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MATHEMATICS

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Simplicial Cutting of a Cubical Set

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ABSTRACT. **To a cubical set Q we assign functorially a simplicial set SQ obtained by canonical "cutting of cubs into simplexes" so that the realizations of Q and SQ are homeomorfic. The corresponding natural chain map $C_\bullet(Q) \to C_\bullet(SQ)$ of the normalized chain complexes which is a *DG*-coalgebras map and, for a monoidal cubical set Q, a *DG*-Hopf algebras map as well, is also constructed.**

Key words: (co)simplicial object, (co)cubical object, tensor product of functors, *DG*-Hopf algebras map.

In this paper we present a construction which assigns to a cubical set Q a simplicial set SQ so that the realizations $|Q|$ and $|SQ|$ are homeomorfic. This construction is functorial and is based on the standard cutting of a cub into simplexes. The construction generalizes this standard cutting process for an arbitrary cubical set. A *DG*-coalgebra map (a map of *DG*-Hopf algebras if Q is a monoidal cubical set) $C_\bullet^N(Q) \to C_\bullet^N(SQ)$ is also constructed. We remark here that the similar chain map $C_\bullet^{cub}(Q) \to C_\bullet^{sim}(SQ)$ from the singular cubical chain complex of a topological space X to the normalized singular simplicial one was considered in [1,2].

The need in such construction arises if we attempt to iterate via cubical set models the modeling process for loop spaces or for total spaces of some fibrationos [1, 3]. Besides, such a construction may have an independent value as well. This construction we use to obtain simplicial models for iterated loop spaces.

For used notions and results from simplicial theory we refer to [4,5].

1. (Co)cubical objects. The notions of cocubical and cubical objects are analogous to the notions of cosimplicial and simplicial objects. Note that the notion of cubical set was introduced in [6].

Definition 1. A cocubical object in a category A is a graded object $K=\{K^i \in obA, i=0,1,\dots\}$ together with operators - morphisms of A $e_i^\varepsilon : K^{n-1} \to K^n$, $\varepsilon = 0,1$; $p_i : K^n \to K^{n-1}$, $i=1,\dots,n$, satisfying the following identities: $e_i^\varepsilon e_j^{\varepsilon_1} = e_{j+1}^{\varepsilon_1} e_i^\varepsilon$ for $i \le j$; $p_j p_i = p_i p_{j+1}$ for $i \le j$ and

$$p_j e_i^\varepsilon = \begin{cases} e_i^\varepsilon p_{j-1} & \text{for } i < j \\ id & \text{for } i = j \\ e_{i-1}^\varepsilon p_j & \text{for } i > j \end{cases}.$$

Example 1. An example of a cocubical topological space is the sequence of cubs $I^\bullet = \{I^0, I^1, I^2, \dots\}$, $I^n = \{(t_1,\dots,t_n) \in R^n, 0 \le t_i \le 1\}$ with operators e_i^ε and p_i which are given by

$$e_i^\varepsilon(t_1,\dots,t_{n-1}) = (t_1,\dots,t_{i-1},\varepsilon,t_i,\dots,t_{n-1}), \quad p_i(t_1,\dots,t_n) = (t_1,\dots,t_{i-1},t_{i+1},\dots,t_n).$$

Example 2. Let V^n be the set of vertexes of the n-cub I^n, i.e. $V^n=\{(\varepsilon_1,\dots,\varepsilon_n), \varepsilon_i=0,1\}$.

Then the operators $e_i^{\varepsilon}: V^{n-1} \to V^n$, $\varepsilon=0,1$; $p_i: V^{n-1} \to V^n$, defined by $e_i^{\varepsilon}(\varepsilon_1,\dots,\varepsilon_{n-1}) = (\varepsilon_1,\dots,\varepsilon_{i-1},\varepsilon,\varepsilon_i,\dots,\varepsilon_{n-1})$, $p_i(\varepsilon_1,\dots,\varepsilon_n) = (\varepsilon_1,\dots,\varepsilon_{i-1},\varepsilon_{i+1},\dots,\varepsilon_n)$ form a cocubical set $V^* = \{V^0, V^1, V^2, \dots;\ e_i^{\varepsilon}, p_i\}$.

Definition 2. A cubical object in a category A is a graded object $Q=\{Q_i \in obA, i=0,1,\dots\}$ together with the face and degeneracy operators - morphisms of A $d_i^{\varepsilon}: Q_n \to Q_{n-1}$, $\varepsilon = 0,1$; $s_i: Q_{n-1} \to Q_n$, $i=1,\dots,n$, satisfying the following identities: $d_j^{\varepsilon} d_i^{\varepsilon_1} = d_i^{\varepsilon_1} d_{j+1}^{\varepsilon}$ for $i \le j$; $s_i s_j = s_{j+1} s_i$ for $i \le j$ and

$$d_i^{\varepsilon} s_j = \begin{cases} s_{j-1} d_i^{\varepsilon} & \text{for } i < j \\ id & \text{for } i = j \\ s_j d_{i-1}^{\varepsilon} & \text{for } i > j \end{cases}.$$

A morphism of (co)cubical objects is a morphism of graded objects $f: Q \to Q'$ which preserves all the structure maps.

An example of cubical set is the singular cubical set of a topological space.

As in the simplicial setting, it is possible to represent cubical and cocubical objects as functors from certain category, which we describe now.

The cocubical set V^* from the example 2 can be extended to a category (which we denote by the same symbol), whose objects are the sets $V^0, V^1, \dots$ and morphisms are maps $V^n \to V^m$ obtained by compositions of operators e_i^{ε}, p_i. As in simplicial setting, any such composition can be represented by canonical form $e_{i_1}^{\varepsilon_1} e_{i_2}^{\varepsilon_2} \dots e_{i_s}^{\varepsilon_s}\ p_{j_1} p_{j_2} \dots p_{j_t}$ with $i_1 > \dots > i_s$, $j_1 < \dots < j_t$. This allows to show that any cocubical object in a category A can be considered as a covariant functor $K: V^* \to A$ and any cubical object Q in the category A can be considered as a contravariant functor $Q: V^* \to A$.

2. Tensor products. Let $\{Q, d_i^{\varepsilon}, s_i\}$ be a cubical set and $\{K, e_i^{\varepsilon}, p_i\}$ be a cocubical set. Let us define their tensor product $Q \otimes K$ as the factor

$$Q \otimes K = \bigcup_n Q_n \times K^n / \sim, \tag{1}$$

where $\sim$ is the equivalence spaned by identifications

$$(d_i^{\varepsilon} q_{n+1}, k^n) = (q_{n+1}, e_i^{\varepsilon} k^n), \quad (s_i q_n, k^{n+1}) = (q_n, p_i k^{n+1}).$$

In fact $Q \otimes K$ is the tensor product of functors $Q: V^* \to$Sets and $K: V^{*op} \to$Sets.

This tensor product is defined also for cubical and cocubical objects $Q: V^* \to A$ and $K: V^{*op} \to B$ whenever all the operations in (1) are given: there is given a paring $A \times B \in C$, there exists $\bigcup_n$ and factorization by $\sim$ in C. In this case $Q \otimes K$ will be an object of the category C. Analogously is defined the tensor product of a simplicial and cosimplicial objects.

Here are some tensor products which arise bellow.

1. If Q is a cubical set and K is a cocubical set, then $Q \otimes K$ is a set.

2. If Q is a cubical set and K is a cocubical topological space, then $Q \otimes K$ is a topological space. Particularly the realization of a cubical set Q is the tensor product $|Q| = Q \otimes I^*$ where I^* is the cocubical topological space from the example 1.

3. If Q is a cubical set and K is a cocubical simplicial set, then $Q \otimes K$ is a simplicial set.

4. If X is a simplicial set and Y is a cosimplicial topological space, then $Q \otimes K$ is a topological space. Particularly the realization of a simplicial set X is the tensor product

$X = X\otimes\Delta^{\bullet}$, where $\Delta^{\bullet} = \{\Delta^0, \Delta^1, \Delta^2, ...; \delta_i, \sigma_i\}$ is the cosimplicial topological space of standard simplexes.

5. If X is a simplicial cocubical set and Y is a cosimplicial topological space, then $X\otimes Y$ is a cocubical topological space.

3. **Simplicial cutting of a cub.** The set $V^n=\{(\varepsilon_1,...,\varepsilon_n),\ \varepsilon_i=0,1\}$ of vertexes of the n-cub I^n is (partially) ordered (see e.g. [2]) as follows: $(\varepsilon_1,..., \varepsilon_n) \leq (\varepsilon'_1,..., \varepsilon'_n)$ if $\varepsilon_i \leq \varepsilon'_i$ for all $1\leq i\leq n$. So to the ordered set V^n of vertexes of I^n we can assignee (by standard manner) a simplicial set $\widetilde{V}_{\bullet}^n = \{\widetilde{V}_0^n, \widetilde{V}_1^n, ...; \partial_i, s_i\}$.

Proposition 1. *The realization* $|\widetilde{V}_{\bullet}^n|$ *is homeomorphic to* I^n.

Proof. By definition $|\widetilde{V}_{\bullet}^n| = \widetilde{V}_{\bullet}^n \otimes \Delta^{\bullet}$ is the factor of $\cup_p \widetilde{V}_p^n \times \Delta^p$ by the equivalence spanned by the identifications $(\partial_i \widetilde{v}_p, t)=(\widetilde{v}_p, \delta_i t)$, $(s_i \widetilde{v}_p, t)=(\widetilde{v}_p, \sigma_i t)$, here Δ^p is the standard p-simplex, $t=(t_0,\ldots,t_p)\in\Delta^p$ and $\widetilde{v}_p=(v_0,\ldots,v_p)\in\widetilde{V}_p^n$. We define a continuous map $\varphi: \cup_p \widetilde{V}_p^n \times\Delta_p \to I^n$ by $\varphi(\widetilde{v}_p, t)=\widetilde{v}_p(t)=\Sigma_i t_i\cdot v_i$. Direct checking shows that φ fits with the above mentioned equivalence and it defines a correct homeomorphism $\Phi^n: |\widetilde{V}_{\bullet}^n| \to I^n$.

Remark. In fact $\widetilde{V}_{\bullet}^n = \widetilde{V}_{\bullet}^1 \times\ldots\times \widetilde{V}_{\bullet}^1$ and $|\widetilde{V}_{\bullet}^1| \approx I^1$. Thus $|\widetilde{V}_{\bullet}^n| = |\widetilde{V}_{\bullet}^1| \times\ldots\times |\widetilde{V}_{\bullet}^1| \approx I^n$.

Note that the maps $e_i^{\varepsilon}: V^{n-1}\to V^n$ and $p_i: V^n \to V^{n-1}$, forming the cocubical set $\{V^n, e_i^{\varepsilon}, p_i\}$ are monotonic. Thus they define simplicial maps $\widetilde{e}_i^{\varepsilon}: \widetilde{V}_{\bullet}^{n-1} \to \widetilde{V}_{\bullet}^n$, $\widetilde{p}_i: \widetilde{V}_{\bullet}^n \to \widetilde{V}_{\bullet}^{n-1}$. The obtained object $\widetilde{V}_{\bullet}^{\bullet} = \{\widetilde{V}_{\bullet}^0, \widetilde{V}_{\bullet}^1, ..., \widetilde{V}_{\bullet}^n, ...\}$ actually is a *cocubical simplicial set* with operators $\widetilde{e}_i^{\varepsilon}$, $\widetilde{p}_i$ from one hand side and a *simplicial cocubical set* with operators ∂_i, s_i from another.

The homeomorphisms $\{\Phi^n\}$ are compatible with operators $\widetilde{e}_i^{\varepsilon}$ and $\widetilde{p}_i$, so $\{\Phi^n : |\widetilde{V}_{\bullet}^n| \to I^n, n=0,1,2,...\}$ is an isomorphism. Thus we obtain the

Proposition 2. *The sequence of homeomorphisms* $\{\Phi^n : |\widetilde{V}_{\bullet}^n| \to I^n, n=0,1,...\}$ *is an isomorphism of cocubical topological spaces.*

4. **Simplicial cutting of a cubical set.** Let $(Q, d_i^{\varepsilon}, s_i)$ be a cubical set. Here we are going to assignee to Q a simplicial set SQ obtained by "cutting of cubs into simplexes".

Let us define $SQ=Q\otimes\widetilde{V}_{\bullet}^{\bullet}$, here Q is a cubical set and $\widetilde{V}_{\bullet}^{\bullet}$ is regarded as a cocubical simplicial set, so the tensor product is a simplicial set. Actually, by definition of the tensor product, $(SQ)_p$ is the factor of $\cup_{n=0}^{\infty} Q_n \times \widetilde{V}_p^n$ by the equivalence relation, spanned by identifications

$$(d_i^{\varepsilon} q_{n+1}, \widetilde{v}_p^n) = (q_{n+1}, e_i^{\varepsilon} \widetilde{v}_p^n), \quad (s_i q_n, \widetilde{v}_p^{n+1}) = (q_n, p_i \widetilde{v}_p^{n+1}).$$

The face and degeneracy operators of SQ are given by

$$\partial_i[(q_n, \widetilde{v}_p^n)] = [(q_n, \partial_i \widetilde{v}_p^n)], \quad s_i[(q_n, \widetilde{v}_p^n)] = [(q_n, s_i \widetilde{v}_p^n)].$$

This assignment is functorial: any morphism of cubical sets $f: Q\to Q'$ determines a map $Sf: SQ\to SQ'$ given by $Sf([(q_n, \widetilde{v}_p^n)]) = [(fq_n, \widetilde{v}_p^n)]$.

Proposition 3. *The realizations of the simplicial set* SQ *and the cubical set* Q *are homeomorphic.*

Proof. It follows from this string of equalities

$$|SQ|=(SQ)\otimes\Delta^{\bullet}=(Q\otimes\tilde{V}_{\bullet}^{\bullet})\otimes\Delta^{\bullet}=Q\otimes(\tilde{V}_{\bullet}^{\bullet}\otimes\Delta^{\bullet})=Q\otimes|\tilde{V}_{\bullet}^{\bullet}|\approx Q\otimes I^{\bullet}=|Q|,$$

which needs some comments. The realization of SQ by definition is the tensor product $(SQ)\otimes\Delta^{\bullet}$. By definition of SQ this is the tensor product $(Q\otimes\tilde{V}_{\bullet}^{\bullet})\otimes\Delta^{\bullet}$ where: Q is a cubical set, $\tilde{V}_{\bullet}^{\bullet}$ is regarded as a cocubical simplicial set and $\Delta^{\bullet}$ is a cosimplicial topological space. This tensor product (a topological space) is homemorphic to $Q\otimes(\tilde{V}_{\bullet}^{\bullet}\otimes\Delta^{\bullet})$ where now $\tilde{V}_{\bullet}^{\bullet}$ is regarded as a simplicial cocubical set. The tensor product $\tilde{V}_{\bullet}^{\bullet}\otimes\Delta^{\bullet}$ is the realization $|\tilde{V}_{\bullet}^{\bullet}|$ which, by the proposition 2, is isomorphic to $I^{\bullet}$. But $Q\otimes I^{\bullet}$ is the realization $|Q|$.

5. Chain complexes. The chain complex $(C_*(Q),d)$ of a cubical set Q is defined as $C_n(Q)=\mathrm{span}\{q_n\in Q_n\}$ and $d(q_n)=\Sigma_i(-1)^i(d_i^0-d_i^1)(q_n)$. The normalized chain complex $(C_{\bullet}^N(Q)/D_*(Q))$ of Q is defined as the quotient $C_*(Q)/D_*(Q)$, where $D_*(Q)$ is the subcomplex of $(C_*(Q),d)$ generated by degenerate elements of Q.

Note also that both $C_*(Q)$ and $C_{\bullet}^N(Q)$ are DG-coalgebras with respect to the comultiplication $\nabla:C_*(Q)\to C_*(Q)\otimes C_*(Q)$ defined in [7]: $\nabla q_n=\Sigma_I d_I^0 q_n\otimes d_{\bar{I}}^1 q_n$ where $I=(i_1,\dots,i_k)\subset(1,2,\dots,n)$, $I\cup\bar{I}=(1,\dots,n)$ and $d_I=d_{i_1}\cdot d_{i_2}\cdot\dots\cdot d_{i_k}$.

Let $f:C_*(Q)\to C_*(SQ)$ be a homomorphism given by

$$f(q_n)=\sum_{\tilde{v}_n^n\in\tilde{V}_n^n}(-1)^{r(\tilde{v}_n^n)}[(q_n,\tilde{v}_n^n)],$$

where $(-1)^{r(\tilde{v}_n^n)}$ is defined as follows. To each $\tilde{v}_n^n=(v_0,\dots,v_n)$ corresponds a permutation $(i_1,\dots,i_n)$ where i_1 is the place of first occurrence of 1 in the vector v_1, i_2 is the place of occurrence of the next 1 in v_2 etc. Then $(-1)^{r(\tilde{v}_n^n)}$ is the sign of this permutation. This map induces $f^N:C_{\bullet}^N(Q)\to C_{\bullet}^N(SQ)$.

The straightforward verification shows that $\nabla f^N=(f^N\otimes f^N)\nabla$ and $\partial f^N=f^N d$ that is f^N is a map of DG-coalgebras.

Let $Q\otimes Q'=\{(Q\otimes Q')_n=\bigcup_{i=0,\dots,n}Q_i\otimes Q'_{n-i}\}$ be the tensor product of two cubical sets Q and Q' (see [6]). A *monoidal cubical set* (see [3]) is defined as a cubical set Q with cubical map $\mu:Q\otimes Q\to Q$ which is associative and has the unit $e\in Q_0$. Clearly, for a monoidal cubical set its chain complex $(C_*^N(Q),d)$ is a DG-Hopf algebra. Similarly, for a simplicial monoid S its chain complex $(C_*^N(S),)$ is a DG-Hopf algebra.

Let now Q be a monoidal cubical set. Then we can define a *simplicial monoid* structure on SQ by setting $[(q_m,\tilde{v}_p^m)]\circ[(q_n,\tilde{v}_p^n)]=[(q_m\circ q_n,\alpha(\tilde{v}_p^m,\tilde{v}_p^n))]$, where the map $\alpha:\tilde{V}_p^m\times\tilde{V}_p^n\to\tilde{V}_p^{m+n}$ is induced by the clear map $V^m\times V^n\to V^{m+n}$ which sends $((\varepsilon_1,\dots,\varepsilon_m),(\varepsilon'_1,\dots,\varepsilon'_n))$ to $(\varepsilon_1,\dots,\varepsilon_m,\varepsilon'_1,\dots,\varepsilon'_n)$. Let the map

$$\beta:\tilde{V}_m^m\times\tilde{V}_n^n\times\{I=(i_1<\dots<i_n)\subset(1,\dots,m+n)\}\to\tilde{V}_{m+n}^{m+n}$$

be given by $\beta=\alpha(s_I\times s_{\bar{I}})$ where $I\cup\bar{I}=(1,\dots,m+n)$ and $s_I=s_{i_n}\cdot\dots\cdot s_{i_1}$. It is possible to check that β is a *bijection*. Using this fact one can easily check that

$f^{\#}: C_{\bullet}^{N}(Q) \to C_{\bullet}^{N}(SQ)$ is multiplicative as well. Thus we obtain the following

Proposition 4. *If Q is a monoidal cubical set then SQ is a simplicial monoid and $f^{\#}: C_{\bullet}^{N}(Q) \to C_{\bullet}^{N}(SQ)$ is a natural map of DG-Hopf algebras.*

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მათემატიკა

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კუბური სიმრავლეების სიმპლიციური დაჭრა

რეზიუმე. სტანდარტული კუბის სიმპლექსებად კანონიკური დაჭრის გამოყენებით ყოველ Q კუბურ სიმრავლეს ფუნქტორიალურად ეთანადება გარკვეული SQ სიმპლიციური სიმრავლე ისე, რომ Q და SQ სიმრავლეების გეომეტრიული რეალიზაციები ჰომეომორფული სივრცეებია. აგებულია ნორმალიზებული ჯაჭვური კომპლექსების ბუნებრივი ჯაჭვური ასახვა $C_*(Q)\to C_*(SQ)$, რომელიც არის DG-კოალგებრების და, როცა Q მონოიდური კუბური სიმრავლეა, აგრეთვე DG-ჰოფის ალგებრების ასახვა.