On Dynkin Gradings in Simple Lie Algebras

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To Tony Joseph, on his 75th birthday

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1 Introduction

In this paper we study Dynkin gradings on simple Lie algebras arising from nilpotent elements. Specifically, we investigate Abelian subalgebras which are degree 1 homogeneous with respect to these gradings.

The study of gradings associated to nilpotent elements of simple Lie algebras is important since the finite and affine classical and quantum W-algebras are defined using these gradings. In order to study integrable systems associated to these W-algebras, it is useful to have their free field realizations. One of the ways to construct them is to use the generalized Miura map [2, 4]. This construction can be further improved by choosing an Abelian subalgebra in the term $\mathfrak{g}_1$ of the grading. That is why the description of such subalgebras, especially those having the maximal possible dimension $\frac{1}{2}\dim\mathfrak{g}_1$, is important.

We show that for each odd nilpotent orbit there always exists a canonically associated "strictly odd" nilpotent orbit, which allows us to reduce our investigations to the latter case. (Strictly odd means that all Dynkin labels are either 0 or 1.) The rest of the paper is devoted to the investigation of maximal Abelian subalgebras in $\mathfrak{g}_1$ for strictly odd nilpotent orbits in simple Lie algebras. For algebras of exceptional type we provide tables with largest possible dimensions of such subalgebras in each

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case. For algebras of classical type, we find expressions for all possible maximal dimensions of Abelian subalgebras in $\mathfrak{g}_1$, and, based on that, characterize those nilpotent orbits for which there exists such subalgebra of half the dimension of $\mathfrak{g}_1$.

2 Recollections

Let us recall the nomenclature for nilpotent elements in a semisimple Lie algebra $\mathfrak{g}$.

Given a nilpotent element e, one chooses an $\mathfrak{sl}_2$-triple (e, h, f) for it, that is, another nilpotent element f such that $[e, f] = h$ is semisimple and the identities $[h, e] = 2e$, $[h, f] = -2f$ hold (Jacobson-Morozov theorem; see, e.g., [1]). The Dynkin grading is the eigenspace decomposition for $\operatorname{ad} h$:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j.$$

Thus, a choice of a nilpotent element e defines a combinatorial object which uniquely describes the orbit of e. It is the *weighted Dynkin diagram* corresponding to e, which is the Dynkin diagram of $\mathfrak{g}$ with numbers assigned to each node. These numbers are the degrees $\alpha_i(h)$ of simple root vectors e_i with respect to the choice of a Cartan and a Borel subalgebra in such a way that h (resp. e) becomes an element of the corresponding Cartan (resp. Borel) subalgebra. The weighted Dynkin diagrams satisfy certain restrictions—for example, the weights can only be equal to 0, 1 or 2; moreover, if $\mathfrak{g}$ is simple of type A, then the weights are symmetric with respect to the center of the diagram, while for types B, C, or D there is no weight 1 occurring to the left of 2.

A nilpotent element is called *even* if there are no 1's in its weighted Dynkin diagram, *odd* if it is not even, and *strictly odd* if there are no 2's.

It is clear that for even nilpotent elements the question about Abelian subspaces in $\mathfrak{g}_1$ is trivial since $\mathfrak{g}_1$ is zero.

We will also need the following fact from [3]:

Proposition 2.1 *The degree* 1 *part* $\mathfrak{g}_1$ *of* $\mathfrak{g}$ *with respect to the grading induced by a nilpotent element* $e \in \mathfrak{g}$ *is generated as a* $\mathfrak{g}_0$*-module by those simple root vectors of* $\mathfrak{g}$ *which have weight* 1 *in the weighted Dynkin diagram corresponding to e.* □

If $\mathfrak{g}$ is a simple Lie algebra of classical type, one can assign to e another combinatorial object—a partition $\lambda_n \geqslant \lambda_{n-1} \geqslant \dots$ which records dimensions of irreducible representations of $\mathfrak{sl}_2$ into which the standard representation of $\mathfrak{g}$ decomposes as a module over its subalgebra (e, h, f). Alternatively, the partition consists of sizes of Jordan blocks in the Jordan decomposition of e as an operator acting on the standard representation of $\mathfrak{g}$. The partitions are restricted in a certain way, according to the type of $\mathfrak{g}$. For type A one may have arbitrary partitions. For types B and D, all even parts must have even multiplicity, while for type C all odd parts must have even multiplicity. These conditions are sufficient as well as necessary, that is, any partition satisfying these conditions corresponds to a nilpotent orbit in a simple Lie algebra of the respective classical type.

Let us recall how one calculates the weighted Dynkin diagram of a nilpotent element defined by a partition $\lambda = (\lambda_n \geqslant \lambda_{n-1} \geqslant \ldots)$ (cf. [6]).

Each element λ_k of the partition λ represents a copy of the λ_k-dimensional irreducible representation of $\mathfrak{sl}_2$, with eigenvalues of h equal to

$$1-\lambda_k, 3-\lambda_k, \ldots, \lambda_k-3, \lambda_k-1.$$

To obtain the weighted Dynkin diagram one collects those eigenvalues for each λ_k , arranges them in decreasing order, and takes consecutive differences.

For example, take the partition 8, 6, 3, 3, 2, 1, 1. This gives the following eigenvalues of h:

$$\begin{array}{ccccccccccc}
-7 & -5 & -3 & & -1 & & 1 & & 3 & 5 & 7 \\
 & -5 & -3 & & -1 & & 1 & & 3 & 5 & \\
 & & & -2 & & 0 & & 2 & & & \\
 & & & -2 & & 0 & & 2 & & & \\
 & & & & -1 & & 1 & & & & \\
 & & & & & 0 & & & & & \\
 & & & & & 0 & & & & &
\end{array}$$

Arranging all numbers from this table in the decreasing order gives

$$7\ 5\ 5\ 3\ 3\ 2\ 2\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ -1\ -1\ -1\ -2\ -2\ -3\ -3\ -5\ -5\ -7.$$

Taking the consecutive differences then gives

$$2\ 0\ 2\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 2\ 0\ 2$$

which is already the weighted Dynkin diagram of the nilpotent in case of type A.

For the types B, C, D one has to leave only the left half of the obtained sequence (which obviously is centrally symmetric); more precisely, for an algebra of rank r, the first $r-1$ nodes of the weighted Dynkin diagram are as stated, while the rightmost node is defined in a specific way, depending on the type. We skip this part, as it will not play any rôle for us; details can be found in, e.g., [1, Section 5.3].

For example, the same partition 8, 6, 3, 3, 2, 1, 1 also encodes a nilpotent orbit in a simple Lie algebra of type C, since all of its odd parts come with even multiplicities. Then, the weighted Dynkin diagram of this nilpotent is

$$2\ 0\ 2\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0.$$

It is easy to see from the above procedure that the resulting weighted Dynkin diagram begins with certain sequence of 0's and 2's; if the largest part of the partition is λ_n with multiplicity m_n, and the parts of the same parity following it are λ_{n-1}

with multiplicity m_{n-1}, λ_{n-2} with multiplicity m_{n-2}, $\dots$, λ_{n-k+1} with multiplicity m_{n-k+1}, while the next part λ_{n-k} has the opposite parity, then the first 1 appears at the $(km_n + (k-1)m_{n-1} + \ldots + 2m_{n-k+2} + m_{n-k+1})$-st place. For the type A the picture is symmetric, so one has the weights 2 and 0 at both ends of the diagram, and the weights 1 and 0 in the middle, while for the types B, C, or D the sequence of weights starts with 0 and 2 followed by a sequence of weights 0 and 1, without any further 2's.

According to the above procedure for assigning to a partition a weighted Dynkin diagram, it is easy to see the following

Proposition 2.2 *A nilpotent element in a simple Lie algebra of classical type is even iff all the parts of the corresponding partition are of the same parity; it is odd iff there are some parts with different parities, and strictly odd iff the largest part and the next largest part differ by* 1. □

3 Important Reduction

Let V and U be finite-dimensional modules over a reductive Lie algebra $\mathfrak{g}$ and let $V \otimes V \to U$ be a $\mathfrak{g}$-module homomorphism. We see this homomorphism as a $\mathfrak{g}$-equivariant algebra structure on V with values in U.

Proposition 3.1 *Suppose that there exists an Abelian subalgebra of dimension d of the algebra V. Then there exists an Abelian subalgebra of the algebra V of dimension d, spanned by weight vectors of V.*

Proof (Proposed by the Referee) It follows from Borel's fixed point theorem. Indeed, the Cartan subgroup acts on the complete variety of d-dimensional Abelian subalgebras of V, hence has a fixed point. □

Using this, in what follows we will assume throughout that for a simple Lie algebra of classical type we are given a basis in the standard representation consisting of weight vectors corresponding to the weights $\pm\varepsilon_i$, $i = 1, \dots, n$ and moreover, for the type B, to the zero weight. In the adjoint representation, accordingly, we will have a basis corresponding to $\pm\varepsilon_i \pm \varepsilon_j$, $i \neq j$ (accounting for tensor products of basis vectors of the standard representation corresponding to $\pm\varepsilon_i$ and to $\pm\varepsilon_j$) and moreover, for the type B only, those corresponding to $\pm\varepsilon_i$ (accounting for tensor product of a basis vector corresponding to $\pm\varepsilon_i$ and that corresponding to the zero weight) and, for C only, corresponding to $\pm 2\varepsilon_i$ (accounting for the tensor product of a basis vector of the standard representation corresponding to $\pm\varepsilon_i$ with itself), $i = 1, \dots, n$.

Proposition 3.2 *For any weighted Dynkin diagram corresponding to a nilpotent element e in a simple Lie algebra $\mathfrak{g}$, consider a subdiagram obtained as a result of erasing all nodes with weight* 2. *Consider the resulting subdiagram together with the remaining weights. Then all connected components of this subdiagram, except* *possibly one of them, have all weights equal to zero. Moreover, this one component (if it exists) is a weighted Dynkin diagram of some strictly odd nilpotent orbit in the diagram subalgebra* $\tilde{\mathfrak{g}} \subseteq \mathfrak{g}$ *of the type determined by the shape of the component.*

Proof For algebras of classical type, this is proved in Lemma 4.6 below. For an algebra of type G_2 this is clear as all nilpotent elements in it are either even or strictly odd. As for the exceptional Lie algebras of types E or F, the assertion can be seen to be true directly from looking at the Tables F4o, E6o, E7o, E8o given in the last section. □

Corollary 3.3 *For any odd nilpotent element* e *in a simple Lie algebra* $\mathfrak{g}$ *there exists a simple diagram subalgebra* $\tilde{\mathfrak{g}} \subseteq \mathfrak{g}$ *and a strictly odd nilpotent element* $\tilde{e} \in \tilde{\mathfrak{g}}$ *such that*

$$\mathfrak{g}_1(e) = \tilde{\mathfrak{g}}_1(\tilde{e}),$$

i.e., the degree 1 homogeneous parts for the grading on $\mathfrak{g}$ *induced by* e *and for the grading on* $\tilde{\mathfrak{g}}$ *induced by* $\tilde{e}$ *coincide. In particular, these degree 1 homogeneous parts have the same Abelian subspaces.*

Proof Let $\tilde{\mathfrak{g}}$ be the subalgebra corresponding to the connected component of the weighted Dynkin diagram of e as described in Proposition 3.2 above. Moreover, let $\tilde{e}$ be a representative of the orbit corresponding to the weights on this connected component—it exists by Proposition 3.2.

By construction, this subalgebra contains all simple root vectors of degree 1, and, moreover, they will be precisely the root vectors of those simple roots of $\tilde{\mathfrak{g}}$ which contribute to the degree 1 part for the grading induced by $\tilde{e}$. From Proposition 2.1 we know that $\mathfrak{g}_1(e)$ is the $\mathfrak{g}_0(e)$-module generated by these root vectors, while $\tilde{\mathfrak{g}}_1(\tilde{e})$ is the $\tilde{\mathfrak{g}}_0(\tilde{e})$-module generated by them.

Now observe that the only removed nodes which connect with an edge to some node in the remaining connected component have weight 2, so that all simple root vectors corresponding to removed nodes with weight 0 commute with every simple root vector in this component.

It follows that the $\mathfrak{g}_0(e)$-module generated by the root vectors corresponding to weight 1 nodes is no larger than the $\tilde{\mathfrak{g}}_0(\tilde{e})$-module generated by them, i. e. $\mathfrak{g}_1(e)$ coincides with $\tilde{\mathfrak{g}}_1(\tilde{e})$. □

Definition 3.4 For the orbit of an odd nilpotent element in a simple Lie algebra $\mathfrak{g}$, call its *strictly odd reduction* the nilpotent orbit in the simple Lie algebra $\tilde{\mathfrak{g}}$ obtained as in Corollary 3.3.

Given a nilpotent element $e \in \mathfrak{g}$ as in Proposition 3.2, one can explicitly construct a nilpotent element $\tilde{e} \in \tilde{\mathfrak{g}}$ from the orbit corresponding to its strictly odd reduction in the sense of Definition 3.4 as follows. The nilpotent element e clearly lies in the degree 2 subspace $\mathfrak{g}_2$ for the corresponding grading. This subspace is a $\mathfrak{g}_0$-module and it decomposes canonically into the direct sum of its submodule $[\mathfrak{g}_1, \mathfrak{g}_1]$ and the submodule $\mathfrak{g}_2(2)$ generated by the root vectors of $\mathfrak{g}$ corresponding to the simple roots of weight 2.

Proposition 3.5 *Given a nilpotent element e, represent it (in a unique way) as a sum* e_1+e_2 *with* $e_1 \in [\mathfrak{g}_1, \mathfrak{g}_1]$ *and* $e_2 \in \mathfrak{g}_2(2)$*. Then the weighted Dynkin diagram of* e_1 *in the subalgebra corresponding to the subdiagram described in Proposition 3.2 is given by the weights on that subdiagram.*

Proof We have a reductive group G_0 corresponding to $\mathfrak{g}_0$ acting on $\mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_1] + \mathfrak{g}_2(2)$, with the element $e = e_1 + e_2$ having an open orbit in $\mathfrak{g}_2$. This means that $[\mathfrak{g}_0, e_1 + e_2] = \mathfrak{g}_2$. But this implies that $[\mathfrak{g}_0, e_1] = [\mathfrak{g}_1, \mathfrak{g}_1]$ (and similarly for e_2). Hence $G_0 e_1$ is an open orbit in $[\mathfrak{g}_1, \mathfrak{g}_1]$.

Let us consider an intermediate subalgebra $\tilde{\mathfrak{g}} \subseteq \mathfrak{g}' \subseteq \mathfrak{g}$ corresponding to the diagram, obtained by erasing the nodes with weight 2, but leaving all other nodes together with their weights intact (this diagram can be disconnected). Proposition 3.2 easily implies that $\mathfrak{g}'$ is a direct sum of $\tilde{\mathfrak{g}}$ and of some simple algebras of type A. Hence e_1, viewed as an element of this direct sum, obviously has zero summands in all these components of type A.

On the other hand, Proposition 3.2 implies that there exists a (strictly odd) nilpotent element $\tilde{e}$ in $[\mathfrak{g}_1, \mathfrak{g}_1]$, which has the needed Dynkin diagram. Then, similarly to e_1, the element $\tilde{e}$ can also be seen as a nilpotent element in $\mathfrak{g}'$, having zero components in all remaining type A components of $\mathfrak{g}'$. It is then clear that this nilpotent element will have the weighted Dynkin diagram obtained as in Proposition 3.2. Moreover, it will have an open G_0-orbit in $[\mathfrak{g}_1, \mathfrak{g}_1]$, hence it coincides with the G_0-orbit of e_1, so $\tilde{e}$ and e_1 have the same weighted Dynkin diagram when viewed as nilpotent elements in $\mathfrak{g}'$. This implies that these elements have the same weighted Dynkin diagram with respect to $\tilde{\mathfrak{g}}$, since the latter is obtained just by throwing out type A components with zero weights only. □

Remark 3.6 It would be convenient to supplement Corollary 3.3 with an explicit construction, assigning to an $\mathfrak{sl}_2$-triple (e, f, h) corresponding to a given nilpotent orbit in $\mathfrak{g}$, an $\mathfrak{sl}_2$-triple $(\tilde{e}, \tilde{f}, \tilde{h})$ for its strictly odd reduction as in Definition 3.4. Since $\tilde{\mathfrak{g}}$ comes with a grading (determined by the weights on the corresponding subdiagram), the semisimple element $\tilde{h}$ of $\tilde{\mathfrak{g}}$ is determined by this grading, while $\tilde{f}$, which we know to exist by Corollary 3.3, is uniquely determined by $\tilde{e}$ and $\tilde{h}$. Thus having an explicit construction of $\tilde{f}$ would provide an alternative proof of Corollary 3.3 that would not require case-by-case analysis of the exceptional types. One possibility that comes to mind is to produce $\tilde{f}$ from f in the same way as we produced $\tilde{e}$ from e in Proposition 3.5—that is, take $\tilde{f} = f_1$ where $f = f_1 + f_2$ is the unique decomposition of $f \in \mathfrak{g}_{-2}$ into a sum of $f_1 \in [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$ and $f_2 \in \mathfrak{g}_{-2}(2)$, the latter being the $\mathfrak{g}_0$-submodule of $\mathfrak{g}_{-2}$ generated by the root vectors corresponding to negatives of the simple roots with weights 2 on the initial weighted Dynkin diagram. However, as the following example shows, this does not give the correct value of $\tilde{f}$ in general.

Example 3.7 For $\mathfrak{g}$ of type D_6, consider the nilpotent orbit corresponding to the weighted Dynkin diagram $2010{}^{1}_{1}$ (and to the partition 5, 3, 2, 2). The following sum of positive root vectors

$$e := e_{1100^0_0} + e_{0111^1_0} + e_{0011^1_0} + e_{0011^0_1} + e_{0001^1_1}$$

where the subscripts denote the linear combinations of simple roots that give the corresponding positive roots, yields a representative of this orbit. The corresponding f in the $\mathfrak{sl}_2$-triple for e is the following combination of negative root vectors:

$$f := 2f_{1000^0_0} + 4f_{1100^0_0} + 2f_{0111^1_0} - 2f_{0111^0_1} + 2f_{0011^1_0} + 4f_{0011^0_1} + f_{0001^1_1},$$

where the subscripts are linear combinations of negative simple roots. Thus $h = [e, f]$ determines the grading corresponding to the above weighted Dynkin diagram. It is straightforward to check that in the degree 2 subspace $\mathfrak{g}_2$, root vectors corresponding to the combinations 1000^0_0 and 1100^0_0 of simple roots span the $\mathfrak{g}_0$-submodule $\mathfrak{g}_2(2) \subseteq \mathfrak{g}_2$ generated by the root vector of 1000^0_0, i. e. of the simple root with weight 2, while the remaining positive root vectors from $\mathfrak{g}_2$ lie in $[\mathfrak{g}_1, \mathfrak{g}_1]$. Thus, according to Proposition 3.5, a strictly odd nilpotent element $\tilde{e} = e_1$ in the diagram subalgebra $\tilde{\mathfrak{g}}$ of type D_5 corresponding to the subdiagram obtained by omitting the node with weight 2 is obtained by omitting in the sum for e the leftmost summand (the one that lies in $\mathfrak{g}_2(2)$). Thus,

$$\tilde{e} = e_{0111^1_0} + e_{0011^1_0} + e_{0011^0_1} + e_{0001^1_1}.$$

Now, if we try to choose for the companion of $\tilde{e}$ in the $\mathfrak{sl}_2$-triple the element f_1 obtained in the same way from f, i. e. by omitting in the sum for f the summands that lie in $\mathfrak{g}_{-2}(2)$, we obtain

$$f_1 = 2f_{0111^1_0} - 2f_{0111^0_1} + 2f_{0011^1_0} + 4f_{0011^0_1} + f_{0001^1_1}.$$

However, it turns out that $[e_1, f_1]$ is not the semisimple element determining the grading of $\tilde{\mathfrak{g}}$. As a matter of fact, this element is not semisimple, rather it has form

$$[e_1, f_1] = h' - e_{0100^0_0}$$

with h' in the Cartan subalgebra of $\tilde{\mathfrak{g}}$. A correct $\tilde{f}$ (the one with $[\tilde{e}, \tilde{f}] = \tilde{h}$ an element in the Cartan subalgebra of $\tilde{\mathfrak{g}}$ which gives the correct grading of $\tilde{\mathfrak{g}}$) is

$$\tilde{f} = 2f_{0111^1_0} - 2f_{0111^0_1} + 2f_{0011^0_1} + f_{0001^1_1}$$

and it is thus not obtained from f by projecting it to $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$ or in any other obvious way.

Let us add that there are also many examples (even for the algebras of type A) when the bracket of the projections $[e_1, f_1]$ of e and f is semisimple but does not induce the required grading on $\tilde{\mathfrak{g}}$.

4 Maximizing Abelian Subspaces

We are interested in Abelian subspaces of $\mathfrak{g}_1$. First of all, one has the following well-known fact.

Proposition 4.1 *Dimension of* $\mathfrak{g}_1$ *is even, and the largest possible dimension of an Abelian subspace in* $\mathfrak{g}_1$ *is at most* $\frac{1}{2}\dim\mathfrak{g}_1$.

Proof Let e be an element of the orbit, and choose an $\mathfrak{sl}_2$-triple (e, h, f) with $e \in \mathfrak{g}_2$, and h inducing the grading. Then one may define a bilinear form on $\mathfrak{g}_1$ via

$$(x, y)_f := \langle f, [x, y]\rangle,$$

where $\langle -, -\rangle$ is the Killing form. It is well known that the skew-symmetric form $(-, -)_f$ is nondegenerate (since $\operatorname{ad} f : \mathfrak{g}_1 \to \mathfrak{g}_{-1}$ is an isomorphism), so that dimension of $\mathfrak{g}_1$ is indeed even. Moreover any commuting elements of $\mathfrak{g}_1$ are orthogonal with respect to this form. Since such a form does not possess isotropic subspaces of more than half dimension of the space, we obtain that there are no Abelian subspaces of more than half dimension of $\mathfrak{g}_1$. □

Remark 4.2 It is known, more generally, that any homogeneous part $\mathfrak{g}_{2i-1}$ of odd degree possesses a nondegenerate skew-symmetric form—see [5, Proposition 1.2]. Thus, each $\dim\mathfrak{g}_{2i-1}$ is even, too.

We now consider the Abelian subalgebras in $\mathfrak{g}_1$, separately for the simple algebras of classical types (right now) and for the algebras of exceptional types (in Sect. 5).

We now consider the simple algebras of classical types. For the type A, it has been proved in [7] that a half-dimensional Abelian subspace in $\mathfrak{g}_1$ exists for any nilpotent orbit.

The central result of this section is the following characterization, in terms of the associated partitions, of those strictly odd nilpotent orbits in types B, C or D admitting an Abelian subspace of half the dimension in $\mathfrak{g}_1$. We will then deduce the general (not necessarily strictly odd) case, using strictly odd reduction as in Definition 3.4.

Theorem 4.3 *Given a strictly odd nilpotent element in a simple Lie algebra* $\mathfrak{g}$ *of type* B, C, *or* D, *there is an Abelian subspace of half dimension in* $\mathfrak{g}_1$ *if and only if the partition corresponding to the nilpotent element satisfies one of the following conditions:*

- *the largest part* μ *of the partition is even and there are no other even parts; moreover if* $\mathfrak{g}$ *is of type* B *then* μ *has multiplicity* 2.
- *the largest part* μ *of the partition is odd, and either there are no other odd parts, or* $\mathfrak{g}$ *is not of type* C, *and the only other parts are* $\mu - 1$ *with multiplicity* 2 *and* 1 *(with any multiplicity).*

In other words, Abelian subspaces of half dimension in $\mathfrak{g}_1$ *occur precisely for those strictly odd nilpotent elements corresponding to the partitions of the following kind:*

$$\begin{array}{rllll}
\textbf{type C}: & \left[1^{2\nu_1}3^{2\nu_3}\cdots(2k-1)^{2\nu_{2k-1}}(2k)^{\nu}\right] & (\nu_{2k-1}\nu\neq 0), & \left[2^{\nu_2}4^{\nu_4}\cdots(2k)^{\nu_{2k}}(2k+1)^{2\nu}\right] & (\nu_{2k}\nu\neq 0);\\
\textbf{type B or D}: & \left[2^{2\nu_2}4^{2\nu_4}\cdots(2k)^{2\nu_{2k}}(2k+1)^{\nu}\right] & (\nu_{2k}\nu\neq 0), & \left[1^{\nu_1}(2k)^2(2k+1)^{\nu}\right] & (\nu_{2k}\nu\neq 0);\\
\textbf{type B}: & \left[1^{\nu_1}3^{\nu_3}\cdots(2k-1)^{\nu_{2k-1}}(2k)^{2}\right] & (\nu_{2k-1}\neq 0), & & \\
\textbf{type D}: & \left[1^{\nu_1}3^{\nu_3}\cdots(2k-1)^{\nu_{2k-1}}(2k)^{2\nu}\right] & (\nu_{2k-1}\nu\neq 0). & &
\end{array}$$

Proof It will be convenient to introduce the following notations: for a partition as above, let m_k be the multiplicity of the number k in it. Moreover let S_k be the h-eigensubspace with eigenvalue k in the standard representation, and let s_k denote dimension of this subspace, i.e. multiplicity of the eigenvalue k for h.

As recalled in Sect. 1 above, the adjoint representation can be identified with the symmetric square of the standard one for type C, and with its exterior square for types B and D.

Because of this, clearly the degree 1 part of the adjoint representation is the direct sum of spaces of the form $S_k^* \otimes S_l$ with $l - k = 1$, $k \geqslant 0$, and

$$\dim \mathfrak{g}_1 = s_0 s_1 + s_1 s_2 + \ldots$$

Now, from the correspondence described in Sect. 2, one has

$$\begin{aligned}
s_0 &= m_1 + m_3 + m_5 + \ldots\\
s_1 &= m_2 + m_4 + m_6 + \ldots\\
s_2 &= m_3 + m_5 + m_7 + \ldots\\
s_3 &= m_4 + m_6 + m_8 + \ldots\\
&\ldots\\
s_{\mu-4} &= m_{\mu-3} + m_{\mu-1}\\
s_{\mu-3} &= m_{\mu-2} + m_{\mu}\\
s_{\mu-2} &= m_{\mu-1}\\
s_{\mu-1} &= m_{\mu}
\end{aligned} \tag{1}$$

Dimension of the subspace $\mathfrak{g}_1$ of grading 1 with respect to the corresponding $\mathfrak{sl}_2$-triple is thus given by

$$\begin{aligned}
s_0 s_1 + s_1 s_2 + s_2 s_3 + s_3 s_4 + \ldots &= \sum_{i,j>0} i m_i m_{i+2j-1}\\
&= m_1 m_2 + 2 m_2 m_3 + m_1 m_4 + 3 m_3 m_4 + 2 m_2 m_5 + \ldots
\end{aligned}$$

Given an Abelian subspace in $\mathfrak{g}_1$, we may assume, using Proposition 3.1, that it has a basis consisting of root vectors. In particular, each of our basis vectors belongs to one of the direct summands $S_k^* \otimes S_{k+1}$.

Note that the elements of $S_{k-1}^* \otimes S_k$ commute with the elements of $S_l^* \otimes S_{l+1}$ for $l > k$; whereas, when $l = k$, we obtain a non-commuting pair as soon as our basis contains an element of the form $x \otimes y \in S_{k-1}^* \otimes S_k$ and $y' \otimes z \in S_k^* \otimes S_{k+1}$ with y and y' mutually dual basis elements. We are thus forced to choose non-intersecting subsets X_k, Y_k in the weight vector bases of S_k and include in the basis of the Abelian subspace only those $x \otimes y$ which satisfy $x \in X_{k-1}$ and $y \in Y_k$. This does not concern $k = \mu - 1$, where $\mu - 1$ is the maximal occurring eigenvalue of h (μ, as above, is the largest part of the corresponding partition): in $S_{\mu-1}$ we may choose arbitrary subset of the basis without affecting Abelianness; and since we are interested in maximal Abelian subspaces, we choose the whole basis of $S_{\mu-1}$.

Moreover, any such choice of non-intersecting subsets X_k, Y_k of bases of S_k gives indeed an Abelian subspace, and we may further assume that $X_k \cup Y_k$ is the whole basis, since otherwise our Abelian subspace would not be maximal.

The case $k = 0$ is special, and depends on the type considered.

Namely, it may happen that two basis vectors, both from $S_0^* \otimes S_1$, do not commute. Two basis elements of this subspace, being the tensor products of basis vectors corresponding to $\pm\varepsilon_i^{(0)} + \varepsilon_j^{(1)}$ and $\pm\varepsilon_k^{(0)} + \varepsilon_l^{(1)}$ respectively, will commute if and only if the sum $\pm\varepsilon_i^{(0)} + \varepsilon_j^{(1)} \pm \varepsilon_k^{(0)} + \varepsilon_l^{(1)}$ is not a root. This implies that the root vector basis of an Abelian subspace in $\mathfrak{g}_1$ cannot contain root vectors corresponding to both $\pm\varepsilon_i^{(0)} + \varepsilon_j^{(1)}$ and $\mp\varepsilon_i^{(0)} + \varepsilon_k^{(1)}$ for $j \neq k$ (since the sum of these is the root $\varepsilon_j^{(1)} + \varepsilon_k^{(1)}$).

This is the only restriction on $S_0^* \otimes S_1$ for type D. For type C, there is an additional restriction that an Abelian subspace of $\mathfrak{g}_1$ cannot contain root vectors corresponding to both $\pm\varepsilon_i^{(0)} + \varepsilon_j^{(1)}$ and $\mp\varepsilon_i^{(0)} + \varepsilon_j^{(1)}$ (since the sum of these is the root $2\varepsilon_j^{(1)}$). For type B, an additional restriction is that an Abelian subspace of $\mathfrak{g}_1$ cannot contain root vectors corresponding to both $(0+)\varepsilon_j^{(1)}$ and $(0+)\varepsilon_k^{(1)}$ for $j \neq k$ (since the sum of these is the root $\varepsilon_j^{(1)} + \varepsilon_k^{(1)}$).

It follows that to obtain a maximal Abelian subspace of $\mathfrak{g}_1$, in addition to splitting the weight vector basis of S_1 into nonintersecting subsets (X_1 and its complement Y_1), for any weights $\varepsilon_j^{(1)}$ and $\varepsilon_k^{(1)}$ corresponding to a weight basis vector in X_1 we have to pick in $S_0^* \otimes S_1$ the root basis elements corresponding either only to $\varepsilon_i^{(0)} + \varepsilon_j^{(1)}$ and $\varepsilon_i^{(0)} + \varepsilon_k^{(1)}$ or only to $-\varepsilon_i^{(0)} + \varepsilon_j^{(1)}$ and $-\varepsilon_i^{(0)} + \varepsilon_k^{(1)}$ for all possible i, but not both. Thus the maximal possible number of basis vectors from $S_0^* \otimes S_1$ which we may include in an Abelian subspace of $\mathfrak{g}_1$ is either $\left[\frac{s_0}{2}\right] x_1$ (if we choose either only $\varepsilon_i^{(0)} + \varepsilon_j^{(1)}$ or only $-\varepsilon_i^{(0)} + \varepsilon_j^{(1)}$ for all possible i and j) or s_0, provided we are not in type C and moreover X_1 consists of a single element (corresponding to some $\varepsilon_j^{(1)}$, and we choose root basis vectors corresponding to $\pm\varepsilon_i^{(0)} + \varepsilon_j^{(1)}$ for all possible i). In addition, if we are in type B, we may add one more root basis vector $v_0 \otimes v_1$ with v_0 a weight basis vector with zero weight and v_1 some weight basis vector from X_1.

Thus, we have the following possibilities for the maximal dimension of the piece of an Abelian subspace corresponding to $S_0^* \otimes S_1$:

	B	C	D
$x_1 = 0$	0	0	0
$x_1 = 1$	s_0	$\frac{s_0}{2}$	s_0
$x_1 > 1$	$\frac{s_0-1}{2}x_1 + 1$	$\frac{s_0}{2}x_1$	$\frac{s_0}{2}x_1$

This results in the following possibilities for the maximal dimension of an Abelian subspace in $\mathfrak{g}_1$:

$$\begin{gathered}\tfrac{s_0-1}{2}x_1 + 1 + (s_1 - x_1)x_2 + (s_2 - x_2)x_3 + \ldots + (s_{\mu-3} - x_{\mu-3})x_{\mu-2} + (s_{\mu-2} - x_{\mu-2})s_{\mu-1} \\ \text{(for type B);} \\ \tfrac{s_0}{2}x_1 + (s_1 - x_1)x_2 + (s_2 - x_2)x_3 + \ldots + (s_{\mu-3} - x_{\mu-3})x_{\mu-2} + (s_{\mu-2} - x_{\mu-2})s_{\mu-1} \\ \text{(for type C or D);} \\ s_0 + (s_1 - 1)x_2 + (s_2 - x_2)x_3 + \ldots + (s_{\mu-3} - x_{\mu-3})x_{\mu-2} + (s_{\mu-2} - x_{\mu-2})s_{\mu-1} \\ \text{(for type B or D).}\end{gathered} \tag{2}$$

where μ is the largest part of the partition.

We thus want to maximize each of these quantities for $0 \leqslant x_k \leqslant s_k$, $k = 1, \ldots, \mu - 2$. Note that each of them is linear in all of the x_k separately, hence any possible maxima are attained when every x_k is either 0 or s_k. In fact, more is true:

Lemma 4.4 *An Abelian subspace of maximal possible dimension in* $\mathfrak{g}_1$ *can be obtained either with* $x_{2j-1} = 0$, $x_{2j} = s_{2j}$ *or with* $x_{2j-1} = s_{2j-1}$, $x_{2j} = 0$ *for all* j.

Proof Looking at the subsum

$$\ldots + (s_{k-2} - x_{k-2})x_{k-1} + (s_{k-1} - x_{k-1})x_k + (s_k - x_k)x_{k+1} + \ldots$$

determining dimension of the Abelian subspace, it is easy to see that each of the following changes:

$$\begin{aligned} x_{k-1} = 0, \quad x_k = 0 &\mapsto x_{k-1} = 0, \quad x_k = s_k, \\ x_{k-1} = s_{k-1}, x_k = s_k &\mapsto x_{k-1} = s_{k-1}, x_k = 0 \end{aligned}$$

does not decrease the dimension of the Abelian subspace.

Indeed, these changes do not affect any other summands except those in the above subsum. The first change transforms

$$\begin{aligned} &\ldots + (s_{k-2} - x_{k-2})0 + (s_{k-1} - 0)0 + (s_k - 0)x_{k+1} + \ldots \\ \mapsto &\ldots + (s_{k-2} - x_{k-2})0 + (s_{k-1} - 0)s_k + 0x_{k+1} + \ldots, \end{aligned}$$

i.e., changes the sum by the amount equal to the change from $s_k x_{k+1}$ to $s_{k-1}s_k$. But $x_{k+1} \leqslant s_{k+1}$, and $s_{k+1} \leqslant s_{k-1}$ by (1), so that indeed the sum does not decrease.

Similarly, the second change transforms

$$\begin{aligned} &\ldots + (s_{k-2} - x_{k-2})s_{k-1} + (s_{k-1} - s_{k-1})s_k + (s_k - s_k)x_{k+1} + \ldots \\ \mapsto &\ldots + (s_{k-2} - x_{k-2})s_{k-1} + (s_{k-1} - s_{k-1})0 + (s_k - 0)x_{k+1} + \ldots, \end{aligned}$$

i.e., changes the sum by the amount equal to the change from 0 to $s_k x_{k+1}$, which is obviously a nondecreasing change.

Now using the above changes we may arrive at one of the needed choices. For simplicity, let us encode a given choice of x's by a sequence of zeroes and ones (at the kth place of the sequence stands zero if $x_k = 0$ and one if $x_k = s_k$). We are allowed to perform "local transformations" of the kind $\cdots 00 \cdots \mapsto \cdots 01 \cdots$ and $\cdots 11 \cdots \mapsto \cdots 10 \cdots$. Using one of these transformations, we can always shift the place of the leftmost occurrence of two consecutive identical symbols to the right: say, if this leftmost occurrence is $\cdots 11 \cdots$ we change it to $\cdots 10 \cdots$ and if it is $\cdots 00 \cdots$, we change it to $\cdots 01 \cdots$, and in the worst case the place of the leftmost occurrence of consecutive identical symbols still shifts to the right by at least one position. Thus, if we keep applying the appropriate transformations to the leftmost occurrence of consecutive identical symbols, we inevitably arrive either at $10101\ldots$ or at $01010\ldots$. □

Applying this in (2), we obtain that the maximal possible dimension of an Abelian subspace in $\mathfrak{g}_1$ can only be equal to one of the following six expressions:

$$\begin{array}{r|rl} \frac{s_0-1}{2}s_1 + 1 + s_2s_3 + s_4s_5 + \ldots & s_1s_2 + s_3s_4 + s_5s_6 + \ldots & \text{(for type B)} \\ \frac{s_0}{2}s_1 + s_2s_3 + s_4s_5 + \ldots & s_1s_2 + s_3s_4 + s_5s_6 + \ldots & \text{(for types C, D)} \\ s_0 + s_2s_3 + s_4s_5 + \ldots & s_0 + (s_1-1)s_2 + s_3s_4 + s_5s_6 + \ldots & \text{(for types B, D)} \end{array}$$

To find out whether there is an Abelian subspace of half the dimension in $\mathfrak{g}_1$ is thus equivalent to finding out whether subtracting from the dimension of $\mathfrak{g}_1$, i. e. from $s_0s_1 + s_1s_2 + \ldots$, one of these sums doubled gives zero, i. e. whether one of the sums

$$\begin{array}{lr|lrl} s_0s_1 + s_1s_2 + \ldots & -2(\frac{s_0-1}{2}s_1 + 1 + s_2s_3 + s_4s_5 + \ldots) & s_0s_1 + s_1s_2 + \ldots & -2(s_1s_2 + s_3s_4 + s_5s_6 + \ldots) & \text{(B)} \\ s_0s_1 + s_1s_2 + \ldots & -2(\frac{s_0}{2}s_1 + s_2s_3 + s_4s_5 + \ldots) & s_0s_1 + s_1s_2 + \ldots & -2(s_1s_2 + s_3s_4 + s_5s_6 + \ldots) & \text{(C, D)} \\ s_0s_1 + s_1s_2 + \ldots & -2(s_0 + s_2s_3 + s_4s_5 + \ldots) & s_0s_1 + s_1s_2 + \ldots & -2(s_0 + (s_1-1)s_2 + s_3s_4 + s_5s_6 + \ldots) & \text{(B, D)} \end{array}$$

is zero.

Simplifying, we obtain respectively

$$\begin{array}{r|rl} s_1 - 2 + s_1s_2 - s_2s_3 + s_3s_4 - s_4s_5 + s_5s_6 - \ldots & s_0s_1 - s_1s_2 + s_2s_3 - s_3s_4 + s_4s_5 - \ldots & \text{(B)} \\ s_1s_2 - s_2s_3 + s_3s_4 - s_4s_5 + \ldots & s_0s_1 - s_1s_2 + s_2s_3 - s_3s_4 + \ldots & \text{(C, D)} \\ -2s_0 + s_0s_1 + s_1s_2 - s_2s_3 + s_3s_4 - \ldots & -2s_0 + 2s_2 + s_0s_1 - s_1s_2 + s_2s_3 - s_3s_4 + \ldots & \text{(B, D)} \end{array}$$

Rewriting this further as

$$\begin{array}{r|rl}
s_1-2+(s_1-s_3)s_2+(s_3-s_5)s_4+(s_5-s_7)s_6+\dots & (s_0-s_2)s_1+(s_2-s_4)s_3+(s_4-s_6)s_5+\dots & \text{(B)}\\
(s_1-s_3)s_2+(s_3-s_5)s_4+(s_5-s_7)s_6+\dots & (s_0-s_2)s_1+(s_2-s_4)s_3+(s_4-s_6)s_5+\dots & \text{(C, D)}\\
s_0(s_1-2)+(s_1-s_3)s_2+(s_3-s_5)s_4+\dots & (s_0-s_2)(s_1-2)+(s_2-s_4)s_3+(s_4-s_6)s_5+\dots & \text{(B, D)}
\end{array}$$

and taking (1) into account this can be rewritten as

$$\begin{array}{r|rl}
s_1-2+m_2s_2+m_4s_4+m_6s_6+\dots & m_1s_1+m_3s_3+m_5s_5+\dots & \text{(B)}\\
m_2s_2+m_4s_4+m_6s_6+\dots & m_1s_1+m_3s_3+m_5s_5+\dots & \text{(C, D)}\\
s_0(s_1-2)+m_2s_2+m_4s_4+\dots & m_1(s_1-2)+m_3s_3+m_5s_5+\dots & \text{(B, D)}
\end{array}$$

Let us now assume that our nilpotent element is strictly odd, which, in terms of the corresponding partition, means that $m_{\mu-1} > 0$ (here, as before, μ is the largest nonzero part of the partition). This then implies that all multiplicities s_i are nonzero. Thus, to obtain an Abelian subspace of half the dimension of $\mathfrak{g}_1$, we have the following possibilities:

$$\begin{array}{r|rl}
s_1=2 \text{ and } m_{2k}=0 \text{ for } 2k<\mu & m_{2k-1}=0 \text{ for } 2k-1<\mu & \text{(B)}\\
m_{2k}=0 \text{ for } 2k<\mu & m_{2k-1}=0 \text{ for } 2k-1<\mu & \text{(C, D)}\\
s_1=2 \text{ and } m_{2k}=0 \text{ for } 2k<\mu & m_1=0 \text{ or } s_1=2, \text{ and } m_{2k-1}=0 \text{ for } 1<2k-1<\mu & \text{(B, D)}
\end{array}$$

We now make the following observations, according to the parity of μ:

- if μ is odd, then the cases in the first column are not realizable, since they require that the partition has no even parts, while, by strict oddity, both $m_{\mu-1}$ and m_μ must be nonzero;
- if μ is even, the cases in the second column are not realizable by exactly the same reason.

Taking this into account, we are left with the following cases: for μ even,

$$\begin{array}{r|l}
m_2=m_4=\dots=m_{\mu-2}=0,\ m_{\mu-1}>0,\ m_\mu=2 & \text{— (B)}\\
m_2=m_4=\dots=m_{\mu-2}=0,\ m_{\mu-1}>0,\ m_\mu>0 & \text{— (C, D)}\\
m_2=m_4=\dots=m_{\mu-2}=0,\ m_{\mu-1}>0,\ m_\mu=2 & \text{— (B, D)}
\end{array}$$

and for μ odd,

$$\begin{array}{r|ll}
\text{—} & m_1=m_3=\dots=m_{\mu-2}=0,\ m_{\mu-1}>0,\ m_\mu>0 & \text{(B)}\\
\text{—} & m_1=m_3=\dots=m_{\mu-2}=0,\ m_{\mu-1}>0,\ m_\mu>0 & \text{(C, D)}\\
\text{—} & m_3=m_5=\dots=m_{\mu-2}=0,\ m_\mu>0 \text{ and either } m_1=0 & \text{(B, D)}\\
 & \qquad \text{or } m_2=m_4=\dots=m_{\mu-3}=0 \text{ and } m_{\mu-1}=2 &
\end{array}$$

Let us also observe the following:

- for μ even, the first case is subsumed by the third one;
- for μ even, the third case is subsumed by the second one for type D;
- for μ odd, the subcase $m_1 = 0$ of the third case is subsumed by the first one for type B, and by the second one for type D.

Taking all of the above into account gives the partitions as described. □

Remark 4.5 Another way to formulate the theorem is the following. In case of type C, there is exactly one parity change along the partition, while in cases B or D there might be either one or two parity changes; but if there are two parity changes, then there must be only parts equal to 1, $\mu - 1$, μ and, moreover, $\mu - 1$ must have multiplicity 2. Moreover, for the type B there is one more restriction in case there is only one parity change: namely, if the largest part is even, its multiplicity must be 2.

We now turn to the not necessarily strictly odd nilpotent orbits, using strictly odd reduction from Definition 3.4. For classical types, its reformulation in terms of partitions is as follows.

Lemma 4.6 *Let $\mathfrak{g}$ be a simple Lie algebra of classical type, and let e be a nilpotent element of $\mathfrak{g}$ corresponding to the partition $[\dots k^{m_k} \ell^{m_\ell} \dots n^{m_n}]$, with $\dots < k < \ell < \dots < n$ such that k and ℓ are of opposite parity while all the larger parts j (those with $\ell \leqslant j \leqslant n$) are of the same parity.*

Then the partition $[\dots k^{m_k}(k+1)^{m_\ell + \dots + m_n}]$ defines a strictly odd nilpotent element in a Lie algebra of the same type, and corresponds to the strictly odd reduction of e, as defined in Definition 3.4.

Proof Let us begin by noting that the modified partition is indeed suitable for the same type: if this requires that all parts of the same parity as k have even multiplicity, then we have not touched them; while if this requires that all parts of the same parity as $k + 1$ are even, then ℓ and all larger parts are of the same parity as $k + 1$, so each of the multiplicities $m_\ell, \dots, m_n$ was even, hence their sum is even too, and we indeed stay with the same type. Moreover, the corresponding nilpotent element is strictly odd since its largest parts are k and $k + 1$.

Let us now reformulate the passage from the original partition to the modified partition in terms of weighted Dynkin diagrams. We get the following procedure: one removes all nodes (and weights) from left to right until no more 2's are left; for the types B, C, D that's all that has to be done; for the type A one has to similarly remove all 2's on the right.

This procedure precisely means leaving the connected component of the weighted Dynkin diagram that contains nonzero weights, as described in Proposition 3.2 above, so that we indeed obtain the strictly odd reduction of e. □

Corollary 4.7 *Given a nilpotent element in a simple Lie algebra* $\mathfrak{g}$ *of classical type* B, C, *or* D, *there is an Abelian subspace of half dimension in* $\mathfrak{g}_1$ *if and only if the partition corresponding to the nilpotent element satisfies the following conditions:*

type C*: there is no more than one parity change along the partition;*
types B *and* D*: there are no more than two parity changes*
and, if there is at least one parity change, then

- *if the largest part of the partition is even, then there is only one parity change, and in the* B *case moreover it must be the unique even part and must have multiplicity* 2*;*
- *if there are two parity changes, then the largest part of the partition is odd, there is a unique even part, it has multiplicity* 2, *and all smaller parts are equal to* 1.

Thus, Abelian subspaces of half dimension in $\mathfrak{g}_1$ *occur precisely for nilpotent elements corresponding to partitions of one of the following kind (with* $k \leqslant \ell$*):*

$$\begin{array}{l|l} \textbf{any type}: & [\cdots(2k-2)^{\nu_{2k-2}}(2k)^{\nu_{2k}}(2\ell+1)^{\nu_{2\ell+1}}(2\ell+3)^{\nu_{2\ell+3}}\cdots]\,; \\ \textbf{type C or D}: & [\cdots(2k-3)^{\nu_{2k-3}}(2k-1)^{\nu_{2k-1}}(2\ell)^{\nu_{2\ell}}(2\ell+2)^{\nu_{2\ell+2}}\cdots]\,; \\ \textbf{type B or D}: & \left[1^{\nu_1}(2k)^2(2\ell+1)^{\nu_{2\ell+1}}(2\ell+3)^{\nu_{2\ell+3}}\cdots\right]; \\ \textbf{type B}: & \left[\cdots(2k-3)^{\nu_{2k-3}}(2k-1)^{\nu_{2k-1}}(2\ell)^2\right], \end{array}$$

Proof This follows from Lemma 4.6. Indeed the latter shows that $\mathfrak{g}_1(e)$ for a nilpotent element e corresponding to some partition has an Abelian subspace of half dimension if and only if $\tilde{\mathfrak{g}}_1(\tilde{e})$, as described in Corollary 3.3, has such a subspace; and this happens if and only if the partition modified as in Lemma 4.6 satisfies conditions of Theorem 4.3.

It remains to note that a partition is of the indicated kind if and only if the partition obtained from it as in Lemma 4.6 satisfies conditions of Theorem 4.3. □

5 Computations

It remains to find out which of the strictly odd nilpotent orbits in simple Lie algebras of exceptional type have an Abelian subspace of half dimension in degree 1.

For that, we used the computer algebra system `GAP`. In the package `SLA` by Willem A. de Graaf included in this system one can compute with nilpotent orbits of arbitrary semisimple Lie algebras. In particular, one obtains canonical bases consisting of root vectors for the homogeneous subspaces of all degrees in the grading of the Lie algebra induced by a nilpotent element.

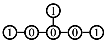
Using Proposition 3.1, we can determine Abelian subspaces in $\mathfrak{g}_1$ as follows. Let B be the basis of $\mathfrak{g}_1$ consisting of positive root vectors. Let us construct a graph with the set of vertices B, where two vertices e_α and e_β are connected with an edge if and only if they do not commute, that is, if and only if $\alpha + \beta$ is a root. Then, by Proposition 3.1, $\mathfrak{g}_1$ has an Abelian subspace of dimension d if and only if the basis consisting of root vectors has a subset of cardinality d consisting of pairwise commuting root vectors.

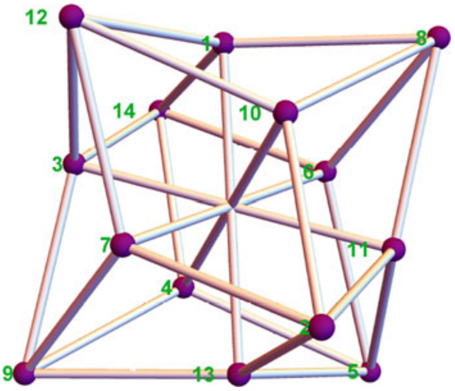
Clearly, this is equivalent to the corresponding graph having an *independent set* of cardinality d—that is, a subset consisting of d vertices such that no two of these vertices are connected by an edge. Hence, describing all possible dimensions of Abelian subspaces in $\mathfrak{g}_1$ reduces to listing all possible cardinalities of independent subsets in the corresponding graph.

There is another package, `GRAPE` by Leonard H. Soicher in `GAP`, which can be used to list all independent sets in a finite graph. Using this package, we determine independent sets of maximal possible cardinality in the graph corresponding to a nilpotent orbit.

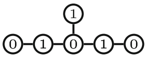
The results are given in the tables below. A `GAP` code for computing maximal dimensions of Abelian subspaces in $\mathfrak{g}_1$ for arbitrary semisimple Lie algebras is available at [8]. In fact, the program can list all subsets of any given cardinality of pairwise commuting elements in the root vector basis.

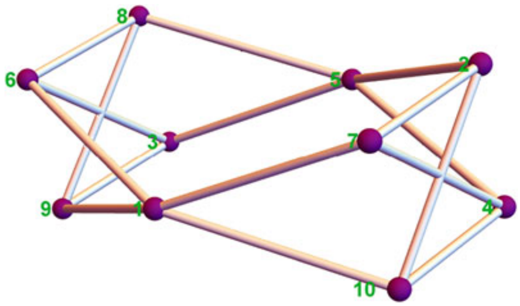
As an illustration, we present below two cases for E_6.

Examples 5.1 The nilpotent orbit with the weighted Dynkin diagram  has $\mathfrak{g}_1$ of dimension 14. The corresponding graph with 14 vertices and edges connecting vertices corresponding to non-commuting root vectors in $\mathfrak{g}_1$ looks as follows:



This graph has independent sets with 6 vertices, e. g. {2, 5, 8, 9, 12, 14}, but any subset on more than 6 vertices contains a pair of vertices connected with an edge, thus for this nilpotent orbit maximal dimension of an Abelian subspace is equal to 6.

Another orbit in E_6, with the diagram , has $\mathfrak{g}_1$ of dimension 10 corresponding to the graph



with 10 vertices. It is easy to find in this graph an independent subset with five elements – e. g. {1, 2, 3, 4, 8}.

Thus, the orbit of the first example has no Abelian subspace of half dimension in $\mathfrak{g}_1$, while that of the second example has.

Tables

Table G2s Strictly odd nilpotent orbits in G_2, all with half-Abelian $\mathfrak{g}_1$

Name	Diagram	dim $\mathfrak{g}_1$
A_1	①≡⓪	4
$\widetilde{A}_1$	⓪≡①	2

Table F4s Strictly odd nilpotent orbits in F_4

With half-Abelian $\mathfrak{g}_1$			Without half-Abelian $\mathfrak{g}_1$		
Name	Diagram	dim $\mathfrak{g}_1$	Name	Diagram	dim $\mathfrak{g}_1$ (largest dimension of an Abelian subspace)
A_1	①—⓪⇐⓪—⓪	14	$\widetilde{A}_1$	⓪—⓪⇐⓪—①	8 (2)
$A_1 + \widetilde{A}_1$	⓪—①⇐⓪—⓪	12	$A_2 + \widetilde{A}_1$	⓪—⓪⇐①—⓪	6 (2)
$C_3(a_1)$	①—⓪⇐①—⓪	6	$\widetilde{A}_2 + A_1$	⓪—①⇐⓪—①	8 (3)

Table E6s Strictly odd nilpotent orbits in E_6

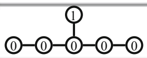
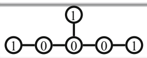
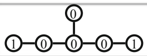
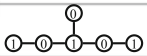
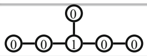
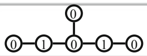
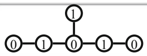
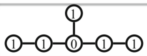
With half-Abelian $\mathfrak{g}_1$			Without half-Abelian $\mathfrak{g}_1$		
Name	Diagram	dim $\mathfrak{g}_1$	Name	Diagram	dim $\mathfrak{g}_1$ (largest dimension of an Abelian subspace)
A_1		20	$A_2 + A_1$		14 (6)
$2A_1$		16	$2A_2 + A_1$		12 (5)
$3A_1$		18			
$A_2 + 2A_1$		12			
$A_3 + A_1$		10			
$A_4 + A_1$		8			

Table E7s Strictly odd nilpotent orbits in E_7

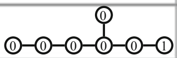
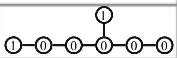
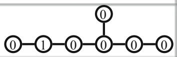
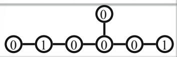
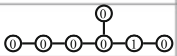
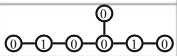
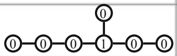
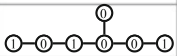
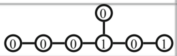
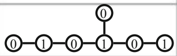
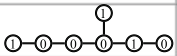
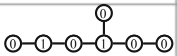
With half-Abelian $\mathfrak{g}_1$			Without half-Abelian $\mathfrak{g}_1$		
Name	Diagram	dim $\mathfrak{g}_1$	Name	Diagram	dim $\mathfrak{g}_1$ (largest dimension of an Abelian subspace)
A_1		32	$4A_1$		26 (11)
$2A_1$		32	$A_2 + A_1$		24 (9)
$3A_1'$		30	$2A_2 + A_1$		20 (8)
$A_2 + 2A_1$		24	$A_3 + 2A_1$		18 (7)
$(A_3 + A_1)'$		18	$A_4 + A_1$		14 (6)
$D_4(a_1) + A_1$		16			
$A_3 + A_2$		16			

Table E8s Strictly odd nilpotent orbits in E_8

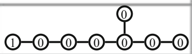
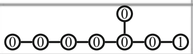
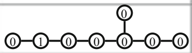
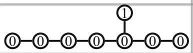
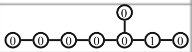
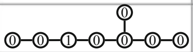
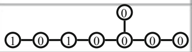
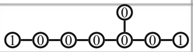
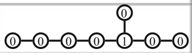
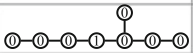
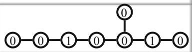
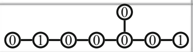
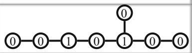
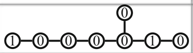
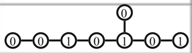
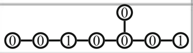
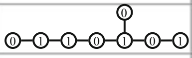
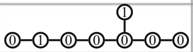
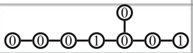
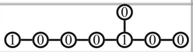
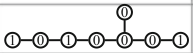
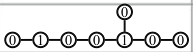
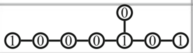
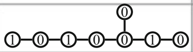
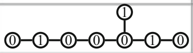
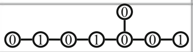
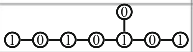
With half-Abelian $\mathfrak{g}_1$			Without half-Abelian $\mathfrak{g}_1$		
Name	Diagram	dim $\mathfrak{g}_1$	Name	Diagram	dim $\mathfrak{g}_1$ (largest dimension of an Abelian subspace)
A_1		56	$2A_1$		64 (22)
$3A_1$		54	$4A_1$		56 (21)
$A_2 + 3A_1$		42	$A_2 + 2A_1$		48 (16)
$A_3 + A_1$		34	$A_2 + A_1$		44 (17)
$A_3 + A_2 + A_1$		30	$2A_2 + 2A_1$		40 (16)
$A_4 + A_2 + A_1$		24	$2A_2 + A_1$		36 (16)
$E_7(a_5)$		18	$A_3 + 2A_1$		36 (15)
$A_6 + A_1$		16	$A_3 + A_2$		32 (13)
A_7		14	$D_4(a_1) + A_1$		32 (12)
			$2A_3$		28 (13)
			$A_4 + 2A_1$		28 (12)
			$A_4 + A_1$		26 (10)
			$A_4 + A_3$		24 (10)
			$A_5 + A_1$		22 (9)
			$D_5(a_1) + A_2$		22 (8)
			$D_6(a_2)$		20 (9)
			$E_6(a_3) + A_1$		20 (8)
			$D_7(a_2)$		16 (7)

Table F4o (Non-strictly) odd nilpotent orbits in F_4, all with half-Abelian $\mathfrak{g}_1$

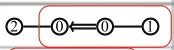
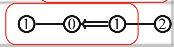
Name	Diagram	Strictly odd piece
B_2		C_3 $(2, 1^4)$
C_3		B_3 $(3, 2^2)$

Table E6o (Non-strictly) odd nilpotent orbits in E_6, all with half-Abelian $\mathfrak{g}_1$

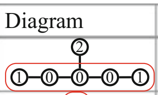
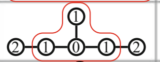
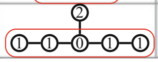
Name	Diagram	Strictly odd piece
A_3		A_5
A_5		$D_4\ (3, 2^2, 1)$
$D_5(a_1)$		A_5

Table E7o (Non-strictly) odd nilpotent orbits in E_7

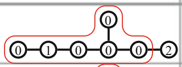
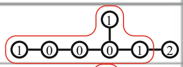
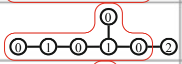
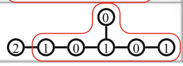
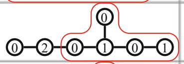
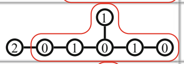
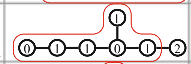
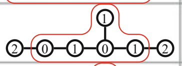
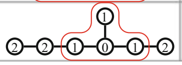
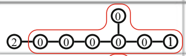
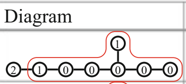
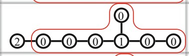
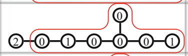
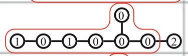
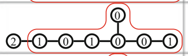
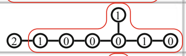
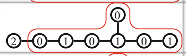
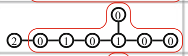
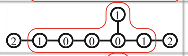
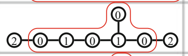
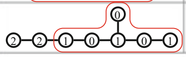
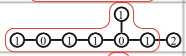
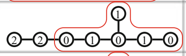
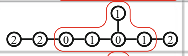
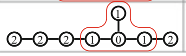
With half-Abelian $\mathfrak{g}_1$			Without half-Abelian $\mathfrak{g}_1$		
Name	Diagram	Strictly odd piece	Name	Diagram	Strictly odd piece
A_3		$D_6\ (2^2, 1^8)$	$D_4 + A_1$		$D_6\ (3, 2^4, 1)$
$D_5(a_1)$		$D_6\ (3^2, 2^2, 1^2)$	$A_5 + A_1$		$E_6\ (2A_2 + A_1)$
A'_5		$D_5\ (3, 2^2, 1^3)$			
$D_6(a_2)$		$E_6\ (A_3 + A_1)$			
$D_5 + A_1$		$D_6\ (4^2, 3, 1)$			
$D_6(a_1)$		$D_5\ (3^2, 2^2)$			
D_6		$D_4\ (3, 2^2, 1)$			

Table E8o (Non-strictly) odd nilpotent orbits in E_8

With half-Abelian $\mathfrak{g}_1$			Without half-Abelian $\mathfrak{g}_1$		
Name	Diagram	Strictly odd piece	Name	Diagram	Strictly odd piece
A_3		$E_7\ (A_1)$	$D_4 + A_1$		$E_7\ (4A_1)$
$D_5(a_1) + A_1$		$E_7\ (A_2 + 2A_1)$	$D_5(a_1)$		$E_7\ (A_2 + A_1)$
A_5		$D_7\ (3, 2^2, 1^7)$	$D_5 + A_1$		$E_7\ (A_3 + 2A_1)$
$D_6(a_1)$		$E_7\ (D_4(a_1) + A_1)$	$E_6(a_1) + A_1$		$E_7\ (A_4 + A_1)$
$E_7(a_4)$		$E_7\ (A_3 + A_2)$	D_6		$D_6\ (3, 2^4, 1)$
$E_7(a_3)$		$D_6\ (3^2, 2^2, 1^2)$	$E_6 + A_1$		$E_6\ (2A_2 + A_1)$
D_7		$D_7\ (5, 4^2, 1)$			
$E_7(a_2)$		$E_6\ (A_3 + A_1)$			
$E_7(a_1)$		$D_5\ (3^2, 2^2)$			
E_7		$D_4\ (3, 2^2, 1)$			

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References

1. D. H. Collingwood, W. M. McGovern, "Nilpotent Orbits In Semisimple Lie Algebras". Van Nostrand Reinhold, New York 1993, xi + 186 pages.
2. A. De Sole, V. G. Kac, D. Valeri, *Structure of classical (finite and affine) W-algebras*. J. Eur. Math, Soc. **18** (2016), 1873–1908.
3. A. G. Elashvili, *Centralizers of nilpotent elements in semisimple Lie algebras*. Proc. Razmadze Math. Inst. **46** (1975), 109–132 (in Russian).
4. V. G. Kac, M. Wakimoto, *Quantum reduction and representation theory of superconformal algebras*. Advances in Math. **193** (2005), 400–458.
5. D. I. Panyushev, *On spherical nilpotent orbits and beyond*. Ann. Inst. Fourier **49** (1999), 1453–1476.
6. T. A. Springer, R. Steinberg, *Conjugacy classes*. In "Seminar on Algebraic Groups and Related Finite Groups", Lecture Notes in Math. **131**, Springer Verlag, New York, 1970, 167–266.
7. T. Shoji, *Shintani descent for special linear groups*. J. Algebra **199** (1998), 175–228.
8. The `GAP` code at https://arxiv.org/src/1806.00893v1/anc/nilorg1.g