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# Moduli Algebras of Some Non-Semiquasihomogeneous Singularities*

**A. G. Elashvili, M. A. Jibladze, and E. B. Vinberg**

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Abstract. Under some additional restrictions we find dimensions and bases of moduli algebras of isolated singularities of polynomials in $n$ variables that are sums of $n$ monomials of equal weighted degrees and one monomial of lower degree.

Key words: isolated singularity, moduli algebra, Milnor number, Brieskorn–Pham singularity, Kouchnirenko's formula.

## Introduction

A singularity (at the origin) of a function $f(x_1,\dots,x_n)$ is called *quasihomogeneous* with respect to positive weights $w_1,\dots,w_n$ of the variables $x_1,\dots,x_n$ if all monomials occurring in the Maclaurin series for $f$ have the same weighted degree (which implies that $f$ is a polynomial). It is called *semiquasihomogeneous* if $f=f_0+f_>$, where $f_0$ has a quasihomogeneous singularity with respect to some weights $w_1,\dots,w_n$ such that the degrees of all monomials in $f_>$ are larger than those in $f_0$. In the latter case, if the singularity of $f_0$ is isolated, then the singularity of $f$ is isolated as well.

By abuse of language, we will sometimes talk about a singularity $f$, having in mind the singularity of the function $f$. In all cases, it will be assumed that $f$ does not contain monomials of the form $x_i$ and $x_ix_j$ with $i\ne j$.

The *moduli algebra* $\mathrm{A}_f$ of a singularity $f$ is defined as

$$\mathrm{A}_f := \mathbb{C}\,[\![x_1,\dots,x_n]\!]\,/J(f), \tag{1}$$

where

$$J(f) := (\partial f/\partial x_1,\dots,\partial f/\partial x_n) \tag{2}$$

is the *Jacobian ideal* of $f$. A singularity is isolated if and only if its moduli algebra is finite-dimensional.

Obviously, one can always choose a basis of the moduli algebra consisting of (residue classes of) monomials. We will call such bases *monomial*.

Quasihomogeneity of a singularity implies the existence of a grading on its moduli algebra, which makes life simpler. For semiquasihomogeneous singularities, the following theorem is known.

**Theorem 0.1** [1, 12.2, Corollary]. *A monomial basis of the moduli algebra $\mathrm{A}_{f_0}$ of the quasihomogeneous part $f_0$ of a semiquasihomogeneous singularity $f=f_0+f_>$ is also a basis of $\mathrm{A}_f$.*

We have been able to find in the literature only few separate explicit descriptions of bases for moduli algebras of isolated singularities which are neither quasihomogeneous nor semiquasihomogeneous.

It is known that under our assumption about the absence of monomials $x_ix_j$ with $i\ne j$, a quasihomogeneous singularity $f_0$ can be isolated only if $f_0$ contains at least $n$ monomials. It is

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thus natural to say that the simplest isolated quasihomogeneous singularities are those which are "minimal," i.e., contain exactly $n$ monomials:

$$f_0(x_1, \dots, x_n) = u_1 + \dots + u_n, \qquad u_i = x_1^{\nu_{i1}} \cdots x_n^{\nu_{in}}. \tag{3}$$

All minimal isolated quasihomogeneous singularities are known. Their description given in Theorem 1.3 of this paper can be extracted from [1] and subsequent papers (see, e.g., [5], [6]). In these subsequent papers some bases of the moduli algebras of the corresponding singularities were constructed too, but some details of the proofs were omitted. For the convenience of the reader, in Section 1 we present a construction of these bases with complete proofs.

The description of minimal isolated quasihomogeneous singularities implies, in particular, that under a suitable numbering of variables one has

$$\nu_{ii} \geqslant 2, \quad \nu_{ij} \leqslant 1 \quad \text{for } i \neq j. \tag{4}$$

In what follows, we will always assume these conditions to be satisfied.

We will investigate what happens if to $f_0$ one adds a monomial $u$ whose (weighted) degree is strictly less than that of $u_1, \dots, u_n$:

$$f(x_1, \dots, x_n) = f_0(x_1, \dots, x_n) + u, \qquad u = x_1^{a_1} \cdots x_n^{a_n}. \tag{5}$$

We will assume that $u$ satisfies the following additional condition.

**Condition 0.2.** Each variable occurs in $u$ to a positive power, and if some variable occurs to a positive power in all of the monomials $u_1, \dots, u_n$, then it occurs in $u$ to a power strictly larger than 1.

Assuming conditions (4) to be satisfied, consider the matrix

$$M_f = \begin{pmatrix} \nu_{11} - a_1 & \dots & \nu_{1n} - a_n \\ \vdots & \ddots & \vdots \\ \nu_{n1} - a_1 & \dots & \nu_{nn} - a_n \end{pmatrix}.$$

Our first result is the following theorem.

**Theorem 0.3.** *Under the above assumptions, the number* $\det M_f$ *is positive and*

$$\dim \mathrm{A}_f = \dim \mathrm{A}_{f_0} - \det M_f. \tag{6}$$

We will refer to the number $\det M_f$ as the *defect* of the singularity $f$ and denote it by $\operatorname{def}(f)$.

In Section 2 we show how formula (6) can be derived with the aid of a formula of Kouchnirenko [3]. The defect of the singularity arises there in the guise of the volume of a certain simplex. Then we will employ a different approach, which provides an additional insight into our formula; in particular, the defect will be interpreted as the order of a certain Abelian group.

In the particular case when $f_0$ is a so-called *Brieskorn–Pham singularity*, i.e.,

$$f_0(x_1, \dots, x_n) = x_1^{p_1} + \dots + x_n^{p_n}, \tag{7}$$

we propose yet another approach, which will eventually yield not only the dimension but also an explicit monomial basis of the algebra $\mathrm{A}_f$ (Theorem 4.1). Finally, in the last section under some additional assumptions we will construct one more basis of the algebra $\mathrm{A}_f$.

We find it worth mentioning that the bases of moduli algebras constructed in the last section, as well as the bases of the moduli algebras of the minimal isolated quasihomogeneous singularities described in Section 1, are *parallelepipedal* in the sense that they can be obtained from a suitable "parallelepiped"

$$P(p_1, \dots, p_n) := \{x_1^{k_1} \cdots x_n^{k_n} \mid 0 \leqslant k_i < p_i,\ i = 1, \dots, n\} \tag{8}$$

by omitting those monomials which belong to $J(f)$. It seems that these very natural bases are different from all Gröbner bases that can be obtained from any admissible monomial orderings.

**Remark 0.4.** Under a monomial we mean a product of some (nonnegative integer) powers of the variables with coefficient 1. However, if the degree vectors of the monomials $u_1, \dots, u_n$ in (3) are linearly independent (which is certainly the case if $f_0$ is an isolated singularity), then, replacing the variables with their suitable nonzero multiples, one can turn $f_0$ into a linear combination of the monomials $u_1, \dots, u_n$ with arbitrary nonzero coefficients. Similarly, if the degree vectors of the monomials $u_1, \dots, u_n, u$ are affinely independent (which is definitely true if Condition 0.2 holds), then, multiplying the variables and the polynomial $f$ itself by some nonzero numbers, one can turn it into a linear combination of $u_1, \dots, u_n, u$ with arbitrary nonzero coefficients. We will sometimes use this observation.

A preliminary version of the paper was published as a preprint [8].

## 1. Minimal Isolated Quasihomogeneous Singularities

Thus, we begin with considering quasihomogeneous polynomials $f_0$ of the form (3). We are interested in those of them which have an isolated singularity at the origin, i.e., are such that the algebra $\mathrm{A}_{f_0}$ is finite-dimensional. Such polynomials were classified in [5]. These are disjoint sums (i.e., sums whose summands have no common variables) of polynomials of two types described in the following examples.

**Example 1.1.** We will refer to a polynomial of the form

$$C = C_{k_1,\dots,k_n}(x_1, \dots, x_n) := x_1^{k_1} x_2 + x_2^{k_2} x_3 + \dots + x_n^{k_n} x_1 \qquad (k_1, \dots, k_n > 1)$$

as a *cycle*. It is quasihomogeneous of degree $d := k_1 \cdots k_n - (-1)^n$ with respect to the following weights of variables:

$$\begin{aligned}
w_1 &= k_2 \cdots k_{n-1} k_n - k_3 \cdots k_{n-1} k_n + k_4 \cdots k_{n-1} k_n - \dots \pm k_n \mp 1, \\
w_2 &= k_3 \cdots k_n k_1 - k_4 \cdots k_n k_1 + k_5 \cdots k_n k_1 - \dots \pm k_1 \mp 1, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
w_n &= k_1 \cdots k_{n-1} - k_2 \cdots k_{n-1} + k_3 \cdots k_{n-1} - \dots \pm k_{n-1} \mp 1.
\end{aligned}$$

Computing partial derivatives, one can see that the moduli algebra of the singularity $C$ is obtained from the algebra of formal power series by imposing the relations

$$\begin{aligned}
x_1^{k_1} &= -k_2 x_2^{k_2-1} x_3, \\
&\dots\dots\dots\dots \\
x_{n-1}^{k_{n-1}} &= -k_n x_n^{k_n-1} x_1, \\
x_n^{k_n} &= -k_1 x_1^{k_1-1} x_2.
\end{aligned}$$

These relations imply that the moduli algebra has finite dimension. Indeed, consider an arbitrary monomial $x_1^{l_1} \cdots x_n^{l_n}$. If it does not lie in the parallelepiped $P(k_1, \dots, k_n)$ (see notation (8)), i.e., $l_i \geqslant k_i$ for some $i$, then one can replace $x_i^{k_i}$ with the right-hand side of the corresponding relation, thus obtaining a monomial of the same (weighted) degree with a coefficient having absolute value strictly greater than one. Repeating this procedure, we will either obtain a scalar multiple of a monomial in $P(k_1, \dots, k_n)$ or twice obtain different scalar multiples of the same monomial. In the latter case, we conclude that the corresponding monomial is equal to zero in the moduli algebra. Thus, in the moduli algebra every monomial becomes either zero or a scalar multiple of a monomial lying in $P(k_1, \dots, k_n)$. Hence the dimension of the moduli algebra does not exceed $k_1 \cdots k_n$. In particular, the singularity is isolated.

On the other hand, the dimension of the moduli algebra can be calculated with the aid of the following well-known formula for the dimension of the moduli algebra $A$ of a quasihomogeneous singularity:

$$\dim A = \prod_{i=1}^{n} \left( \frac{d}{w_i} - 1 \right)$$

(see, e. g., [7] or [1]). In our case, this gives $k_1 \cdots k_n$.

Thus, all monomials in $P(k_1, \ldots, k_n)$ constitute a basis of the moduli algebra.

**Example 1.2.** Following [5], we will refer to a polynomial of the form

$$T = T_{k_1,\ldots,k_n}(x_1, \ldots, x_n) := x_1^{k_1} x_2 + x_2^{k_2} x_3 + \cdots + x_{n-1}^{k_{n-1}} x_n + x_n^{k_n} \qquad (k_1, \ldots, k_n > 1)$$

as a *chain*. It is quasihomogeneous of degree $d := k_1 \cdots k_n$ with respect to the following weights of variables:

$$w_i = k_1 \cdots k_{i-1}(k_{i+1} \cdots k_n - k_{i+2} \cdots k_n + k_{i+3} \cdots k_n - \cdots \pm k_n \mp 1).$$

Computing partial derivatives, we find that the moduli algebra of $T$ is obtained from the algebra of formal power series by imposing the relations

$$\begin{aligned} 0 &= -k_1 x_1^{k_1-1} x_2, \\ x_1^{k_1} &= -k_2 x_2^{k_2-1} x_3, \\ &\cdots\cdots\cdots\cdots \\ x_{n-2}^{k_{n-2}} &= -k_{n-1} x_{n-1}^{k_{n-1}-1} x_n, \\ x_{n-1}^{k_{n-1}} &= -k_n x_n^{k_n-1}. \end{aligned}$$

Taking a suitable linear combination of these relations multiplied, respectively, by $x_1, \ldots, x_n$, we obtain the relation

$$x_n^{k_n} = 0.$$

As in the previous example, this implies that a monomial basis of the moduli algebra can be chosen in $P(k_1, \ldots, k_n)$. However, this time, this basis will not coincide with $P(k_1, \ldots, k_n)$. Namely, from the relations listed above we successively obtain that the monomials

$$\begin{aligned} &x_1^{k_1-1} x_2, \\ &x_1^{k_1-1} x_3^{k_3-1} x_4, \\ &x_1^{k_1-1} x_3^{k_3-1} x_5^{k_5-1} x_6, \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{9}$$

become equal to zero in the moduli algebra. Let us prove that those monomials in $P(k_1, \ldots, k_n)$ which are not divisible by any of the monomials in (9) constitute a basis of the moduli algebra.

The monomials in $P(k_1, \ldots, k_n)$ which are divisible by one of the monomials in (9) decompose into the pairwise disjoint sets

$$\begin{aligned} &\{x_1^{k_1-1} x_2^{i_2} x_3^{i_3} \cdots x_n^{i_n} \mid 0 < i_2 < k_2,\ 0 \leqslant i_3 < k_3,\ \ldots,\ 0 \leqslant i_n < k_n\}, \\ &\{x_1^{k_1-1} x_3^{k_3-1} x_4^{i_4} x_5^{i_5} \cdots x_n^{i_n} \mid 0 < i_4 < k_4,\ 0 \leqslant i_5 < k_5,\ \ldots,\ 0 \leqslant i_n < k_n\}, \\ &\{x_1^{k_1-1} x_3^{k_3-1} x_5^{k_5-1} x_6^{i_6} x_7^{i_7} \cdots x_n^{i_n} \mid 0 < i_6 < k_6,\ 0 \leqslant i_7 < k_7,\ \ldots,\ 0 \leqslant i_n < k_n\}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots . \end{aligned}$$

The cardinality of the union of these sets is

$$(k_2 - 1)k_3 \cdots k_n + (k_4 - 1)k_5 \cdots k_n + (k_6 - 1)k_7 \cdots k_n + \ldots,$$

where the sum ends with $k_n - 1$ if $n$ is even and with 1 if $n$ is odd.

On the other hand, employing the above formula of [1], we see that

$$\prod_{i=1}^{n} \left( \frac{d}{w_i} - 1 \right) = k_1 \cdots k_n - (k_2 - 1)k_3 \cdots k_n - (k_4 - 1)k_5 \cdots k_n - (k_6 - 1)k_7 \cdots k_n - \ldots,$$

so that all remaining monomials form a basis of the moduli algebra.

**Theorem 1.3** [5]. *Let a polynomial $f_0(x_1, \dots, x_n)$ contain at most $n$ monomials and have an isolated singularity at the origin. Then it contains exactly $n$ monomials and decomposes into a disjoint sum of cycles and chains.*

**Remark 1.4.** Theorem 1.3 implies that if $f$ has the form (5), then Condition 0.2 almost always reduces simply to the requirement that all variables occur in the monomial $u$ to positive powers. An additional restriction is needed only if the number of variables equals 2 and $f_0$ is either a chain $x_1^{k_1}x_2 + x_2^{k_2}$ (then $u$ must be divisible by $x_2^2$) or a cycle $x_1^{k_1}x_2 + x_1x_2^{k_2}$ (in this case, $u$ must be divisible by $x_1^2x_2^2$).

## 2. Calculations Using Kouchnirenko's Formula

Let us remind that the Newton polyhedron $P(f)$ of a formal series $f$ is the convex hull of the cones $p + \mathbb{R}_+^n$, where $p$ ranges over the support of the series $f$ (the set of degree vectors of its members), or, which is the same thing, the union of cones $p + \mathbb{R}_+^n$, where $p$ ranges over the convex hull of the support of the series $f$.

Under certain nondegeneracy conditions on the $(n-1)$-dimensional faces $\Delta_1, \dots, \Delta_m$ of the Newton polyhedron of an isolated singularity $f$, the Milnor number $\mu(f) := \dim \mathrm{A}_f$ can be calculated by Kouchnirenko's formula [3, 1.10 (ii)]

$$\mu(f) = n!\, V_n - (n-1)!\, V_{n-1} + \dots + (-1)^{n-1} 1!\, V_1 + (-1)^n,$$

where $V_n$ is the sum of volumes of the pyramids $\Delta_i^-$ over the faces $\Delta_i$ with vertex at the origin and $V_k$ for $k < n$ is the sum of $k$-dimensional volumes of the intersections of the union of these pyramids with all $k$-dimensional coordinate planes. The nondegeneracy condition is definitely satisfied for any simplicial face which intersects the support of the singularity $f$ only in its vertices [4, remark 9]. As we will see, in our case, all faces of the Newton polyhedron are precisely of this kind.

Consider now a singularity $f$ of the form (5) satisfying Condition 0.2. Let us assume that the variables are numbered in such a way that conditions (4) hold and denote the degree vectors of the monomials $u_1, \dots, u_n, u$ by $p_1, \dots, p_n, p$, respectively.

Obviously, the polyhedron $P(f_0)$ has a unique compact face, namely, the simplex $\Delta$ with vertices $p_1, \dots, p_n$. To find the compact faces of $P(f)$, we need Theorem 3 of [2] in a slightly modified form (see Theorem 2.2 below), as well as the following definition.

**Definition 2.1.** We will say that a square matrix $A$ is *irreducible* if it does not preserve any nontrivial coordinate subspace, i.e., one cannot obtain a corner of zeros with vertex on the diagonal by using any permutation of rows and the same permutation of columns.

**Theorem 2.2.** *Let $A$ be an irreducible matrix in which all off-diagonal entries are nonpositive. Suppose that there exists a column vector $W > 0$ with $AW > 0$. Let $K$ denote the coordinate orthant $\{X \in \mathbb{R}^n \mid X \geqslant 0\}$. Then the cone $AK$ strictly contains $K$, and the same holds for the transpose matrix $A'$.*

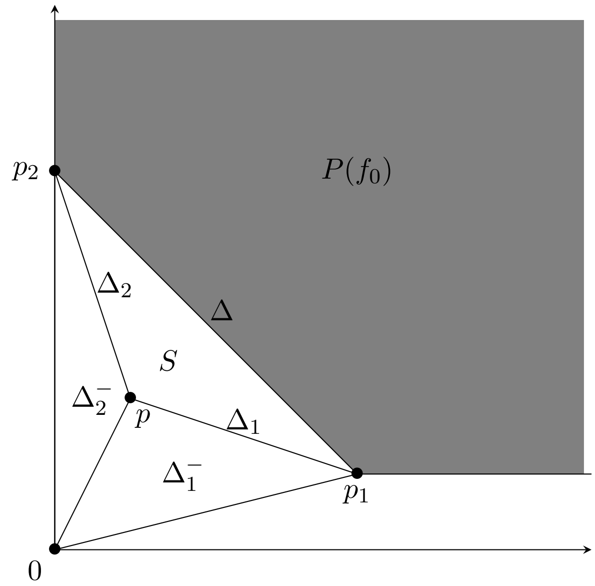
**Proof.** Irreducibility easily implies (see Lemma 11 in [2]) that $K \cap A^{-1}K \subset K^\circ \cup \{0\}$. (Here $A^{-1}K$ is the full inverse image of the cone $K$, and $K^\circ$ is its interior.) Since $A^{-1}K$ contains the vector $W \in K^\circ$, we have $A^{-1}K \subset K^\circ \cup \{0\}$, and, thus, $K \subset AK^\circ \cup \{0\}$, i.e., the cone $AK$ strictly contains $K$.

Switching to the conjugate cones and taking into account the fact that the cone $K$ is self-conjugate and the cone $AK$ is conjugate to $(A')^{-1}K$, we see that $A'K$ strictly contains $K$, as required. $\square$

**Corollary 2.3.** *If a matrix $A$ satisfies the conditions of the theorem, then its determinant is positive.*

**Proof.** Since $AK \supset K$, it follows that the matrix $A$ is nondegenerate. Further, it is obvious that, for any $t \geqslant 0$, the matrix $A + tE$ also satisfies all conditions of the theorem. It is clear from continuity considerations that the sign of its determinant does not depend on $t$; but $\det(A + tE)$ is a polynomial in $t$ with leading coefficient 1, which is surely positive for sufficiently large $t$. $\square$

Let us apply Theorem 2.2 to the matrix $M_f$ with rows $p_1 - p, \ldots, p_n - p$. It follows from Condition 0.2 that the off-diagonal entries of $M_f$ are nonpositive. At the same time, the explicit form of the function $f$ shows that each row of this matrix can contain at most one zero. Hence, $M_f$ can be reducible only for $n = 2$. But in this case, irreducibility is also ensured by Condition 0.2. Furthermore, the condition that the weighted degree of the monomial $u$ is strictly less than that of each of the monomials $u_1, \ldots, u_n$ means that the linear combination of the columns of $M_f$ with coefficients equal to the weights of the variables has positive coordinates. Thus, the conditions of the theorem hold (with $W$ as the vector of weights) and, hence, the simplicial cone $p + K$ with vertex at $p$ whose edges pass through the points $p_1, \ldots, p_n$ strictly contains the corner $p + \mathbb{R}^n_+$ without the point $p$. This implies that the polyhedron $P(f)$ is the union of the polyhedron $P(f_0)$ and the simplex $S$ with vertices at the points $p_1, \ldots, p_n, p$, and, moreover, all faces $\Delta_1, \ldots, \Delta_n$ of this simplex different from $\Delta$ are not continuations of faces of $P(f_0)$. Thus, the compact faces of the polyhedron $P(f)$ are $\Delta_1, \ldots, \Delta_n$. (See the picture for the case $f = x^4y + y^5 + xy^2$.)




Furthermore, the cone $p - K$ opposite to the one considered above strictly contains the corner $p - \mathbb{R}^n_+$ without $p$ and, in particular, the origin. This means that the point $p$ lies strictly inside the pyramid $\Delta^-$, which implies that the intersection of this pyramid with any coordinate plane cannot contain interior points of the simplex $S$ or of its face $\Delta$ and is thus contained in the union of pyramids $\Delta_i^-$.

It follows from all of the above that the difference between the Milnor numbers of the singularities $f_0$ and $f$ calculated by Kouchnirenko's formula is equal to $n!$ times the volume of the simplex $S$, i.e., to the determinant of the matrix $M_f$. Theorem 0.3 is thus proved.

## 3. Algebraic Approach

Retaining the assumptions and notation of the previous section, we are going to employ the algebra

$$\mathrm{P}_f := \mathbb{C}[x_1, \ldots, x_n]/J_{\mathrm{pol}}(f)$$

obtained by replacing the algebra of formal power series with the algebra of polynomials in the definition of $\mathrm{A}_f$. Here $J_{\mathrm{pol}}(f)$ is the ideal of $\mathbb{C}[x_1, \ldots, x_n]$ generated by the partial derivatives of the polynomial $f$, so that $J(f)$ is the closure of $J_{\mathrm{pol}}(f)$ in the formal topology of the algebra $\mathbb{C}[\![x_1, \ldots, x_n]\!]$.

Note that, for an isolated quasihomogeneous singularity $f_0$, the algebra $\mathrm{P}_{f_0}$ coincides with $\mathrm{A}_{f_0}$, since the ideal $J_{\mathrm{pol}}(f)$ contains all quasihomogeneous polynomials of sufficiently large degree.

As for the nonquasihomogeneous case, the following assertion is valid.

**Proposition 3.1.** *Suppose that an ideal $I = (f_1+g_1, \dots, f_n+g_n)$ of the algebra $\mathbb{C}[x_1, \dots, x_n]$ is generated by polynomials $f_1+g_1, \dots, f_n+g_n$, where the polynomials $f_1, \dots, f_n$ are quasihomogeneous with respect to some positive weights and*

$$\deg(g_i) < \deg(f_i), \qquad i = 1, \dots, n,$$

*with respect to these weights. Suppose also that the algebra $\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_n)$ is finite-dimensional. Then the algebra $\mathbb{C}[x_1, \dots, x_n]/I$ is finite-dimensional of the same dimension, and, moreover, any basis of the former algebra consisting of quasihomogeneous polynomials is also a basis of the latter.*

**Proof.** Finite-dimensionality of the algebra $\mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_n)$ means that the (quasihomogeneous) polynomials $f_1, \dots, f_n$ form a regular sequence in $\mathbb{C}[x_1, \dots, x_n]$, which implies that $\mathbb{C}[x_1, \dots, x_n]$ is a free module over its subalgebra $\mathbb{C}[f_1, \dots, f_n]$ generated by the polynomials $f_1, \dots, f_n$. Further, let $H \subset \mathbb{C}[x_1, \dots, x_n]$ be a quasihomogeneous subspace complementary to the ideal $(f_1, \dots, f_n)$. Then any basis (over $\mathbb{C}$) of $H$ is a basis of the algebra $\mathbb{C}[x_1, \dots, x_n]$ as a module over $\mathbb{C}[f_1, \dots, f_n]$. In other words, any element of the algebra $\mathbb{C}[x_1, \dots, x_n]$ can be uniquely represented as a polynomial in $f_1, \dots, f_n$ with coefficients in $H$. Considering successively leading terms, one ascertains that all this remains true if one replaces the polynomials $f_1, \dots, f_n$ with $f_1 + g_1, \dots, f_n + g_n$. This means that the subspace $H$ is complementary to the ideal $I$ as well. $\square$

**Corollary 3.2.** *Let $f$ be a polynomial of the form $f_0 + f_<$, where $f_0$ is an isolated quasihomogeneous singularity and $f_<$ consists of monomials of degree strictly less than that of $f_0$. Then the algebras $\mathrm{P}_f$ and $\mathrm{P}_{f_0} = \mathrm{A}_{f_0}$ have equal dimensions*; *moreover, any monomial basis of $\mathrm{P}_{f_0}$ is also a basis for $\mathrm{P}_f$.*

**Remark 3.3.** If one tries to use a similar trick for terms of lowest rather than highest degree, then one has to deal with a decreasing filtration. The completeness of the algebra is sufficient for obtaining the desired result in this case, so that this can be done in the algebra of formal power series. Thus, it is also true that if $\deg(g_i) > \deg(f_i)$, $i = 1, \dots, n$, then any monomial basis of the algebra $\mathbb{C}\llbracket x_1, \dots, x_n \rrbracket/(f_1, \dots, f_n)$ is also a basis of $\mathbb{C}\llbracket x_1, \dots, x_n \rrbracket/(f_1 + g_1, \dots, f_n + g_n)$. As a corollary, one obtains, in particular, Theorem 0.1.

Unlike the algebra $\mathrm{A}_f$, which is obviously local, the algebra $\mathrm{P}_f$ may have several maximal ideals. Like every finite-dimensional algebra, it is a direct sum of local algebras, one of which is obviously $\mathrm{A}_f$. In our particular case, $\mathrm{P}_f$ has a special feature, which will be crucial in this section.

**Lemma 3.4.** *Suppose that a polynomial $f$ satisfies Condition* 0.2. *Then in any local quotient algebra $A$ of $\mathrm{P}_f$ either all variables become nilpotent* (*so that $A$ is a quotient algebra of $\mathrm{A}_f$*) *or all variables become invertible.*

**Proof.** It suffices to show that, for any homomorphism $\chi\colon \mathrm{P}_f \to \mathbb{C}$, if $\chi(x_i) = 0$ for some variable $x_i$, then all variables map to zero, too.

Note that if $\chi(x_i) = 0$, then by Condition 0.2 we have $\chi(\partial u/\partial x_k) = 0$ for all $k \neq i$ and hence $\chi(\partial f_0/\partial x_k) = \chi(\partial f/\partial x_k) = 0$. Therefore, if $\chi(x_i) = \chi(x_j) = 0$ for some distinct $i$ and $j$, then $\chi(\partial f_0/\partial x_k) = 0$ for all $k$. Since $f_0$ is an isolated singularity, it follows that $\chi(x_k) = 0$ for all $k$. We arrive at the same conclusion if $\chi(x_i) = 0$ and $x_i$ occurs in $u$ to a power greater than 1.

Suppose now that $f_0$ is a disjoint sum of (quasihomogeneous) polynomials $f_1$ and $f_2$ and $\chi(x_i) = 0$ for some variable $x_i$ which occurs in $f_1$. Then, for any variable $x_j$ occurring in $f_2$, we have $\chi(\partial f_2/\partial x_j) = \chi(\partial f_0/\partial x_j) = \chi(\partial f/\partial x_j) = 0$. Since $f_2$ is an isolated singularity, this implies that $\chi(x_j) = 0$ for all such $j$, and, in virtue of what we have proved, for any $j$ whatever.

It thus remains to prove the statement formulated at the beginning of the proof in the case when $f_0$ is a cycle or a chain. Moreover, one may assume that either $n > 2$ or $i = 1$ and $f_0$ is a chain of length 2, since otherwise $f_0$ is divisible by $x_i$ and, according to Condition 0.2, $u$ is divisible by $x_i^2$.

If $f_0$ is a cycle of length exceeding 2 and $\chi(x_i) = 0$, then, considering the indices of variables modulo $n$ and differentiating with respect to $x_{i+1}$, we obtain

$$\chi(\partial f/\partial x_{i+1}) = \chi(\partial f_0/\partial x_{i+1}) = \chi(k_{i+1} x_{i+1}^{k_{i+1}-1} x_{i+2}) = 0,$$

which implies that either $\chi(x_{i+1}) = 0$ or $\chi(x_{i+2}) = 0$. In any case, according to the above discussion, $\chi(x_k) = 0$ for all $k$.

A similar argument works also in the case when $f_0$ is a chain and $i < n$. (For $i = n-1$, the variable $x_{i+2}$ is absent.) If $f_0$ is a chain of length exceeding 2 and $i = n$, then, differentiating with respect to $x_{n-1}$, we obtain

$$\chi(\partial f/\partial x_{n-1}) = \chi(\partial f_0/\partial x_{n-1}) = \chi(x_{n-2}^{k_{n-2}}) = 0,$$

which implies that $\chi(x_{n-2}) = 0$ and, hence, $\chi(x_k) = 0$ for all $k$. □

Thus, we obtain

$$\mathrm{P}_f = \mathrm{A}_f \oplus \mathrm{P}_f[x_1^{-1}, \dots, x_n^{-1}],$$

where $\mathrm{P}_f[x_1^{-1}, \dots, x_n^{-1}]$ is the localization of the algebra $\mathrm{P}_f$ obtained by inverting all variables.

Note that since all variables $x_i$ in the algebra $\mathrm{P}_f$ are invertible, the relation $\partial f/\partial x_i = 0$ in it is equivalent to $x_i \partial f/\partial x_i = 0$. Hence the algebra $\mathrm{P}_f[x_1^{-1}, \dots, x_n^{-1}]$ can be obtained by imposing the relations

$$\sum_{j=1}^{n} \nu_{ji} u_j + a_i u = 0, \qquad i = 1, \dots, n.$$

Since the degree vectors of the monomials $u_1, \dots, u_n$ are linearly independent, it follows that the matrix $(\nu_{ji})_{1 \leqslant j,i \leqslant n}$ is nonsingular. Considering the above relations as a system of linear equations with unknowns $u_1, \dots, u_n$, we obtain a unique solution of the form $u_i = \lambda_i u$, $i = 1, \dots, n$, with $\lambda_1, \dots, \lambda_n \in \mathbb{R}^*$. Multiplying the variables $x_1, \dots, x_n$ by suitable nonzero numbers, one can make all these coefficients $\lambda_i$ equal to 1.

Consequently, under the normalization of variables specified above, the defining relations of the algebra $\mathrm{P}_f[x_1^{-1}, \dots, x_n^{-1}]$ have the form $u_i = u$, $i = 1, \dots, n$. This means that $\mathrm{P}_f[x_1^{-1}, \dots, x_n^{-1}]$ is the group algebra of the finite Abelian group $G$ with generators $x_1, \dots, x_n$ and defining relations $u_1 u^{-1} = \dots = u_n u^{-1} = 1$. The dimension of this algebra is equal to the order of the group $G$, which, in its turn, is equal to the absolute value of the determinant of the matrix of defining relations, i.e., of the matrix $M_f$. We thus have obtained another, purely algebraic, proof of Theorem 0.3.

## 4. An Explicit Basis for the Brieskorn–Pham Case

Let us now present yet another proof of the defect formula for the particular case when $f_0$ is the Brieskorn–Pham singularity (7). As a byproduct, we will produce two monomial bases of $\mathrm{A}_f$, one of which lies in the fundamental parallelepiped (8).

Using Remark 0.4, we may assume that

$$f = \frac{a_1}{p_1} x_1^{p_1} + \dots + \frac{a_n}{p_n} x_n^{p_n} - x_1^{a_1} \cdots x_n^{a_n}. \tag{10}$$

Then

$$\frac{x_i}{a_i} \frac{\partial f}{\partial x_i} - \frac{x_j}{a_j} \frac{\partial f}{\partial x_j} = x_i^{p_i} - x_j^{p_j} \in J(f),$$

and we can factor the algebra $\mathbb{C}[\![x_1, \dots, x_n]\!]$ by $J(f)$ in two steps: first we factor it by the ideal $I(f)$ generated by the differences $x_i^{p_i} - x_j^{p_j}$, and then we factor the obtained algebra by $J(f)/I(f)$.

Let us first study the algebra $\mathrm{P}_f = \mathbb{C}[x_1, \dots, x_n]/J_{\mathrm{pol}}(f)$ defined in the previous section. It can be obtained in two steps: first we factor the algebra $\mathbb{C}[x_1, \dots, x_n]$ by the ideal $I_{\mathrm{pol}}(f)$ generated by the differences $x_i^{p_i} - x_j^{p_j}$, and then we factor the obtained algebra by the ideal $J_{\mathrm{pol}}(f)/I_{\mathrm{pol}}(f)$.

It is convenient to embed the algebra $\mathbb{C}[x_1, \dots, x_n]$ into the algebra $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of Laurent polynomials. Let us denote the ideals of $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ generated by $J_{\mathrm{pol}}(f)$ and $I_{\mathrm{pol}}(f)$ with $J_{\mathrm{Lau}}(f)$ and $I_{\mathrm{Lau}}(f)$, respectively.

The algebra

$$\mathrm{L}_f^\infty = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/I_{\mathrm{Lau}}(f)$$

can be viewed as the group algebra of the (infinite) Abelian group $G^\infty$ with generators $g_1, \dots, g_n$ and defining relations $g_i^{p_i} = g_j^{p_j}$.

Adding one more generator $g_0$, we write the defining relations of $G^\infty$ in the form

$$g_0 = g_1^{p_1} = \dots = g_n^{p_n}. \tag{11}$$

Then a canonical form of an element $u = g_0^{k_0} g_1^{k_1} \cdots g_n^{k_n} \in G^\infty$ is

$$g_0^q g_1^{r_1} \cdots g_n^{r_n} \quad \text{with } 0 \leqslant r_1 < p_1, \dots, 0 \leqslant r_n < p_n. \tag{12}$$

The numbers $r_i$ $(i = 1, \dots, n)$ are determined by $k_i = q_i p_i + r_i$ and $q = k_0 + q_1 + \dots + q_n$.

Similarly, the algebra

$$\mathrm{P}_f^\infty = \mathbb{C}[x_1, \dots, x_n]/I_{\mathrm{pol}}(f).$$

is the semigroup algebra of the Abelian semigroup (with a unit) $G_+^\infty$ given by the generators $\tilde{g}_1, \dots, \tilde{g}_n$ and defining (semigroup) relations $\tilde{g}_i^{p_i} = \tilde{g}_j^{p_j}$.

Adding an auxiliary generator $\tilde{g}_0$, we write the defining relations of $G_+^\infty$ in the form

$$\tilde{g}_0 = \tilde{g}_1^{p_1} = \dots = \tilde{g}_n^{p_n}.$$

Then a canonical form of elements of this semigroup is

$$\tilde{g}_0^q \tilde{g}_1^{r_1} \cdots \tilde{g}_n^{r_n} \qquad \text{with } q \geqslant 0,\ 0 \leqslant r_1 < p_1, \dots, 0 \leqslant r_n < p_n. \tag{13}$$

We see that the natural homomorphism $G_+^\infty \to G^\infty$ taking $\tilde{g}_i$ to $g_i$ is an embedding. Its image consists of the monomials $g_0^{k_0} g_1^{k_1} \cdots g_n^{k_n}$ with $k_0, k_1, \dots, k_n \geqslant 0$, which we will call *positive*. In what follows, we will identify the semigroup $G_+^\infty$ with the subsemigroup of $G^\infty$ formed by the positive monomials (and the elements $\tilde{g}_i$ will be identified with $g_i$).

Accordingly, the natural homomorphism $\mathrm{P}_f^\infty \to \mathrm{L}_f^\infty$ is an embedding whose image is spanned by the positive monomials; thus, we will identify the algebra $\mathrm{P}_f^\infty$ with the subalgebra of $\mathrm{L}_f^\infty$ spanned by the positive monomials.

Let us now note that the algebra

$$\mathrm{L}_f = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/J_{\mathrm{Lau}}(f)$$

is the group algebra of the finite Abelian group $G$ obtained from $G^\infty$ by imposing the additional relations

$$g_i^{p_i - 1} = g_1^{a_1} \cdots g_{i-1}^{a_{i-1}} g_i^{a_i - 1} g_{i+1}^{a_{i+1}} \cdots g_n^{a_n}. \tag{14}$$

Since these relations imply the relations $g_i^{p_i} = g_j^{p_j}$, the order $\mathrm{def}(f)$ of the group $G$ is equal to the determinant of the matrix

$$M_f = \begin{pmatrix} p_1 - a_1 & -a_2 & \dots & -a_n \\ -a_1 & p_2 - a_2 & \dots & -a_n \\ \vdots & \vdots & \ddots & \vdots \\ -a_1 & -a_2 & \dots & p_n - a_n \end{pmatrix}.$$

Note that relations (14) reduce to the single relation $g_0 = g_1^{a_1} \cdots g_n^{a_n}$ in the group $G^\infty$, so that adding them means factoring by the (infinite) cyclic subgroup $\langle T \rangle$ generated by the element

$$T = g_0 g_1^{-a_1} \cdots g_n^{-a_n}.$$

We will refer to the cosets of $\langle T \rangle$ in $G^\infty$ as $T$-lines, so that the elements of $G$ can be viewed as $T$-lines.

Similarly, the algebra $\mathrm{P}_f$ is the semigroup algebra of the Abelian semigroup $G_+$ obtained from $G_+^\infty$ by imposing the additional relations (14). However, this time, these relations, when regarded as semigroup relations, no longer reduce to one relation.

We say that two elements $u, v \in G_+^\infty$ are equivalent and write $u \sim v$ if relations (14) imply (in the semigroup sense) $u = v$. A necessary condition for this is that $u$ and $v$ belong to the same

$T$-line. An element $u \in G_+^\infty$ written in the canonical form (13) is equivalent to $Tu$ if and only if one of the relations in (14) is applicable to $u$, which means that

$$ug_1^{-a_1} \cdots g_{i-1}^{-a_{i-1}} g_i^{-a_i+1} g_{i+1}^{-a_{i+1}} \cdots g_n^{-a_n} \in G_+^\infty$$

for some $i$. This can be reformulated as the arithmetic condition

$$\#\{j \mid r_j < a_j\} \leqslant \begin{cases} q+1 & \text{if } r_j = a_j - 1 \text{ for some } j, \\ q & \text{if } r_j \neq a_j - 1 \text{ for all } j. \end{cases} \tag{15}$$

It is obvious that two elements of a $T$-line are equivalent if and only if any two consecutive elements of the interval between them are equivalent. This implies that the equivalence classes lying on a given $T$-line are some intervals (possibly consisting only of one point) whose union is the intersection of this line with $G_+^\infty$. Let us call them $T$-intervals. The elements of the semigroup $G_+$ can be viewed as $T$-intervals.

Since the degree of $T$ is positive, the degree of monomials linearly increases along any $T$-line. With respect to the natural ordering on a $T$-line, all sufficiently small elements do not belong to the semigroup $G_+^\infty$, while all sufficiently large elements belong to it and are pairwise equivalent. Thus, there are finitely many $T$-intervals on any given $T$-line, one of which is infinite and all others are finite.

Under the passage from $\mathrm{P}_f$ to $\mathrm{A}_f$, infinite $T$-intervals go to zero and the images of finite intervals constitute a basis of $\mathrm{A}_f$. Thus, we arrive at the equality

$$\dim \mathrm{A}_f = \dim \mathrm{P}_f - |G| = \dim \mathrm{P}_f - \operatorname{def}(f),$$

which again gives the defect formula.

Moreover, the finite intervals are exactly those having a (unique) largest element, and these largest elements are exactly those to which none of the relations in (14) is applicable. Thereby, we come to the following conclusion.

**Theorem 4.1.** *The images of positive monomials* (13) *not satisfying condition* (15) *constitute a basis of the algebra* $\mathrm{A}_f$.

## 5. Parallelepipedal Basis for the Brieskorn–Pham Case

Another possibility is to use the smallest elements of $T$-intervals instead of the largest ones. An element $u$ of the semigroup $G_+^\infty$ written in the canonical form (13) is the smallest one in its $T$-interval if and only if none of the relations in (14) is applicable to $u$ in the opposite direction, that is, $ug_i^{-(p_i-1)} \notin G_+^\infty$ for all $i$; this means that $0 \leqslant r_i < p_i - 1$ for $i = 1, \dots, n$. In this way we obtain a parallelepipedal basis for the algebra $\mathrm{P}_f$. It remains to determine which of these elements belong to infinite $T$-intervals. We cannot do this in the general case. There is, however, one case when this can be done.

Namely, the following assertion holds.

**Proposition 5.1.** *Suppose that*

$$p_1 \geqslant na_1, \dots, p_n \geqslant na_n. \tag{16}$$

*Then a monomial* $x_1^{k_1} \cdots x_n^{k_n}$ *with* $0 \leqslant k_i < p_i - 1$ *for* $1 \leqslant i \leqslant n$ *is zero in* $\mathrm{A}_f$ *if and only if it is divisible by one of the* $n!$ *monomials*

$$x_1^{\sigma(1)a_1 - 1} \cdots x_n^{\sigma(n)a_n - 1},$$

*where* $\sigma$ *ranges over all permutations of the set* $\{1, \dots, n\}$.

For brevity, let us agree to denote the element of $G^\infty$ represented in the canonical form (12) by the symbol $(q; r_1, \dots, r_n)$.

**Proof.** According to the proof of Theorem 4.1, we have to show that, for some permutation $\sigma$, the operator $T$ described above can be applied to a symbol of the form $(0;\sigma(1)a_1-1,\dots,\sigma(n)a_n-1)$ infinitely many times without leaving $A_\infty^+$. Indeed, one easily sees that

$$\begin{aligned}
&T(0;\sigma(1)a_1-1,\dots,\sigma(n)a_n-1)\\
&\quad=(0;(\sigma(1)-1)a_1-1,\dots,p_{\sigma^{-1}(1)}-1,\dots,(\sigma(n)-1)a_n),\\
&T^2(0;\sigma(1)a_1-1,\dots,\sigma(n)a_n-1)\\
&\quad=(0;(\sigma(1)-2)a_1-1,\dots,p_{\sigma^{-1}(1)}-a_{\sigma^{-1}(1)}-1,\dots,p_{\sigma^{-1}(2)}-1,\dots,(\sigma(n)-2)a_n),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&T^k(0;\sigma(1)a_1-1,\dots,\sigma(n)a_n-1)=(0;c_{1k},\dots,c_{nk}),
\end{aligned}$$

where

$$c_{ik}=\begin{cases}(\sigma(i)-k)a_i-1, & \sigma(i)>k,\\ p_i-(k-\sigma(i))a_i-1, & \sigma(i)\leqslant k\end{cases}$$

for $k<n$, so that $T$ still can be applied after each of these steps, and

$$T^n(0;\sigma(1)a_1-1,\dots,\sigma(n)a_n-1)=(0;p_1-(n-\sigma(1))a_1-1,p_n-(n-\sigma(n))a_n-1).$$

But the latter symbol satisfies the inequalities

$$p_1-(n-\sigma(1))a_1-1\geqslant\sigma(1)a_1-1,\dots,p_n-(n-\sigma(n))a_n-1\geqslant\sigma(n)a_n-1,$$

so that $(0;\sigma(1)a_1-1,\dots,\sigma(n)a_n-1)$ indeed represents zero in $\mathrm{A}_f$. □

The converse statement is a direct corollary of the following lemma.

**Lemma 5.2.** *There exists a permutation $\sigma$ for which $k_i\geqslant\sigma(i)a_i-1$, $1\leqslant i\leqslant n$, if and only if, for any nonempty subset $S\subseteq\{1,\dots,n\}$ of cardinality $m$, there is an $i\in S$ with $k_i\geqslant ma_i-1$.*

**Proof.** If $k_i\geqslant\sigma(i)a_i-1$ for all $i$, then any subset $S\subset\{1,\dots,n\}$ contains an element $i$ for which $\sigma(i)\geqslant m$, and for this $i$, the inequality $k_i\geqslant ma_i-1$ holds.

Conversely, if the condition of the lemma is satisfied, then, taking $S=\{1,\dots,n\}$, we find $i$ for which $k_i\geqslant na_i-1$. We set $\sigma(i)=n$. Further, in $S=\{1,\dots,n\}\setminus\{i\}$ there is a $j$ for which $k_j\geqslant(n-1)a_j-1$. We set $\sigma(j)=n-1$. Next, we consider $\{1,\dots,n\}\setminus\{i,j\}$, and so on. Evidently, this will produce a permutation with the required properties. □

**Proposition 5.3.** *Under the conditions of Proposition 5.1 (i.e., if inequalities (16) hold), a basis of the algebra $\mathrm{A}_f$ can be composed of those monomials $x_1^{k_1}\cdots x_n^{k_n}$ with $0\leqslant k_i\leqslant p_i-2$ for which either $k_i\leqslant a_i-2$ for some index $i$, $k_i\leqslant 2a_i-2$ for two indices, $k_i\leqslant 3a_i-2$ for three indices, …, or $k_i\leqslant na_i-2$ for all $i$.*

**Proof.** We use the previous proposition and lemma. Proposition 5.1 implies that the monomial $x_1^{k_1}\cdots x_n^{k_n}$, where $0\leqslant k_i<p_i-1$ for $1\leqslant i\leqslant n$, is nonzero in $\mathrm{A}_f$ if and only if there exists an $m\geqslant 1$ such that $k_i<ma_i-1$ for $m$ distinct indices $i$.

Thus, it suffices to prove that if an $n$-tuple $(k_1,\dots,k_n)$ does not satisfy the equivalent conditions of Lemma 5.2, then the operator $T$ can be applied to the symbol $(0;k_1,\dots,k_n)$ only finitely many times. In other words, one can find indices $1\leqslant i_1<\dots<i_m\leqslant n$ for which $k_{i_1}<ma_{i_1}-1,\dots,k_{i_m}<ma_{i_m}-1$.

Indeed, we have

$$T^{m-1}(0;k_1,\dots,k_n)=(m-1;k_1-(m-1)a_1,\dots,k_n-(m-1)a_n),$$

and since we know that there are at least $m$ distinct indices $i$ for which $k_i-(m-1)a_i<a_i-1$, we conclude that condition (15) is violated, so that the operator $T$ can no longer be applied. □

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A. Razmadze Mathematical Institute, Georgian Academy of Sciences
e-mail: alela@rmi.ge

A. Razmadze Mathematical Institute, Georgian Academy of Sciences
e-mail: jib@rmi.ge

M. V. Lomonosov Moscow State University
e-mail: evinberg@gmail.com


*Translated by M. A. Jibladze*