\widetilde{B}$-construction $\widetilde{B}(H(A,\{m_i\})$ so that the diagram of dg coalgebra maps

$$\begin{array}{ccc}
\widetilde{B}\big(H(A),\{m_i\}\big)\otimes\widetilde{B}\big(H(A),\{m_i\}\big) & \xrightarrow{\widetilde{B}(E_{*,*})} & \widetilde{B}\big(H(A),\{m_i\}\big) \\
{\scriptstyle \widetilde{B}(f)\otimes\widetilde{B}(f)}\downarrow & & \downarrow{\scriptstyle \widetilde{B}(f)} \\
BA\otimes BA & \xrightarrow[B(E)]{} & BA
\end{array}$$

commutes up to homotopy. Thus the dg coalgebra map

$$\widetilde{B}(f):\widetilde{B}\big(A,\{m_i\}\big)\to BA$$

is multiplicative up to homotopy.

Finally we have the following assertion.

Theorem 3. *Let* $(A,d,\mu,\{E_{1,k}\})$ *be a hGa. Then on its homology* $H(A)$ *there exists a structure of* B_∞*-algebra* $(H(A),\{m_i\},\{E_{p,q}\})$ *such that homology algebras*

$$H\big(\widetilde{B}\big(H(A),\{m_i\},\{E_{p,q}\}\big)\big),\quad \textit{and}\quad H\big(B\big(A,d,\mu,\{E_{1,k}\}\big)\big)$$

are isomorphic.

For a hGa $(A,d,\mu,\{E_{1,k}\})$, the twisting cochain $E: BA\otimes BA\to A$ satisfies the additional conditions (6) which guarantee that the induced multiplication on BA is associative. The twisting cochain $E_{*,*}$ we have obtained satisfies only Brown's condition, but not that condition for associativity, so the obtained multiplication

$$\widetilde{B}(f):\widetilde{B}\big(H(A),\{m_i\}\big)\otimes\widetilde{B}\big(H(A),\{m_i\}\big)\to\widetilde{B}\big(H(A),\{m_i\}\big)$$

is a chain map, but nonassociative generally. Thus the bar construction $\widetilde{B}(H(A),\{m_i\})$ is a nonassociative bialgebra. We expect that this nonassociative multiplication will be a part of a certain A_∞ algebra structure on $\widetilde{B}(H(A),\{m_i\})$, which will allow us to iterate the process.

Acknowledgment. This work was partially supported by FP7–IRSES grant No. 317721 of the European Commission and the Shota Rustaveli National Science Foundation (projects CF/19/5-113/3 and DI/27/5-103/12).

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